Precautionary Demand for Money in a Monetary-Search Business Cycle Model *

Preliminary and Incomplete

Irina A. Telyukova
University of California, San Diego

Ludo Visschers
Simon Fraser University

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Abstract

We investigate the quantitative implications of precautionary demand for money for business cycle dynamics of velocity of money and other nominal aggregates. There is a standing challenge in monetary macroeconomics to account for business cycle dynamics of nominal variables, as previous business cycle models that have tried to incorporate demand for money have failed to generate realistic predictions in this regard. Our stance is that part of this failure results from the fact that demand for money in those previous models is deterministic, since agents in them face only aggregate risk, whereas we believe idiosyncratic risk to be important as well. We conduct the exercise inside a monetary search model, as our additional goal is to put to the test a recent generation of such frictional models to examine whether their quantitative predictions are realistic, and whether they generate additional insight into business cycle data that non-search models of money cannot generate. On the first question, our results suggest that precautionary demand for money plays a substantial role in accounting for business cycle behavior of velocity of money and other nominal variables such as inflation and nominal interest rates. On the second question, our preliminary results indicate that the search frictions generate discernible, though small, quantitative effects, helping the model account for relevant properties of the data, but not very significantly. The biggest quantitative potential for search frictions in this setting is in the dynamic implications they generate for inventories and markups in the retail sector. Assessment of this potential is in progress.

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1 Introduction

In this paper, we study, theoretically and quantitatively, aggregate business cycle implications of precautionary demand for money. It is an outstanding challenge in the literature to account for business cycle behavior of nominal aggregates, their interaction with real aggregates, as well as to account for seemingly substantial real effects of monetary policy. Business cycle models that have tried to incorporate money through, for example, cash-in-advance constraints, have done so by assuming that agents face only aggregate risk, which has resulted in deterministic demand for money, and found little interaction between real and nominal variables (see, e.g., Cooley and Hansen, 1995). Yet precautionary motive for holding liquidity seems to be strong in the data (Telyukova, 2008), implying that *idiosyncratic* risk may play a key role for money demand, and thus that its aggregate implications are important to investigate. The first goal of this paper is to test this hypothesis. Thus the first set of questions we want to answer is: (a) What are the aggregate implications of precautionary demand for money? (b) Can it help account for business cycle dynamics of velocity of money, interest rates and inflation? (c) Can it help to account for real effects of monetary policy?

Moreover, we conduct our investigation in a microfounded monetary search model with a productive real sector. While it is not the minimal needed model in which we could address the questions we outlined above, in that a stochastic version of a cash-credit good model could get us some of the answers, it is a natural way in which to incorporate both real and nominal activity. Search models of money in the style of Lagos and Wright (2005) combine frictionless and frictional trade in the same economy, thus allowing for a natural incorporation of both idiosyncratic risk and business cycles. However, quantitative implications of monetary search models, including those that incorporate a real sector and business cycle fluctuations are not yet well known. Our second goal is to expand the knowledge of the quantitative properties of these models by investigating ours in a disciplined calibration exercise. The second set of questions that we seek to answer is: (d) Is our monetary search model capable of generating realistic aggregate dynamics? (e) What does the explicit modeling of frictions in trade add to our understanding of the data, relative to reduced-form models of money?

Starting with a model that combines a productive real sector with a competitive-search monetary retail sector, we incorporate uninsurable idiosyncratic preference risk which creates a precautionary motive for holding liquidity. Telyukova (2008) demonstrates that this preference uncertainty is significant in the data, which in turn creates sizeable precautionary balance holdings, absent in most standard models. In a standard deterministic-demand setting, cash-credit good models, for example, calibrated to aggregate data cannot account for aggregate facts such as variability of velocity of money, correlation of velocity with output growth or money growth, correlation of inflation with nominal interest rates, and others, as Hodrick, Kocherlakota and Lucas (1991) have shown. The reason is that in such models, agents' money demand is almost entirely deterministic, because the only type of uncertainty the households face in these models is aggregate uncertainty, and the amount of aggregate uncertainty in the data is not large enough to generate significant precautionary motives for holding money in the model. Then, the cash-in-advance constraint almost always binds and money demand is made equivalent to cash-good consumption. This also implies that volatility of money demand is tightly linked to volatility of aggregate consumption. Aggregate consumption is not volatile enough in the data to generate enough volatility of money demand or other nominal aggregates.

We show that incorporating precautionary demand for money generated by unpredictable idiosyncratic variation, in combination with aggregate uncertainty, can help account for monetary issues mentioned above, by breaking the link between money demand and aggregate consumption. Agents generally hold more money than they spend, and money demand is no longer linked to *average* aggregate consumption, but rather to consumption of agents whose preference shock realizations make them constrained (i.e. they spend all of their balances) in trade. We show that, as a result, velocity of money can be significantly more volatile in this heterogeneous-agent setting, compared to standard deterministic cash-in-advance models.

We study this link quantitatively in a model that combines, in each period, both centralized and decentralized trade in a sequential manner, as in, for example, Lagos and Wright (2005). Agents' utility function in the centralized market features linear preferences on labor, which allows us to simplify the model by not having to keep track of the distribution of agents, but other features of the model will still render the model analytically intractable, which leads us to rely on computational methods to solve the model. The centralized market is otherwise much like a standard real business cycle model, except that trade in this market can be conducted using either money or credit, and agents have to decide how much money to carry out of the market. The production function in this market is subject to aggregate productivity shocks.

Like in Rocheteau and Wright (2005), instead of modeling bilateral trade in the decentralized market via bargaining, we focus on a competitive search setting, where the terms of trade in bilateral meetings are posted on "islands" ahead of the decentralized market, and agents choose which island to go to. At the start of the decentralized market, agents are subject to preference shocks which determine how much they want to consume during the subperiod, but the realization of the shock is not known at the time that agents make their portfolio decisions. This generates precautionary motive for holding liquidity. One additional feature of our model that is not standard in search models of money is that we allow capital, which exists in the centralized market, to be productive for the decentralized market, in the following way: we assume that the goods that are sold in decentralized trade are made in the centralized markets, using capital goods purchased in the same centralized markets. This introduces an explicit link between the real and monetary sectors of the economy.

One of the contributions of our work is the calibration of the model. Relatively few models in monetary search literature have been calibrated, and those that have (i.e. Aruoba, Waller and Wright, 2008, Wang and Shi, 2006) have been calibrated to aggregate targets. Because of this, such models tend to suffer from the problem of free parameters (such as bargaining weights) which are hard to pin down using aggregate data. The structure of our model, which is frictional but avoids inefficiencies that bargaining creates, helps us discipline it with the use of survey data in addition to aggregate targets. For example, the search setting will give us predictions about inventories and markups in the retail sector, which we can use as additional steady-state calibration targets; inventories and markups would not exist in a non-search setting. In addition, we will use micro data on liquid consumption from the Consumption Expenditure Survey, like in Telyukova (2008), to calibrate idiosyncratic preference risk in the decentralized market.

Once calibrated, we solve the model computationally to investigate the effects of real productivity shocks and monetary policy shocks. On our first set of questions, we find that precautionary demand for money alone makes a dramatic difference for the model in terms of helping it account for a variety of dynamic moments related to nominal aggregates in the data. We test these results by also computing a version of the model where we shut down the idiosyncratic risk, and find that without it, the model is incapable of reproducing any of the key moments in the data. On the second set of questions pertaining to the implications of search frictions, our preliminary findings are that they have discernible, but small, effects in helping the model account for the properties of the data in question, as well as in amplifying the real response of the economy to monetary policy. Qualitatively, search frictions do add wedges relative to a non-search model, but these do not seem quantitatively large, given our current calibration. The most significant potential for the search frictions seems to be in terms of the predictions that they generate for markups and inventories in the retail sector – these would be absent in a non-search model – in terms of their size and dynamic behavior. The quantitative investigation of these implications is currently work in progress, as we complete our calibration.

This paper is related to several strands of literature. On the subject of precautionary demand for liquidity, Hagedorn (2008) has demonstrated that strong liquidity effects that translate into significant aggregate implications can arise when precautionary demand for money is taken into account in an otherwise standard cash-credit good model. His setting is quite different from ours, in that he generates the liquidity effects using banks, which we abstract from; on the other hand, his model has no real sector and no aggregate uncertainty. Faig and Jerez (2006) also model precautionary demand for liquidity using preference shocks, in a setting reminiscent of, but different from, Rocheteau and Wright (2005). They solve for the long-run steady state relationship between velocity of money and the nominal interest rate, and estimate it. In their model, there is no productive capital or aggregate productivity shocks and their estimation does not constrain the preference shocks, and hence the nature of precautionary liquidity demand, to match specific data targets. In these major ways, their work is different from ours. However, the mechanisms at work in their model are closely related to our model, and their empirical insights are complimentary to ours.

Several papers have looked at quantitative implications of monetary search models in various settings, among them Wang and Shi (2006), Chiu and Molico (2008), Aruoba, Waller and Wright (2008), Aruoba and Schorfheide (2007). Wang and Shi (2006) investigate the business cycle properties of velocity of money in a monetary search model, but the main mechanism in their model that generates the fluctuations is connected to search intensity, rather than to precautionary liquidity demand. Chiu and Molico (2008) are studying aggregate properties of a monetary search model with both aggregate and idiosyncratic risk. However, their focus is on studying distributional properties of inflation and monetary policy as well as on optimal monetary policy rules in a setting where shocks propagate slowly through the heterogeneous households. Naturally, they do not have a degenerate distribution of money holdings. Aruoba, Waller and Wright (2008) have capital in the model, like we do, but no significant precautionary motive for money holding, and no aggregate productivity risk. Aruoba and Schorfheide (2007) introduce nominal rigidities in a search-based monetary model and estimate it using Bayesian methods to study the properties of the model relative to money-in-the-utility-function specifications.

The paper is organized as follows. Section 2 describes the model and characterizes the equilibrium. The following section, to be added, describes a simplified version of the model and analyzes how search frictions and precautionary demand for liquidity affect the dynamics of nominal and real aggregates in the model. Section 3 describes our preliminary calibration strategy, and section 4 details the solution algorithm. Section 5 presents our quantitative results and discusses the role of the frictions that we model. Section 6 concludes. Many of the proofs, and some computation details, are relegated to the appendix.

2 Model

The economy is populated by a measure 1 of households, who live infinitely in discrete time. The households rent labor and capital to firms, consume goods bought from the firms, and save. There are two types of markets: a Walrasian centralized market, and a decentralized market, characterized by search frictions. There are also two types of firms in the economy. Production firms use capital and labor as inputs in production, and sell their output in the centralized market. This output is used for centralized-market consumption and capital investment, but also bought up by retailers, who bring it into the decentralized market and sell these retail goods there.

In the centralized market, all parties involved in transactions are known and all trades can be enforced; intertemporal trade and asset trading are possible. In the decentralized market, on the other hand, agents are anonymous. In addition, in our setup there is no scope for barter in the decentralized market – the buyers have no goods to offer. As a result, no retail trade at all would occur in the absence of money, so that a medium of exchange – here, fiat money – is essential.

We model the two markets as occurring sequentially within each period. First, the centralized market opens in the first subperiod, closes, and the decentralized market opens in the second subperiod. As mentioned above, we model the latter as a market where search frictions prevent buyers and sellers from meeting at a centralized point; instead, buyers and sellers either succeed in meeting in pairs, or they fail to meet at all. The setup of the decentralized market is as a competitive search market (see Moen 1997, Shimer 1996). In this setup, price-quantity combinations (q, d) are posted in advance in a submarket, and have to be honored by all those, buyers and sellers, who choose to come to the submarket. The buyers and sellers who come are matched according to a matching function, which relates the matching probability matching for either party to the ratio of buyers and sellers that have shown up. Let us denote the seller-buyer ratio as $s/b \equiv n$. The matching function exhibits constant returns to scale, and thus we can write the buyer matching rate, $a(n)$, and the seller matching rate, $a(n)/n$, as a function of n only. Let η denote the elasticity of the buyer matching rate with respect to n; thus, $a(n) = n^{\eta}$. We assume, for the purposes of the theoretical exposition, that this elasticity is constant.

For each submarket, we assume that it is buyers who post price-quantity vectors in advance; formulating the problem such that sellers post prices will lead to identical equilibrium outcomes, but this formulation will give us a cleaner way of dealing with deviations off the equilibrium path.[1]

2.1 Households

Households maximize lifetime expected discounted utility,

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t v_t(c_t, q_t, h_t, \vartheta_t), \tag{1}$$

[1]Moreover, we assume that these price-quantity combinations are posted when the centralized market is still open. This assumption helps avoid complications due to coordination failures. For example, retail firms might take only a few goods to the market, expecting only low quantities to be posted, while buyers bring only low amounts of money, expecting only to be able to secure low amounts of goods; both parties would rather buy and sell more, if more goods and money were available.

where $0 < \beta < 1$. v_t, the utility achieved in each period, depends on consumption in the centralized market, c_t, and consumption in the decentralized market, q_t, time spent working h_t, and the consumption preference shock ϑ_t. Subscripting our variables by t is an abuse of notation, as we mean that each variable at time t is actually chosen conditional on the entire history the household has experienced up to that period, in principle. Moreover, the expectation is taken with respect to all possible histories.

In the first subperiod, utility follows the Hansen-Rogerson specification of indivisible labor with lotteries, and is given in reduced form by

$$U(c_t) - Ah_t. \tag{2}$$

Conditional on matching and receiving q_t, which is consumed instantly, second-subperiod utility is

$$\vartheta_t u(q_t). \tag{3}$$

The taste shock ϑ_t realizes when the centralized market is already closed and money holdings can no longer be adjusted, as described below. This will lead to precautionary demand for money. The taste shock comes from a distribution with finite support. In this section, the probability of a realization ϑ is implied by the expectation operator, but in later sections we refer to it by $\mathbb{P}(\vartheta)$. The choice of d_t together with q_t imply a probability $a(n(q_t, d_t))$ of meeting. We assume the usual properties on the $U(c_t)$ and $u(q_t)$ components of the utility function: they are continuous, strictly increasing, strictly concave, $\lim_{c_t \downarrow 0} U'(c_t) = \infty$, $\lim_{q_t \downarrow 0} u'(q_t) = \infty$. We also assume these functions are bounded.[2] Note that it is uncertain whether the household succeeds in acquiring q_t because of the meeting frictions.

Households own capital k_t, hold money $\hat{m}$ for future trade, which they may spend in the decentralized market if they meet a firm, and own nominal and real bonds, b_n and b_r, which they acquire from retail firms, as described below.[3] We normalize the price of the centralized good to one. We also normalize the household's money holdings by the aggregate money holdings. Given a measure 1 of households, m is then this normalized measure of money holdings, with $\tilde{m}$ as the normalized counterpart of $\hat{m}$ above. Let wage, capital rent (net of depreciation), value of one unit of normalized money, and share prices for the nominal and real bonds be, respectively, $w_t, r_t, \phi_t, P_{bn,t}, P_{br,t}$. The resulting budget constraint is

$$\phi_t m_t + (1 + r_t)k_t + w_t h_t + b_{n,t} + b_{r,t} = \\ c_t + \phi_t \tilde{m}_t + k_{t+1} + P_{bn,t} b_{n,t+1} + P_{br,t} b_{r,t+1}. \tag{4}$$

[2] If storage would depreciate, this is without loss of generality.

[3] In principle, households can hold shares of firms as well. We will see that in our formulation all firms make zero profits, so share holding is irrelevant. Alternatively, we can formulate the economy with firms selling shares instead of bonds; this leads to equivalent allocations of resources, but involves more notation.

Moreover, the household chooses its money holdings $\tilde{m}$ in the centralized market before ϑ realizes, and needs to bring at least d of it to complete a transaction in the market with posted (q, d), so that

$$d_t \leq \tilde{m}_t. \tag{5}$$

Finally, hours worked are constrained,

$$h \in [0, 1]. \tag{6}$$

The full household problem is thus specified as

$$\max \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \Big(U(c_t) - Ah_t + \vartheta_t a(n_t(q_t, d_t)) u(q_t) \Big) \right] \tag{7}$$

with respect to $(c_t, h_t, \{q_t(\vartheta_t), d_t(\vartheta_t)\}, k_{t+1}, \tilde{m}_t, b_{n,t+1}, b_{r,t+1})$, taking as given ($w_t$, r_t, $P_{br,t}$, $P_{bn,t}$, ϕ_t, $n_t(q_t, d_t)$), and subject to constraints (4)-(6) and the evolution of money holdings

$$\begin{aligned} m_{t+1} &= \tilde{m}_t - d_t(\vartheta) + \varpi_t M_t \text{ if matched at } (q_t(\vartheta), d_t(\vartheta)), \\ m_{t+1} &= \tilde{m}_t + \varpi_t M_t \text{ if not matched,} \end{aligned} \tag{8}$$

where ϖM_t refers to injections of money by the monetary authority, consistent with the rate of growth of money supply, to be discussed in more detail below. The terms of trade that households post, (q_t, d_t), imply a seller-buyer ratio n_t and a meeting rate with firms $a(n_t)$, with $0 \leq n_t(q_t, d_t) \leq 1$. In the sequential household's problem $n_t(q_t, d_t)$, just as the sequence of prices, is taken as given.[4]

Each period, the amount of utility that can be achieved is bounded from above by $\sup_x U(x) + \sup_q u(q)$. The value (supremum) of the household's problem in the sequential formulation is finite. Bellman's Principle of Optimality holds, and we can equivalently formulate the problem recursively, which we will do below.

2.2 Production Firms

The problem of the production firm is static and completely standard – to maximize its profits in each period. Given a constant returns to scale production function

$$y_t = e^{z_t} f(k_t, h_t)$$

, where z_t is the stochastic productivity level described in more detail below, the problem is

$$\max_{k_t, h_t} e^{z_t} f(k_t, h_t) - w_t h_t - r_t k_t - \delta k_t,$$

[4]Below, we will employ the assumption that the household knows the equilibrium zero-profit/free-entry condition, so it can predict n_t.

and the solution is characterized by the usual first order conditions

$$r_t = e^{z_t} f_k(k_t, h_t) - \delta \tag{9}$$

$$w_t = e^{z_t} f_h(k_t, h_t) \tag{10}$$

Given constant returns to scale, these firms make zero profit. This implies that their shares will not constitute any wealth for households; the only share price consistent with zero profits is zero.

2.3 Retail Firms

The retail firms face a more complicated problem. They decide in the first subperiod to buy q, to bring into an island of their choice in the second subperiod. Then for any amount q that is taken to an island, they might match with probability $a(n)/n$, in which case they will get d units of money. Or, with probability $(1-(a(n)/n))$ they do not match, in which case they sell the q units in the next centralized market. We assume that retail firms sell nominal and real bonds, b_n and b_r, to be paid out next period, to raise the financial means to pay for q. When firms either have a large enough portfolio of q's or they are able to pool the matching risk among themselves, they can sell $(1-(a(n)/n))q$ worth of real bonds, and $(a(n)/n)d$ of nominal bonds. The retail firm's maximization problem for period t becomes

$$\Pi_{rt} = \max_{(q_t, d_t, n_t)} \left[P_{bnt} \frac{a(n_t)}{n_t} d_t + P_{br} \left(1 - \frac{a(n_t)}{n_t}\right) q_t - q_t \right]. \tag{11}$$

The retail firms' choice variables denote that they choose the island to go to, with terms of trade having been posted by households. Notice that once again, this maximization problem is essentially static: retail firms try to make a profit at time t, by buying q_t and, simultaneously, selling claims (bonds) to next period's money and goods, left over from decentralized trade. Moreover, free entry of retail firms implies that n_t will increase until no economic profit is left: $\Pi_{rt} = 0$. Households posting (q_t, d_t)-vectors will anticipate the free entry, and use (11) to predict their chance of matching on their island.[5]

2.4 Monetary Policy and Aggregate Shocks

The monetary authority follows an interest rate feedback rule

$$\frac{1+i_{t+1}}{1+\bar{i}} = \left(\frac{1+i_t}{1+\bar{i}}\right)^{\rho_i} \left(\frac{1+\pi_t}{1+\bar{\pi}}\right)^{\rho_\pi} \left(\frac{y_t}{\bar{y}}\right)^{\rho_y} \exp(\varepsilon_{t+1}^{mp}). \tag{12}$$

[5] As we mentioned above, it is outcome-equivalent to let sellers post. It is commonly assumed that the set of markets is complete: it is not possible to post a different price-quantity vector such that households get a higher expected utility than before, while the retailers make at least as much profit. With buyer posting, this is directly implied.

The term ε^{mp} denotes a stochastic monetary policy shock which realizes at the beginning of the period. Consistent with the movement in interest rates, the rate of money supply growth ϖ_t adjusts, and the changes in money supply are transferred to households via lump-sum injections $\varpi_t M_t$, where ϖ_t thus refers to money supply growth from period t to $t + 1$.

In addition, we will assume that the monetary policy shock ε^{mp} and the stochastic productivity shock z are correlated, so that the interest rate responds to changes in the productivity levels and vice versa. We will let the aggregate shock processes follow a finite-state Markov process.

2.5 Recursive Formulation of the Household Problem

We focus on Markov (payoff-relevant) decision strategies. The constraint set of the household depends only on the state variables discussed below, and we assume (and later prove) that all prices only depend on the aggregate state variables.

From now on, we will conserve notation by omitting time subscripts, and using primes to denote $t + 1$. There are the following aggregate state variables in this economy: the aggregate capital stock, K, the total amount of unsold retail goods carried over from the second subperiod due to matching frictions, which we denote as G, the technology shock z, the previous interest rate in the economy i_{-1}, and the term $(1 + \varpi_{-1})\phi_{-1}$, which denotes the previous period's post-injection real value of money, and which households need to know in order to determine the current rate of inflation in the economy.

In this economy, retail firms are subject to idiosyncratic matching risk. Households are subject to the matching risk, as well as idiosyncratic preference risk. The individual state variables at the beginning of the centralized market are normalized money holdings m, goods g, and capital holdings k, where the money and goods holdings are measured after the nominal and real bonds are paid out. Recall that individual money holdings m are defined relative to total money stock M: if a household holds the average stock of money, then $m = 1$. This renders the money holdings stationary. We define ϕ as the value of one unit of *normalized* money, which implies $\phi = M/P$, the real value of the total money stock. At the beginning of the centralized market in the next period, the monetary policy shock realizes, and each household receives an injection of ϖM units of money accordingly. If a household has $\tilde{m}$ units of normalized money left at the end of the decentralized subperiod, its money holdings at the beginning of the next period, *before* the bonds are paid out, and normalized by next period's money stock M' are given by

$$m' = \frac{\tilde{m}}{1 + \varpi} + \frac{\varpi}{1 + \varpi}.$$

Instead of writing this as a problem with separate value functions for centralized and decentralized subperiods, we can rewrite the household's problem as a more transparent full-period problem. This means that in the first subperiod the household can make the choices for the second subperiod, contingent on its information at the start of the second subperiod. In our environment, the information that does not become known until the second subperiod is the realization of the preference shock ϑ_t specific to each household, and what (q_t, d_t) vectors are posted in islands. As mentioned before, the households can infer the buyer-seller ratio $n_t(q_t, d_t)$ from the zero-profit condition for the firms; instead of defining an equilibrium functional for this relationship, we carry the retail firms' zero-profit condition as an additional constraint, and consider n to be part of the posted vector, (q, d, n). We allow households to post this set of $\{(q(\vartheta), d(\vartheta), n(\vartheta))\}$ themselves, so that they can decide which island to go to after the realization of ϑ. To conserve notation, we will abbreviate the vector of posted quantities as $\{q, d, n\}$, and an element of this vector $(q_\vartheta, d_\vartheta, n_\vartheta)$, where it should not be forgotten that these quantities also depend on the aggregate states of the economy, and possibly on the individual state variables of the household.

Thus we can write all choices to be made within the two subperiods as occurring in the first subperiod, when the household chooses m' and a vector $\{q, d, n\}$ to post in the decentralized market, given the state variables. In the second subperiod, all decentralized purchases are subject to $d_\vartheta \leq m'$ for each ϑ. In sum, we have the following recursive maximization problem:

$$V(k, m, g, K, G, z, i_{-1}, (1+\varpi_{-1})\phi_{-1}) = \max_{\{c,h,\tilde{m},k',b'_n,b'_r,\{q,d,n\}\}} \Big\{ U(c) - Ah + \mathbb{E}_\vartheta \Big[a(n_\vartheta) \Big(\vartheta u(q_\vartheta) + \beta \mathbb{E}[V(k', m' - \frac{d_\vartheta}{1+\varpi}, g', K', G', z', i, (1+\varpi)\phi)] \Big) + (1 - a(n_\vartheta))\beta \mathbb{E}[V(k', m', g', K', G', z', i, (1+\varpi)\phi)] \Big] \Big\} \tag{13}$$

subject to

$$\phi m + g + (1+r)k + wh = c + \phi\tilde{m} + k' + P_{bn}b'_n + P_{br}b'_r \tag{14}$$

$$d_\vartheta \leq \tilde{m} \ \forall \ \vartheta \tag{15}$$

$$q_\vartheta = \left(\frac{a(n_\vartheta)}{n_\vartheta}\right) P_{bn} d_\vartheta + \left(1 - \frac{a(n_\vartheta)}{n_\vartheta}\right) P_{br} q_\vartheta, \quad \forall (q_\vartheta, d_\vartheta, n_\vartheta), \vartheta \tag{16}$$

$$m' = \frac{\tilde{m}}{1+\varpi'} + \frac{\varpi'}{1+\varpi'} + \frac{b'_n}{1+\varpi'} \tag{17}$$

$$g' = b'_r \tag{18}$$

$$K'(K, G, z, i_{-1}, (1+\varpi_{-1})\phi_{-1}), G'(K, G, z, i_{-1}, (1+\varpi_{-1})\phi_{-1}) \tag{19}$$

$$[z', i'] = \Xi[z, i, \pi, y] + [\varepsilon'_1, \varepsilon'_2] \tag{20}$$

$$h \in [0, 1] \tag{21}$$

Note that we have defined b'_n to be the payoff *tomorrow* in *today's* normalized units. As a result the nonnormalized units of money (i.e. money in the conventional sense) paid out tomorrow is known today, however, its value is not. The shock processes in (20) are not necessarily independent, i.e. $corr(\varepsilon_1, \varepsilon_2)$ can be different from zero. The shocks are defined on a finite set of states. The term Ξ refers to a 2×4 matrix.

Proposition 1. *The household problem, taking as given prices and aggregate laws of motion that only depend on the aggregate states, can be solved recursively, with* (13) *as the value function, which is strictly increasing the individual state variables, and concave in k and g.*

The proof is in the appendix.

From now on, we denote the aggregate state variables by S, where $S = (K, G, z, i_{-1}, (1+\varpi_{-1})\phi_{-1})$, and the individual variables by $s = (k, m, g)$. Further, denote the policy functions of the household's problem by $\alpha(s, S)$, with $\alpha_x(.)$ as the policy function for the choice variable x.

2.6 Equilibrium

Definition 1. *A Symmetric Stationary Monetary Equilibrium is a set of pricing functions $\phi(S)$, $w(S)$, $r(S)$, $P_{br}(S)$, $P_{bn}(S)$, $b_n(S)$, $b_r(S)$; a set of laws of motions $K'(S)$, $G'(S)$, value function $V(s, S)$ and policy functions $c(s, S)$, $h(s, S)$, $k(s, S)$, $g(s, S)$, $b_n(s, S)$, $b_r(s, S)$, $m(s, S)$, $\{n_\vartheta(s, S), q_\vartheta(s, S), d_\vartheta(s, S)\}$, all ϑ, such that:*

1. *The value function solves the household optimization, in* (13)*, with associated policy functions, given prices and laws of motion;*
2. *Production and retail firm optimize, given prices and laws of motion, as in sections 2.2 and 2.3.*
3. *Free entry of retailers: $\Pi_{rt} = 0$.*
4. *Consistent expectations: aggregate laws of motion follow from the sum of all individual decisions (index individual households by i) –*

$$K'(S) = \int_0^1 k^i(s, S)di$$

$$G'(S) = \int_0^1 \left[\sum_\vartheta \mathbb{P}(\vartheta)(1 - a(n^i_\vartheta(s, S)))q^i_\vartheta(s, S) \right] di.$$

5. *Market clears in all centralized markets (capital, labor, general goods, money,*

financial markets):

$$\int_0^1 m^i(s,S)di = 1$$
$$\int_0^1 b_r^i(s,S)di = g = \int_0^1 \left[\sum_{\vartheta} \mathbb{P}(\vartheta)(1-a(n_\vartheta^i(s,S)))q_\vartheta^i(s,S)\right] di$$
$$\int_0^1 b_n^i(s,S)di = \int_0^1 \left[\sum_{\vartheta} \mathbb{P}(\vartheta)a(n_\vartheta^i(s,S))d_\vartheta^i(s,S)\right] di$$
$$\int_0^1 h^i(s,S)di = H$$
$$(1-\delta)K + G + e^z f(H,K) = C + K' + \int_0^1 \left[\sum \mathbb{P}(\vartheta)n_\vartheta^i(s,S)q_\vartheta^i\right] di \qquad (22)$$

2.7 Optimization and Market Clearing in Centralized Markets

In this section, we study the household decisions in equilibrium in more detail, and put these together with market clearing and equilibrium prices, to work towards a set of equations that will characterize the equilibrium allocation.

2.7.1 Centralized Market creates Homogeneity

For general utility functions, different realizations of the idiosyncratic matching and preference risks could lead to a nontrivial distribution of wealth (with, for example, those who have recently matched with high ϑ's being poorer).[6] In turn, households with different wealth could make different portfolio decisions, and hence the *distribution* of individual state variables would likely be relevant for the equilibrium prices.

However, the quasi-linear specification of the problem allows equilibria in which all heterogeneity created in the second subperiod washes out in the centralized market.[7] This occurs if the boundary conditions of h are never hit. Our strategy is to solve the problem assuming that optimal choices of h are interior, and check (in our calibrated equilibrium) whether this is indeed the case.

[6] See Molico (2005) and Chiu and Molico (2008), who investigate the distributional impact of heterogeneity on e.g. the welfare effects of monetary policy when markets have search frictions.

[7] This result arises in other models that combine centralized and decentralized trade, such as Lagos and Wright (2005) and Rocheteau and Wright (2005). An alternative achieving homogeneity is 'families' who pool their resources (Shi 1997).

Let us substitute the budget constraint for h into the household's value function:

$$V(s,S) = \max_{c,h,\tilde{m},k',b'_n,b'_r,\{q,d,n\}} \Bigg\{ U(c) - A\left(\frac{c + \phi\tilde{m} + k' + P_{br}b'_r + P_{bn}b'_n - \phi m - g - (1+r)k}{w}\right) + \mathbb{E}_\vartheta\Big[a(n_\vartheta)\Big(\vartheta u(q_\vartheta) + \beta\mathbb{E}[V(s',S')]\Big) + (1-a(n_\vartheta))\beta\mathbb{E}[V(s',S')]\Big]\Bigg\}, \tag{23}$$

given all price and expectation functions that were written down in system (13). Note that we can split the value function into two parts

$$V(s,S) = A\left(\frac{\phi m + (1+r)k + g}{w}\right) + \max_{\dots} \Bigg\{ U(c) - A\left(\frac{c + \phi\tilde{m} + k' + P_{bn}b'_n + P_{br}b'_r}{w}\right) + \mathbb{E}_\vartheta\Big[a(n_\vartheta)\Big(\vartheta u(q_\vartheta) + \beta\mathbb{E}[V(s',S')]\Big) + (1-a(n_\vartheta))\beta\mathbb{E}[V(s',S')]\Big]\Bigg\}, \tag{24}$$

where the first term on the right-hand side gives the assets of the household carried over from the previous period. Note that the maximization is not affected by this term, which is why we could take it outside the expression that is to be maximized. Consequently, the choices over which the objective is maximized, $(c, h, \tilde{m}, k', b'_n, b'_r, \{q, d, n\})$, do not depend on $s = (m, k, g)$ as long as the implied h is interior, which we assumed.

Moreover, in an equilibrium with a finite H, there exists an allocation of capital investment that maximizes all households' utility, while all households invest the same in k', buy the same number of bonds, and for a value function strictly concave in $\tilde{m}$ (to be investigated below), take out the same amount of cash.

Denoting the maximizing choices of the controls, given the aggregate state variables, as $(c^*, h^*, \tilde{m}^*, k^{*\prime},\ {b'_n}^*, {b'_r}^*,\ \{q^*, d^*, n^*\})$, we can rewrite the value function as

$$V(s,S) = W^*(S) + A\left(\frac{\phi m + (1+r)k + g}{w}\right) \tag{25}$$

where

$$\begin{aligned} W^*(S) = U(c^*) - A\left(\frac{c^* + \phi\tilde{m}^* + k^{*\prime} + P_{br}b_r' + P_{bn}b_n'}{w}\right) \\ + \mathbb{E}_\vartheta\Big[a(n^*(\vartheta))\Big(\vartheta u(q^*(\vartheta)) + \beta\mathbb{E}[V(s^{*\prime}, S')]\Big) \\ + (1 - a(n_\vartheta))\beta\mathbb{E}[V(s^{*\prime}, S')]\Big], \end{aligned} \tag{26}$$

The value function $V(.)$ is differentiable in k, m, g (Benveniste-Scheinkman applies trivially). The envelope conditions are, then,

$$V_m(s,S) = \frac{A\phi}{w(S)} \tag{27}$$

$$V_k(s,S) = \frac{A(1 + r(S))}{w(S)} \tag{28}$$

$$V_q(s,S) = \frac{A}{w(S)} \tag{29}$$

At this stage, we can write down the Euler equation with respect to capital and the first-order condition with respect to labor (the problem is weakly concave in these variables, and the solution is interior, as we assumed that (21) is not binding); we will discuss the other choice variables below.

$$U'(c^*(S)) = \beta\mathbb{E}[U'(c^*(S'))(1 + r(S'))] \tag{30}$$

$$U'(c^*(S)) = \frac{A}{w(S)} \tag{31}$$

Consumption is a function only of aggregate state variables, because of the quasi-linearity, as discussed in the section above.

2.7.2 Equilibrium price of bonds

Let us reiterate that all financial instruments can be traded *between centralized markets*. In our model, a nontrivial amount of bonds will be traded, and households will hold positive amounts of both nominal and real bonds in every period. The Euler equation for nominal bonds is given by

$$U'(c^*(S))P_{bn}(S) = \beta\mathbb{E}[U'(c^*(S'))\frac{\phi'(S')}{1 + \varpi'}] \tag{32}$$

$$\iff P_{bn}(S) = \beta\mathbb{E}\left[\frac{\phi'(S')}{1 + \varpi'}\frac{w(S)}{w(S')}\right] = \frac{\phi}{1 + i}, \tag{33}$$

where we used the fact that $dm'/db_n = (1+\varpi')^{-1}$. This condition must hold in order for the choice of nominal bond holdings to be in the interior. The quasi-linearity implies that when this condition is satisfied at one particular interior b_n, it is satisfied at all interior b_n, as long as h is not at a corner. Moreover, as the choice of b_n, b_r does not depend on individual states, we assume that every household chooses exactly the same bond portfolio. This, incidentally, will greatly facilitate our computation. Market clearing then implies that, at this price, $b_n = \sum_\vartheta \mathbb{P}(\vartheta)a(n_\vartheta)d_\vartheta$. A similar exercise with real bonds yields the Euler equation for real bonds, and the implied market-clearing price,

$$U'(c^*(S))P_{br}(S) = \beta\mathbb{E}[U'(c^*(S'))] \tag{34}$$

$$\iff P_{br}(S) = \beta\mathbb{E}\left[\frac{w(S)}{w(S')}\right]. \tag{35}$$

At this price, the bond market clears: $b_r = g = \sum_\vartheta \mathbb{P}(\vartheta)(1-a(n_\vartheta))q_\vartheta$. To abbreviate notation, we will denote

$$\mathbb{E}\left[\frac{\phi'(S')}{1+\varpi'}\frac{w(S)}{w(S')}\right] \equiv \widetilde{\mathbb{E}\phi'}, \quad \mathbb{E}\left[\frac{w(S)}{w(S')}\right] \equiv \widetilde{\mathbb{E}}.$$

The free-entry condition for retail firms, which the household takes into account when deciding to post $(q_\vartheta, d_\vartheta)$, which implicitly will return the seller/buyer ratio n_ϑ, thus can be written as

$$q_\vartheta = \frac{a(n_\vartheta)}{n_\vartheta}\beta\widetilde{\mathbb{E}\phi'}d_\vartheta + \left(1 - \frac{a(n_\vartheta)}{n_\vartheta}\right)\beta\widetilde{\mathbb{E}}q_\vartheta \tag{36}$$

2.7.3 Market Clearing

Household i's labor decision h_i is given by their initial holdings m_i, k_i, g_i at the beginning of the period, which differ among households only in m_i (so that we drop the i-index for k, g). In addition, the decisions $c, \tilde{m}, \{q_\vartheta, d_\vartheta, n_\vartheta\}, k', b_n', b_r'$ are identical across all households, such that

$$h_i = (1/w)(\phi m_i + g + (1+r)k - \phi\tilde{m} - k' - b_n' - b_r').$$

In the aggregate $\int_I h(i) = H$. It must then follow that

$$\int_I h_i = H = (1/w)\left(\phi\left(\int_I m_i\right) + g + (1+r)k - \phi\tilde{m} - k' - b_n' - b_r'\right).$$

Factor market clearing implies the usual,

$$w(S) = e^z f_H(K,H); \quad r(S) = e^z f_K(K,H) - \delta.$$

Market clearing dictates $\tilde{m} = 1$, and therefore $\int m_i = \sum_\vartheta \mathbb{P}(\vartheta)\Big(a(n_\vartheta)(\tilde{m} - d_\vartheta) + a(n_\vartheta)d_\vartheta\Big) = 1$, while $b'_n + b'_r = \sum_\vartheta \mathbb{P}(\vartheta)n_\vartheta q_\vartheta$. By Walras Law we then have that the last remaining market - the goods market - clears as well:

$$G + e^z f(K, H) + (1-\delta)K = C + K' + \sum_\vartheta \mathbb{P}(\vartheta)q_\vartheta n_\vartheta$$

Finally, we have to check whether our 'working' assumption that $h_i \in [0, 1]$ is correct for each i. We can do so for each type of the household individually in our calibrated model. However, a weak sufficient condition for this that only employs aggregate information is

$$H - \frac{\phi}{w} > 0,$$

while an upper bound is given by $H - h_i < (1/w)\phi\tilde{m}$.

2.8 Properties of Decentralized Market Trade

Above we have discussed the Euler equations that link consumption, capital investment and bond investment between periods. What remains is to study the decisions that are made for decentralized market trade. Importantly, in general equilibrium, the monetary and real sectors do not dichotomize, as the marginal utility of consuming (or working) in the centralized market still enters the decentralized market problem. However, it enter in a way that allows us to study the decentralized problem in relative isolation.

Households are subjected to a draw of a preference shock ϑ from a non-trivial distribution. As a result, there will be submarkets for households with different realized shocks in the decentralized subperiod, as households with different shocks ϑ prefer different vectors $(q_\vartheta, d_\vartheta, n_\vartheta)$. The previous analysis concluded, without loss of generality, that all households who have the same ϑ realization will choose the same $(q_\vartheta, d_\vartheta, n_\vartheta)$, if the problem is concave. The maximization problem thus involves choosing, along with money holdings $\tilde{m}$, the right $(q_\vartheta, d_\vartheta, n_\vartheta)$ for each realization ϑ. We rewrite the household problem to isolate these decisions:

$$\begin{aligned}
V(s,S) = {} & A\left(\frac{\phi m + (1+r)k + g}{w}\right) \\
& + \max_{k', b'_n, b'_r, c}\left\{ - A\left(\frac{c + k' + P_{bn}b'_n + P_{br}b'_r}{w}\right) + u(c) + \beta\mathbb{E}\left[A\frac{k'(1+r') + b'_r + \frac{\phi'}{1+\varpi'}b'_n}{w'(S')}\right]\right\} \\
& \quad + \max_{\tilde{m}, \{q_\vartheta, d_\vartheta, n_\vartheta\}}\left\{ - \frac{A\phi\tilde{m}}{w} + \mathbb{E}_\vartheta\left[a(n_\vartheta)\Big(\vartheta u(q_\vartheta) + \beta\mathbb{E}\left[A\frac{\phi'(S')(\tilde{m} - d_\vartheta + \varpi')}{w'(S')(1+\varpi')}\right]\right]\right. \\
& \qquad \left. + (1 - a(n_\vartheta))\beta\mathbb{E}\left[A\frac{\phi'(S')(\tilde{m} + \varpi')}{w'(S')(1+\varpi')}\right]\right\} + \beta\mathbb{E}[W^*(S')] \qquad (37)
\end{aligned}$$

Given prices, we can separate the decisions pertinent to the decentralized market $(q_\vartheta, d_\vartheta, n_\vartheta), \tilde{m}$ from the decisions about centralized market assets k', b'_n, b'_r and current consumption c. The labor supply h will be chosen to balance the budget, given decisions c^*, b_n, b_r, $(q_\vartheta, d_\vartheta, n_\vartheta), \tilde{m}$ and k', as long as these choices imply that the bounds on h are not hit.

2.8.1 Solving the Decentralized Market Maximization Problem

Thus for the decentralized market, it is only relevant to concentrate on the second maximization problem in (37). Without loss of generality, we can premultiply the maximization by the constant $\frac{w(S)}{A}(= (U'(c))^{-1})$, which does not affect the maximization, to get

$$\max_{(q_\vartheta, d_\vartheta, n_\vartheta), \tilde{m}} -\phi\tilde{m} + \sum_\vartheta \mathbb{P}(\vartheta)\Big(a(n_\vartheta)\Big(\frac{\vartheta u(q_\vartheta)}{U'(c)} - \beta\widetilde{\mathbb{E}\phi'}d_\vartheta\Big)\Big) + \beta\widetilde{\mathbb{E}\phi'}\tilde{m} \tag{38}$$

subject to

$$-q_\vartheta + \left(\frac{a(n_\vartheta)}{n_\vartheta}\beta\widetilde{\mathbb{E}\phi'}d_\vartheta + \left(1 - \frac{a(n_\vartheta)}{n_\vartheta}\right)\beta\widetilde{\mathbb{E}}q_\vartheta\right) = 0 \ \ \forall \ \ \vartheta \tag{39}$$

$$d_\vartheta \leq \tilde{m} \ \ \forall \ \ \vartheta \tag{40}$$

Denote $\nu_\vartheta, \mu_\vartheta$ as the multipliers of (39) and (40), for each ϑ.

Proposition 2. *If $\eta < 0.5$ and the coefficient of relative risk aversion (at the q^*_ϑ satisfying the equations below) is larger than some number ς, where ς is close enough to, but below, 1, the* ***necessary and sufficient conditions*** *for the globally optimal interior choice in the problem (38)-(40) are given by*

$$\eta\left(\vartheta\frac{u(q_\vartheta)}{U'(c)} - \beta\widetilde{\mathbb{E}\phi'}d_\vartheta\right)\beta\widetilde{\mathbb{E}\phi'}d_\vartheta - (1-\eta)\beta\left(\widetilde{\mathbb{E}\phi'}d_\vartheta - \widetilde{\mathbb{E}}q_\vartheta\right)\frac{\vartheta u'(q_\vartheta)q_\vartheta}{U'(c)} = 0 \ \forall\vartheta. \tag{41}$$

$$\sum_\vartheta\left(\mathbb{P}(\vartheta)a(n_\vartheta)\frac{\vartheta u'(q_\vartheta)q_\vartheta}{U'(c)d_\vartheta} - \mathbb{P}(\vartheta)a(n_\vartheta)\beta\widetilde{\mathbb{E}\phi'}\right) + \beta\widetilde{\mathbb{E}\phi'} - \phi = 0 \tag{42}$$

$$\left(\frac{\vartheta u'(q_\vartheta)q_\vartheta}{U'(c)d_\vartheta} - \beta\widetilde{\mathbb{E}\phi'}\right)(\tilde{m} - d_\vartheta) = 0, \frac{\vartheta u'(q_\vartheta)q_\vartheta}{U'(c)d_\vartheta} \geq \beta\widetilde{\mathbb{E}\phi'}, \text{ and } d_\vartheta \leq \tilde{m}. \ \forall \ \ \vartheta, \tag{43}$$

Very loosely, we can call equation (41) a "bargaining condition", (42) a "money demand equation", and (43) the complementary slackness condition.

As far as we know, sufficient conditions to locate the optimal choices in money search setting (in particular settings based on Lagos and Wright 2005) have been restrictive, while the sufficiency established above provides the result under relatively weak conditions.[8] In Lagos and Wright, for example, this is derived for a bargaining power of 1, or close to 1, for the buyer. Alternatively, utility exhibiting increasing

[8] This stands in contrast, e.g. to the existence proof for monetary steady states in the deterministic case. Wright (2008) finds that generically, there is a unique monetary steady state. However,

absolute risk aversion, the value function is concave and the first order conditions therefore locate the maximum. While the proposition finds sufficiency under weaker conditions, it is specific to the setting of our model,with retailers moving products to the decentralized markets. However, it is indicative of generalizations that are possible with respect to sufficiency in the Lagos-Wright setting. (See Visschers 2008 for a proof of sufficiency in Lagos-Wright itself, under similar conditions (η being the bargaining power in that particular case). To show sufficiency, the proof establishes the *pseudo*-concavity of the problem (i.e. when the first order conditions equal zero at a certain point, the objective function is concave at that point), not global concavity.

Intuitively, having a high enough relative risk aversion (RRA) implies that utility is 'very concave', and the concavity of the instantaneous utility function then helps with the pseudo-concavity of the problem in general. This intuition is, however, not everything: one of the derived comparative statics in Nash Bargaining[9] is that the person with a lower *absolute* risk aversion gets relatively more out of the bargaining, all other things equal. A constant relative risk aversion means that absolute risk aversion is decreasing in c, so as the buyer brings more money, he improves his bargaining position as he becomes less risk-averse in the absolute sense. Without looking in more detail, we cannot rule out that the improvement in the bargaining from less risk-aversion outdoes the decrease of marginal utility as more goods are being transacted. We avoid this issue with the increasing absolute risk aversion, or alternatively put log-concavity on u', condition in Lagos and Wright (and when the bargaining power is 1 or close to 1, the gains of improving one's 'position' in bargaining are small–hence the other condition). In the proof of sufficiency, which is in the appendix, it is shown however, that whenever we restrict ourselves to points at which the first order conditions hold, the objective function is locally concave, under the above conditions.

Under the assumptions of the proposition we derive that the first order conditions of the maximization are in fact sufficient to find the global maximizers $\{q, d, n\}$, given any ϕ and $\widetilde{\beta\mathbb{E}\phi'}$, $\phi > \widetilde{\beta\mathbb{E}\phi'}$. Moreover, decentralized quantities, meeting rates and most importantly, today's value of money depend continuously on the expected value of money tomorrow.

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the generality of the proof provides no guarantee that steady state variables are continuous in parameters in this proof, something that could create some trouble for our setting. Moreover, the equivalence of the first-order conditions in his setting by themselves do not always isolate the optimal choice of the agents: they could isolate local maxima and minima instead. While issues of existence and uniqueness are settled for the deterministic case, the proof does not spell out a constructive way to isolate the global maximum (as a function of $\widetilde{\mathbb{E}\phi'}$ (or i, in his setting)) in a system of equations. For purposes of comparison, we can state our result in Wright (2008) language: our conditions guarantee a decreasing $\ell(i)$.

[9] The competitive search allocation is equivalent to a bargaining solution with constrained-efficient bargaining power.

Lemma 1. *Given RRA > 1, (16), (41)-(43) pin down uniquely* ϕ, $\{q, d, n\}$ *as functions of* $\beta\widetilde{\mathbb{E}\phi'}$.

The proof is in the appendix.

This lemma has important implications not only for the theoretical investigation here, but also for our computational model. As we will discuss later, our chosen computation method relies on approximating the expectation terms in the characterizing equations of the problem with polynomials. The continuity shown above is essential to claim that our approximating functions are appropriate. Moreover, our result is that the first order conditions find uniquely the global maximum for the entire range of relevant values of parameters and also the relative range of tomorrow's values (co-states), e.g. of money. This means that we also avoid complications with local maxima that cease to be global maxima for certain ranges of parameter and co-state values. Since our model is stochastic, a non-trivial range of these values will occur at different points in time, making this an issue of practical relevance.

Let us now derive the first order conditions, and complementary slackness of the problem (38)–(40) as

$$-\mu_\vartheta - \mathbb{P}(\vartheta)a(n_\vartheta)\beta\widetilde{\mathbb{E}\phi'} + \nu_\vartheta \frac{a(n_\vartheta)}{n_\vartheta}\beta\widetilde{\mathbb{E}\phi'} = 0 \quad (44)$$

$$\mathbb{P}(\vartheta)(a(n_\vartheta)\frac{u'(q_\vartheta)}{U'(c)}) + \nu_\vartheta(-1 + \beta\widetilde{\mathbb{E}} - \beta\frac{a(n_\vartheta)}{n_\vartheta}\widetilde{\mathbb{E}}) = 0 \quad (45)$$

$$\mathbb{P}(\vartheta)a'(n_\vartheta)(\frac{\vartheta u(q_\vartheta)}{U'(c)} - \beta\widetilde{\mathbb{E}\phi'}d_\vartheta) + \nu_\vartheta\frac{a'(n_\vartheta)n_\vartheta - a(n_\vartheta)}{n_\vartheta^2}\beta(\widetilde{\mathbb{E}\phi'}d_\vartheta - \widetilde{\mathbb{E}}q_\vartheta) = 0 \quad (46)$$

$$\sum_\vartheta \mu_\vartheta + \beta\widetilde{\mathbb{E}\phi'} - \phi = 0 \quad (47)$$

$$\mu_\vartheta(\tilde{m} - d_\vartheta) = 0, d \leq \tilde{m},\ \mu_\vartheta \geq 0\ \forall\ \vartheta \quad (48)$$

Given $\widetilde{\mathbb{E}\phi'}, \widetilde{\mathbb{E}}$, we can solve this system of equations. Substituting out the ν_ϑ in (44) and (45), we find a system with (47), (48), and

$$\mathbb{P}(\vartheta)\left(a(n_\vartheta)\frac{\vartheta u'(q_\vartheta)q_\vartheta}{U'(c)d_\vartheta}\right) - \mathbb{P}(\vartheta)a(n_\vartheta)\beta\widetilde{\mathbb{E}\phi'} - \mu_\vartheta = 0\ \forall\vartheta \quad (49)$$

$$\eta\left(\vartheta\frac{u(q_\vartheta)}{U'(c)} - \beta\widetilde{\mathbb{E}\phi'}d_\vartheta\right)\beta\widetilde{\mathbb{E}\phi'}d_\vartheta - (1-\eta)\beta\left(\widetilde{\mathbb{E}\phi'}d_\vartheta - \widetilde{\mathbb{E}}q_\vartheta\right)\frac{\vartheta u'(q_\vartheta)q_\vartheta}{U'(c)} = 0\ \forall\vartheta. \quad (50)$$

Result 1. *As long as the money constraint is not binding for a realization of* ϑ, q_ϑ *and* n_ϑ *are independent of* $\beta\widetilde{\mathbb{E}\phi'}$.

If the money constraint does not bind, then $\mu_\vartheta = 0$, and from (44) and (45) it follows that

$$\frac{\vartheta u'(q_\vartheta)q_\vartheta}{U'(c)} = \beta\widetilde{\mathbb{E}\phi'}d_\vartheta \quad (51)$$

and using (51) with (50) one more time, yields

$$\eta \left(\vartheta \frac{u(q_\vartheta)}{U'(c)} - \vartheta \frac{u'(q_\vartheta)}{U'(c)} q_\vartheta \right) - (1-\eta) \left(\vartheta \frac{u'(q_\vartheta)}{U'(c)} q_\vartheta - \beta \widetilde{\mathbb{E}} q_\vartheta \right) = 0, \tag{52}$$

Solving (50) for q_ϑ and plugging it into (51), we find q_ϑ, d_ϑ for the unconstrained case. Moreover, from (51) substituted into the free entry condition, we find that the latter only depends on the unchanged q_ϑ (keeping $\widetilde{\mathbb{E}}$ constant), hence n_ϑ is unchanged as well. Note, however, that a change in the *real* interest will change both q_ϑ and n_ϑ.

In case all $d_\vartheta \leq \tilde{m} \ \forall \ \vartheta$ from the solution of (50) in (51), the second-subperiod money holding constraint is not binding in equilibrium. Moreover, we can then conclude that $\phi = \beta \widetilde{\mathbb{E}\phi'}$, from (47). For $d_\vartheta > \tilde{m}$ in the unconstrained case, the constraint $d \leq \tilde{m} (=1)$ is actually binding, and instead we solve

$$\eta \left(\vartheta \frac{u(q_\vartheta)}{U'(c)} - \beta \widetilde{\mathbb{E}\phi'} \right) \beta \widetilde{\mathbb{E}\phi'} - (1-\eta)\beta \left(\widetilde{\mathbb{E}\phi'} - \widetilde{\mathbb{E}} q_\vartheta \right) \vartheta \frac{u'(q_\vartheta)}{U'(c)} q_\vartheta = 0, \tag{53}$$

and from (49), we find $\mu_\vartheta = \mathbb{P}(\vartheta) a(n) \frac{\vartheta u'(q)}{U'(c)} - \mathbb{P}(\vartheta) a(n) \frac{\beta \widetilde{\mathbb{E}\phi'}}{q}$. The price of money ϕ is then given by $\beta \widetilde{\mathbb{E}\phi'} + \sum_\vartheta \mu_\vartheta$. The total amount of goods taken to the decentralized market (nq in the resource constraint) is $\sum_\vartheta \mathbb{P}(\vartheta) n_\vartheta q_\vartheta$, as the total number of buyers with preference shock ϑ is $\mathbb{P}(\vartheta)$, and the total number of sellers serving these buyers is $\mathbb{P}(\vartheta)$, each bringing q_ϑ to the decentralized market.

2.8.2 Precautionary Demand for Money under Multiplicative Preference Uncertainty

In this section, we study the effect of idiosyncratic preference shocks, ϑ, premultiplying decentralized utility $u(q)$. Buyers post $(q_\vartheta, d_\vartheta, n_\vartheta)$ for each possible realization of ϑ before the actual ϑ realizes, and subsequently visit the appropriate market. We show that for sufficient risk aversion, multiplicative preference shocks imply 'monotone' behavior: higher preference shocks raise buyer meeting rates, quantity bought and money paid across the board, for nonbinding shocks. If a shock implies a binding cash constraint, the cash constraint is binding for all shocks that are even higher. Moreover, the value of money (on top of $\beta \widetilde{\mathbb{E}\phi'}$) is determined by marginal value of consumption for binding shocks, where higher shocks contribute more to the value of money (per unit of probability). This both facilitates the understanding of the behavior of the agents in the model, and helps us by setting up an efficient solution algorithm in the computational exercise.

In this analysis we use the notation $\beta\phi'$ for $\beta \widetilde{\mathbb{E}\phi'}$, and β for $\beta \widetilde{\mathbb{E}}$, as we keep these future variables constant, while varying the realization of ϑ.

Lemma 2. *When the money constraint does not bind, q_ϑ,d_ϑ and n_ϑ (and d_ϑ/q_ϑ) are increasing in ϑ if $RRA > (1-\eta)$.*

The proof is in the appendix.

Given these results, in particular d_ϑ (weakly) increasing in ϑ, it is now straightforward to deduce the following result,

Corollary 1. *If $RRA > 1-\eta$, and if a shock ϑ leads to $d_\vartheta = 1$, then for all shocks $\vartheta_i > \vartheta$, $d_{\vartheta_i} = 1$.*

Finally, if more than one shock is binding, it will not necessarily be the case that q_ϑ is increasing in ϑ among all shocks that result in bind constraints. Intuitively, if for a lower ϑ it comes out of the competitive search solution that the buyer surrenders all his money in exchange for q_ϑ, then the more desperate buyer cannot do more then surrendering all his money; he can either increase q_ϑ at the cost of n_ϑ or the other way around. Below we show that for larger degrees of relative risk aversion, those with high ϑ prefer a higher matching probability.

Lemma 3. *If two or more shocks $\vartheta_l, \dots, \vartheta_h$ result in binding cash constraints, n_ϑ is increasing in ϑ, but q_ϑ is decreasing in ϑ, if $RRA > 1$. Furthermore,*

$$\frac{\mu_{\vartheta_h}}{\mathbb{P}(\vartheta_h)} > \frac{\mu_{\vartheta_l}}{\mathbb{P}(\vartheta_l)}.$$

The proof is in the appendix.

2.8.3 Efficiency

Let us solve the efficient allocation of resources that maximizes the ex ante expected utility of each individual in the centralized market. We assume that everybody is identical in the first centralized market, and solve for the equally weighted Pareto problem; however, as utility is linear in hours, utility is transferable as long as we don't hit the bounds on h, and the allocation is not affected by any chance in the distribution of Pareto weights (except for different distribution h's adding up to the same H).

The planner's maximization problem can be formulated recursively, along the same lines as above.

$$V(S) = \max_{C,H,K',\{q_\vartheta,n_\vartheta\}} u(C) - AH + \sum_\vartheta \mathbb{P}(\vartheta) a(n_\vartheta)(\vartheta u(q_\vartheta)) + \beta \mathbb{E}[V(S')] \tag{54}$$

subject to

$$G + f(K,H) + (1-\delta)K = C + K' + \sum_\vartheta \mathbb{P}(\vartheta) n_\vartheta q_\vartheta$$

$$G' = \sum_\vartheta \mathbb{P}(\vartheta)\big(n_\vartheta - a(n_\vartheta)\big) q_\vartheta$$

It follows that

$$u'(c) = \beta \mathbb{E}_{z'}[u'(c')(1 + e^{z'} f_K(K', H') - \delta)] \tag{55}$$

$$u'(c) = \frac{A}{e^z f_H(K, H)} \tag{56}$$

With respect to q_ϑ, we derive

$$0 = -\, u'(c) + \frac{a(n_\vartheta)}{n_\vartheta}(\vartheta u'(q_\vartheta) - \beta \mathbb{E}_{z'}[u'(c')]) + \beta \mathbb{E}_{z'}[u'(c')] \tag{57}$$

$$\iff \frac{a(n_\vartheta)}{n_\vartheta}(\vartheta u'(q_\vartheta) - \beta \mathbb{E}[u'(c')]) = \beta \mathbb{E}[(e^z f_k(K', H') - \delta) u'(c')] \tag{58}$$

Substituting (57) into the first order condition with respect to n_ϑ, we find,

$$\eta(\vartheta u(q_\vartheta) - \vartheta u'(q_\vartheta) q_\vartheta) - (1 - \eta)(\vartheta u'(q_\vartheta) q_\vartheta - \beta \mathbb{E}[u'(c')] q_\vartheta) = 0, \tag{59}$$

Condition (58) is very intuitive, it says that the return on a unit invested in the decentralized market should equal the return when it is invested as capital.

From the proof of Lemma 1, we know that nonbinding q_ϑ can be chosen by a high enough $\widetilde{\mathbb{E}\phi'}$ or low enough $1+\varpi$. Choosing $1+\varpi'$ such that the cash constraint will not bind when the money market clears at $\tilde{m} = 1$ and $d_{\vartheta_h} = 1$ for the highest shock ϑ_h, is our version of the Friedman rule[10] What we want to show is that this indeed yields the same allocation as the social planner's problem solved above. Thus, the Friedman rule in our model dictates

$$\vartheta u'(q_{\vartheta_h}) q_{\vartheta_h} = \beta \mathbb{E}\left[\frac{\phi'}{1+\varpi'} u'(c')\right] d_{\vartheta_h} = \beta \mathbb{E}\left[\frac{\phi'}{1+\varpi'} u'(c')\right],$$

where q_{ϑ_h} follows from (52). Equations (52) and (105) are equivalent by construction. Note that $\phi u'(c) = \beta \mathbb{E}\left[\frac{\phi'}{1+\varpi'} u'(c')\right]$. What is left to show is that (57) is equivalent to the zero profit condition of retail firms, which follows from dividing (57) by $u'(c)$.

Result 2. *Constrained-efficiency is reached at the Friedman rule.*

Result 3. $(q_\vartheta, n_\vartheta)$ *is constrained-efficient for all non-binding shocks* ϑ

This result, a strengthening of result 1, states that as long as the cash constraint does not bind for a shock ϑ, in equilibrium q_ϑ, n_ϑ are constrained-efficient; they are not affected by the value of $\widetilde{\beta\mathbb{E}\phi'}$ as long as the cash constraint stays non-binding. Any change in $\widetilde{\beta\mathbb{E}\phi'}$ is offset by a proportional change in d_ϑ. The result follows from the fact that equation (105) holds for every non-binding shock.

[10] Choosing consistently a level of $1+\varpi$ even lower than the one dictated here is inconsistent with equilibrium, as the value of money will explode and exceed feasibility at some point. This situation is avoided if we choose ϖ' such that the highest shock is not binding (in the sense that if an agent had more money he would not spend one bit more), but $d_{\vartheta_h} = 1$.

2.8.4 Nominal Wedge and Search Wedge: Differences from First-Best

The cash constraint in the model that stems from the idiosyncratic shock to liquidity need, as well as the search frictions, create wedges relative to the unconstrained first-best allocation. Search frictions imply that there is a chance that goods will not be sold in the decentralized market, but only in the next centralized market. The expected value today of a good in the centralized market tomorrow is $1/(1+r_t)$. The constrained-efficient allocation takes into account this possibility of not matching.

Rewriting (57) in terms of (expected) interest rates, we find in the constrained-efficient case:

$$\frac{\vartheta u'(q_\vartheta)}{u'(c)} = \frac{1}{1+r} + \frac{n_\vartheta}{a(n_\vartheta)}\frac{r}{1+r} \tag{60}$$

We call the right-hand side of this expression the *search wedge*. The term $\frac{n_\vartheta}{a(n_\vartheta)}$ is 1, when the retail firms are certain to match. In this case, $\vartheta u'(q_\vartheta) = u'(c)$, and the constrained-efficient allocation coincides with the first-best allocation (in the absence of frictions). The lower the probability of matching for retail firms, the higher $\frac{n_\vartheta}{a(n_\vartheta)}$, the inverse of the firm's matching rate, and the higher the ratio $\frac{\vartheta u(q_\vartheta)}{q_\vartheta}$.

Now, consider the case of positive nominal interest rates. From the firm's zero profit condition, we derive

$$\frac{d_\vartheta}{q_\vartheta} = \left(\frac{1+i}{\phi}\right)\left(\frac{n_\vartheta}{a(n_\vartheta)}\frac{r}{1+r} + \frac{1}{1+r}\right). \tag{61}$$

Here, the price paid per decentralized market product equals the price per product paid in the centralized market times the nominal interest rate – which is the *nominal wedge* created by the possibility of binding liquidity shocks – times the search wedge. (Notice that if $1+i=1$, then the only remaining wedge is the search wedge.) If we now substitute this condition into (42) and (43), we see first, for nonbinding shocks that q_ϑ is such that

$$\vartheta\frac{u(q_\vartheta)q_\vartheta}{u'(c)} = \beta\widetilde{\mathbb{E}\phi'}d_\vartheta = \frac{\phi d_\vartheta}{1+i},$$

which leads to

$$\frac{\vartheta u'(q_\vartheta)}{u'(c)} = \frac{1}{1+r} + \frac{n_\vartheta}{a(n_\vartheta)}\frac{r}{1+r}, \tag{62}$$

hence the wedge equals the constrained efficient wedge for nonbinding shocks. Notice that for non-binding shocks, the nominal wedge does not play a role. For binding shocks, from (42), we get, in the case of one binding shock,[11]

$$\frac{\vartheta u(q_\vartheta)}{u'(c)} = \left(1 + \frac{i}{\mathbb{P}(\vartheta)a(n_\vartheta)}\right)\left(\frac{n_\vartheta}{a(n_\vartheta)}\frac{r}{1+r} + \frac{1}{1+r}\right). \tag{63}$$

[11] This maps directly into predictions one would get for a non-search version of this model, i.e. for a stochastic cash-credit good model, where for a non-binding shock, the MRS between cash and credit goods would be 1, and for binding shocks the MRS would be $1 + i/\mathbb{P}(\vartheta)$.

2.8.5 Dynamic Behavior of the Value of Money

An important dimension of a monetary model, even more so in a model with fiat money, is how the value of money today is affected by the value of money in the future. Empirically, the pattern is that expectations of future inflation lead to (weakly) higher prices in the present. In monetary models, this monotonicity is not a guaranteed outcome; indeed, it is relatively easy to construct models where higher value of money tomorrow lead to a higher value today, which in turn would lead to a lower value of money yesterday etc.[12] In this section we investigate the impact of changes in next period's value of money on today's value of money.

Lemma 4. *If RRA is close to or greater than 1, for realizations of ϑ where the cash constraint binds, the elasticity of q_ϑ with respect to small changes in tomorrow's value of money $\widetilde{\mathbb{E}\phi'}$, $\varepsilon_{q_\vartheta,\beta\widetilde{\mathbb{E}\phi'}}$, lies between 0 and 1.*

The proof is in the appendix.

Let us look further at the effect of an increase in the value of money tomorrow on the value of money today. This is given by

$$\phi = \sum_{\{\vartheta\}^b} \Big(\mathbb{P}(\vartheta) a(n_\vartheta) \left(\vartheta u'(q_\vartheta) q_\vartheta - \beta\phi' \right) \Big) + \beta\phi', \tag{64}$$

for binding ϑ's, $\vartheta \in \{\vartheta\}^b$, from (42). A look at this equation tells us that the behavior of ϕ in response to $\beta\phi'$ will depend on the effect of the latter on $u'(q_\vartheta)q_\vartheta$. Now, $\varepsilon_{u'(q)q,q} = 1 - RRA(q)$, and since $\varepsilon_{q,\beta\phi'} > 0$, it follows that

$$\varepsilon_{u'(q_\vartheta)q_\vartheta,\beta\phi'} < 0, \text{ if } RRA(q_\vartheta) > 1.$$

This means that for utility functions with $RRA > 1$, if there was certainty about matching, and about facing a shock leading to a binding constraint, the value of today's money would fall in response to a rise in the value of money tomorrow. The reason is that decreasing marginal utility is strong enough to outdo the fact that one unit of money will buy more consumption (where all inframarginal units of money buy more consumption as well). However, because of both the match uncertainty and the idiosyncratic preference uncertainty, money is not used with a given probability, in which case an increase in tomorrow's value directly translates into an increase in value today. Thus, we have two opposing forces here: the marginal value of money when used decreases, but the value of money when not used increases.

[12] This can give rise to a high volatility of velocity, but yields unrealistic time series of inflation and nominal interest rates (see e.g. Hodrick et al. 1991). Therefore a theory which aims to account, for example, for the volatility of velocity in the data should not be built on this non-monotonicity.

2.9 Equilibrium System of Equations

To summarize the above discussion, the system of equations that characterizes the equilibrium of this model is given by (41), (42), (43), together with (31),(30), the free entry condition for retail firms (16), the law of motion for G, and the exogenous shock process (19), while the aggregate feasibility constraint (22) holds, and all individual h_i's are feasible. In equilibrium, aggregate variables, which we capitalized before, equal many of the individual variables; we proceed to use the lower case for them. We also now use the last equality in the bond pricing equation (33) to simplify the system further, by replacing the stochastic pricing kernel of nominal bonds, $\widetilde{\beta\mathbb{E}\phi'}$, with $\phi/(1+i)$. Finally, we abbreviate notation by making implicit the dependence of all endogenous variables on the states of the model.
The system of equations to be solved is thus:

$$U'(c) = \beta\mathbb{E}[U'(c')(1 + e^{z'} f_k(k', h') - \delta)] \tag{65}$$

$$U'(c) = \frac{A}{e^z f_h(k, h)} \tag{66}$$

$$\eta\left(\vartheta\frac{u(q_\vartheta)}{U'(c)} - \frac{\phi}{1+i}d_\vartheta\right)\frac{\phi}{1+i}d_\vartheta - (1-\eta)\left(\frac{\phi}{1+i}d_\vartheta - \beta\widetilde{\mathbb{E}}q_\vartheta\right)\frac{\vartheta u'(q_\vartheta)q_\vartheta}{U'(c)} = 0 \;\forall\vartheta \tag{67}$$

$$\sum_\vartheta\left(\mathbb{P}(\vartheta)a(n_\vartheta)\frac{\vartheta u'(q_\vartheta)q_\vartheta}{U'(c)d_\vartheta} - \mathbb{P}(\vartheta)a(n_\vartheta)\frac{\phi}{1+i}\right) + \frac{\phi}{1+i} - \phi = 0 \tag{68}$$

$$\left(\frac{\vartheta u'(q_\vartheta)q_\vartheta}{U'(c)d_\vartheta} - \frac{\phi}{1+i}\right)(\tilde{m} - d_\vartheta) = 0, \frac{\vartheta u'(q_\vartheta)q_\vartheta}{U'(c)d_\vartheta} \geq \frac{\phi}{1+i}, \text{ and } d_\vartheta \leq \tilde{m}, \;\forall\;\; \vartheta \tag{69}$$

$$q_\vartheta = \left(\frac{a(n_\vartheta)}{n_\vartheta}\right)\frac{\phi}{1+i}d_\vartheta + \left(1 - \frac{a(n_\vartheta)}{n_\vartheta}\right)\beta\widetilde{\mathbb{E}}q_\vartheta, \quad \forall(q_\vartheta, d_\vartheta, n_\vartheta), \tag{70}$$

$$g' = \sum_\vartheta \mathbb{P}(\vartheta)(n_\vartheta - a(n_\vartheta)q_\vartheta \tag{71}$$

$$g + e^z f(k, h) + (1-\delta)k = c + k' + \sum_\vartheta \mathbb{P}(\vartheta)n_\vartheta q_\vartheta \tag{72}$$

$$[z', i'] = \Xi[z, i, \pi, y] + [\varepsilon_1', \varepsilon_2'] \tag{73}$$

3 Calibration

The model period is a quarter. The functional forms that we choose are as follows:

$$
\begin{aligned}
U(c) &= \frac{c^{1-\sigma}}{1-\sigma} \\
u(q) &= \frac{x_1 q^{1-\sigma}}{1-\sigma} + x_2 \\
f(k,h) &= k^{\theta} h^{1-\theta} \\
m(s,b) &= \frac{sb}{(s^{\kappa} + b^{\kappa})^{\frac{1}{\kappa}}}
\end{aligned}
$$

Note that this choice of the matching function, due to Den Haan, Ramey and Watson (2000), guarantees that matching probabilities cannot exceed 1, but at the cost of a varying matching elasticity η.

In terms of parameterizing the model, the ultimate goal is to employ information not only in aggregate data, but also in micro data, to calibrate the model. For example, the model gives us predictions on retail markups and inventories - thanks to the fact that we model the retail sector with explicit search frictions - that we want to exploit in mapping it to the data. We will do this calibration by simulated method of moments. As the full estimation exercise is in itself involved, we begin with a more basic calibration, where we target the steady state aggregate properties of the economy as is standard in the RBC literature. The complication comes from the addition of the nominal side, which requires taking some stands on parameters as described below.

In total, we need to calibrate the following parameters, given our functional form choices: on the real side, we have β, σ, A, θ, δ. On the nominal side, the parameters to calibrate are x_1, x_2 and κ. We also have to calibrate the process for the idiosyncratic shock ϑ. Finally, Ξ, which is a 2×4 matrix, the standard deviations σ_{ε_1} and σ_{ε_2}, as well as the covariance of the two shocks, $\sigma_{\varepsilon_1,\varepsilon_2}$ have to be calibrated to parameterize the real and monetary aggregate shock processes.

We parameterize the model as follows. $\beta = 0.9901$ matches the annual capital-output ratio of 3. $\sigma = 2$ is chosen within the standard range of 1.5-3 in the literature. $A = 34$ is chosen to match aggregate labor supply of 0.3, given the other parameter choices. $\theta = 0.36$ is the capital share of output as measured in the data. $\delta = 0.02$ gives quarterly depreciation rate of 2%, consistent with existing estimates in the data.

The constant $x_1 = 6$ is chosen to set the size of the retail market at 75% of total consumption, consistent with the data, found in Telyukova(2008), that roughly 75% of consumer transactions, in terms of value, take place using liquid payments methods - cash, checks, and debit cards. Parameters x_2 and κ jointly influence the buyer-seller ratio (and thus, the matching probabilities) in the decentralized market,

Table 1: Calibration

β	σ	A	θ	δ	x_1	x_2	κ
0.990	2	34	0.36	0.02	6	35	2

Table 2: Preference Shock Process

ϑ_1	ϑ_2	ϑ_3	ϑ_4	ϑ_5	$\mathbb{P}(\vartheta_1)$	$\mathbb{P}(\vartheta_2)$	$\mathbb{P}(\vartheta_3)$	$\mathbb{P}(\vartheta_4)$	$\mathbb{P}(\vartheta_5)$
0.34	0.58	1	1.73	2.98	0.07	0.24	0.38	0.24	0.07

as well as the values of q and d, and hence the markups in the decentralized market. We fix $\kappa = 2.0$ for now, and set $x_2 = 35$. These give us steady-state markups of 3-8% in the retail submarkets, which is on the low side of the 18-45% range that surveys of retailers find (see, for example, Faig and Jerez (2006)). The understatement of the size of the markups will dampen the results for the dynamics of nominal variables, thus giving us the idea of the lower bound of what the model can do. In final calibration (SMM), we will set the parameters to match micro estimates of markups (and micro data properties of inventories) to pin down the parameters above.

To calibrate the process for the preference shock, as a first pass, we use micro-data estimates from Telyukova (2008), which estimates a similar preference shock process by matching time series properties of survey data on liquid household expenditures. In that paper, liquid consumption is measured in the Consumer Expenditure Survey (CEX). Its time series properties (average monthly autocorrelation and standard deviation across households) are computed in the data. These moments are then used as a target for corresponding simulated moments in the model. We will re-estimate the process within our own model later on. For now, we will assume that the shocks are i.i.d., with the shock realizations and probabilities given by the stationary distribution of shocks as calibrated in Telyukova(2008). The above discussion is summarized in tables 1 and 2.

Finally, we calibrate technology and monetary policy shocks. We model these as a joint stochastic process, and parameterize it by estimating a VAR of the following form:

$$\begin{aligned} z_t &= \xi_{zz} z_{t-1} + \xi_{zi} \ln\left(\frac{1+i_{t-1}}{1+\bar{i}}\right) + \xi_{z\pi} \ln\left(\frac{1+\pi_{t-1}}{1+\bar{\pi}}\right) + \varepsilon_1 \\ \ln\left(\frac{1+i_t}{1+\bar{i}}\right) &= \xi_{ii} \ln\left(\frac{1+i_{t-1}}{1+\bar{i}}\right) + \xi_{i\pi} \ln\left(\frac{1+\pi_{t-1}}{1+\bar{\pi}}\right) + \xi_{iy} \ln\left(\frac{y}{\bar{y}}\right) + \varepsilon_2 \end{aligned}$$

where the Solow residual is measured in the standard way, and we take out the linear trend from both the Solow residual and the output series. The variables with

Table 3: VAR Estimates of the Aggregate Shock Process, 1984-2007

ξ_{zz}	ξ_{zi}	$\xi_{z\pi}$	ξ_{ii}	$\xi_{i\pi}$	ξ_{iy}	σ_{ε_1}	σ_{ε_2}	$\sigma_{\varepsilon_1,\varepsilon_2}$
0.937	-0.144	0.060	0.780	0.120	0.008	0.004	0.001	-0.0000002

bars over them capture long-term averages of the respective variables in our sample period, as is standard in estimating central banks' targets in policy rules. The sample of data on which we estimate this process is from 1984 until 2007, to capture the period when the Federal Reserve is perceived to have begun using (implicit) inflation targeting. Notice that our productivity process and the interest rate rule both depend on endogenous variables. We use the Federal Funds rate as the measure of choice of interest rates in the data. The resulting VAR coefficients are in table 3.

4 Computation

We employ the Parameterized Expectations Approach (PEA) to solve the model. The main idea of the method is to approximate the expectations terms in our Euler equation system (65)-(73) - two in total - by polynomial functions of the state variables. The coefficients of the polynomials form the basis of an iterative approach pioneered by Den Haan and described in our context below. In order for the algorithm to converge, we need a good first guess of these coefficients, which we will derive by using a version of homotopy - that is, by first solving a second-order approximation of the model and deriving the first set of polynomial coefficients from the solution.

In particular, let χ_t denote the state variables of the problem (known at time t that help predict the expectations terms), ζ_t denote all the endogenous and exogenous variables that appear inside the expectation terms, and u_t denote the shocks of the problem. We have

$$\begin{aligned}
\chi_t &= \{g_t, k_t, z_t, i_{t-1}, \phi_{t-1}(1+\varpi_{t-1})\} \\
\zeta_t &= \{k_t, h_t, n_t, q_t, d_t, \phi_t, g_t, z_t\} \\
u_t &= z_t, i_t.
\end{aligned}$$

Note that c_t does not appear in ζ_t because it can be determined from the other variables and the budget constraint. The variables we are solving for are $\{c_t, k_{t+1}, h_t, n_t, q_t, d_t, \phi_t, g_t\}$.

The approximating functions for the expectation terms are always just functions of x_t. We choose the following forms:

$$\mathbb{E}\left[(c')^{-\sigma}(1+e^{z'}\theta(k')^{\theta-1}(h')^{1-\theta}-\delta)\right] = \psi^1(\chi,;\gamma^1) \quad (74)$$
$$\mathbb{E}\left[\frac{1}{w'}\right] = \frac{\widetilde{\mathbb{E}}}{w} = \psi^2(\chi;\gamma^2)$$

where, for example,

$$\psi^j(\chi;\gamma^j) = \gamma_1^j \exp(\gamma_2^j \log g + \gamma_3^j \log k + \gamma_4^j z + \gamma_5^j \log i_{-1} + \gamma_6^j \log[\phi_{-1}(1+\varpi_{-1})])$$

The accuracy of approximation can be increased by raising the degree of approximating polynomials above. We now substitute the expressions in (74) into the system of Euler equations, to obtain the system of equations that we use in solving the model. The full iterative algorithm is described in the appendix.

5 Results

5.1 Contributions of Preference Shocks

In order to demonstrate what precautionary demand for money does for dynamics of nominal variables in our model, we compute two versions of the model - the one described above, and a version where we shut down the idiosyncratic shocks, thus shutting down precautionary motive for holding money. We focus for now on the dynamics of two measures of velocity of money - production and consumption velocities. We measure both based on the M2 aggregate in the data, to follow the previous literature on the subject. In the data, the measures are

$$V_y = \frac{NGDP}{M2}$$
$$V_c = \frac{PC}{M2},$$

where we measure NGDP and nominal consumption of nondurables and services from NIPA data. In the model, the corresponding measures take into account production and value added, as well as consumption, in both centralized and decentralized markets as follows:

$$V_y = \frac{y}{\phi} + \sum_{i=1}^{5} \mathbb{P}(\vartheta_i)a(n_{\vartheta_i})(d_{\vartheta_i} - \frac{q_{\vartheta_i}}{\phi})$$
$$V_c = \frac{c}{\phi} + \sum_{i=1}^{5} \mathbb{P}(\vartheta_i)a(n_{\vartheta_i})d_{\vartheta_i}$$

Table 4 summarizes the results concerning the dynamic properties of some key nominal variables. It is important to emphasize that we are not targeting any of

Table 4: Dynamic Properties of the Model - Comparison of Model With and Without Preference Shocks

Moment	Data	No-Shock Model	Full Model
$\mathbb{E}(V_y)$	1.897	1.560	1.156
$\mathbb{E}(V_c)$	1.120	1.210	0.882
$\sigma(V_y)$	0.017	0.009	0.015
$\sigma(V_c)$	0.014	0.0002	0.012
$\sigma(y)$	0.009	0.011	0.010
$\sigma(1+i)$	0.003	0.001	0.002
$corr(V_y, y)$	0.638	0.997	0.595
$corr(V_c, y)$	0.447	-0.823	0.005
$corr(V_y, g_y)$	0.059	0.260	0.106
$corr(V_c, g_y)$	-0.094	-0.288	-0.068
$corr(V_y, 1+i)$	0.714	-0.072	0.821
$corr(V_c, 1+i)$	0.688	-0.420	0.998
$corr(1+\pi, 1+i)$	0.529	0.778	0.760

All data logged and BP filtered. Data period: 1984-2007. Velocity moments calculated based on M2. Interest rate is the Fed Funds rate. Inflation measured based on GDP deflator. g_y refers to output growth. "No-Shock" model is the version of the model with idiosyncratic preference shocks shut down.

these moments in choosing the parameters of the model. As is clear from the table, introducing precautionary motive for holding money into our model makes an enormous difference for the performance of the model: without it, the model is not able to capture any of the moments in the data, while introducing precautionary demand makes the model align quite successfully on nearly all of the dimensions listed. We over-predict correlation of consumption velocity with gross nominal interest rates, and, on a related note, underpredict volatility of nominal interest rates, but we do better on these moments with the preference shocks than we would without, and aside from these moments, our model is at least in the ballpark of the data numbers. This is encouraging, as our calibration is not complete, so we do not expect to be capturing all of the moments yet.

The main contribution of the preference shocks in the model - that is, of introducing the precautionary motive for holding money - is that it adds dynamics to velocity of money that would not be there otherwise. This is what leads to the vast difference in the results of the two models in the table above, especially in moments like volatility of velocity and correlation of velocity with interest rates. This is further clarified by the impulse response functions generated by the model, shown in figures 1 through 4 below. These figures show the response of market tightness, consumption and money spent in the decentralized market, as well as of real variables in the centralized market, to, respectively, an orthogonalized monetary policy shock (figures 1,2) and an orthogonalized productivity shock (figure 3, 4). Notice that the discussion that follows of impulse responses can be linked to our the discussion of the nominal wedge in section 2.8.4, which only affects consumption of agents who are constrained in their preference shock realization.

We have two types of agents in the model - those who face binding preference shocks (ϑ_5, with corresponding consumption levels q_5 and money spending d_5 in the decentralized market), and those who face non-binding shocks (in the model, shocks $\vartheta_1 - \vartheta_4$). Those whose shocks are binding always spend all of their money. Thus, in response to a decrease in the value of money (through a positive nominal interest rate shock, for example, figure 1), when prices of consumption goods increase, these agents are constrained in terms of how much money they spend - they cannot respond to increased prices by spending more money, so their consumption responds, just as the nominal wedge would predict, while their money spending does not. Hence, the fall in the impulse response for q_5, with the flat line (not shown) for d_5. In contrast, those who are not constrained (all remaining shocks) always attain their optimal consumption levels (they have no nominal wedge affecting their marginal rate of substitution between centralized and decentralized market goods). Thus, even with a price increase that results from a monetary policy shock, they can maintain their consumption levels, but respond by spending more of their money. Hence, there is no response on impact in $q_1 - q_4$, but a pronounced increase on impact in $d_1 - d_4$. This response in d contributes to a response of velocity of money to a change in

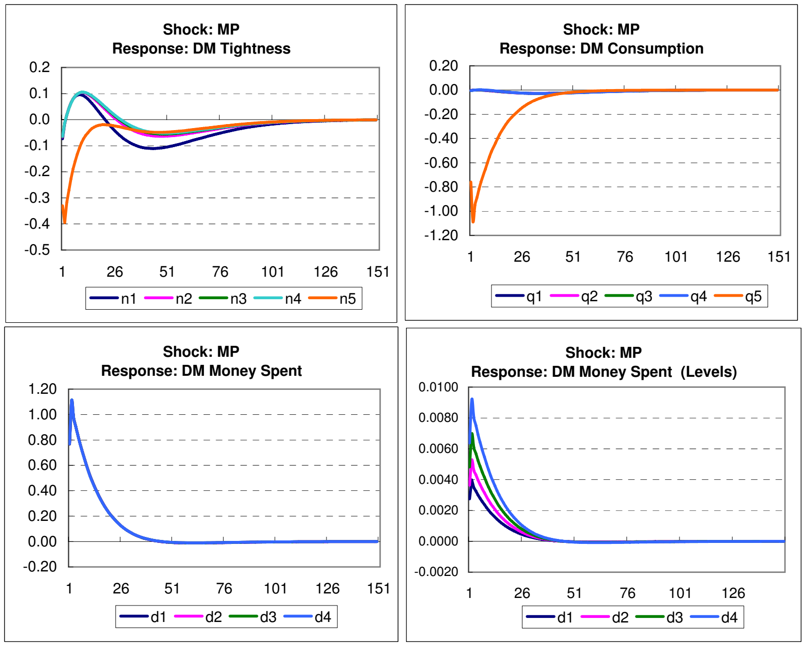

Figure 1: Decentralized Market Impulse Responses to a Monetary Policy Shock, % of Steady State

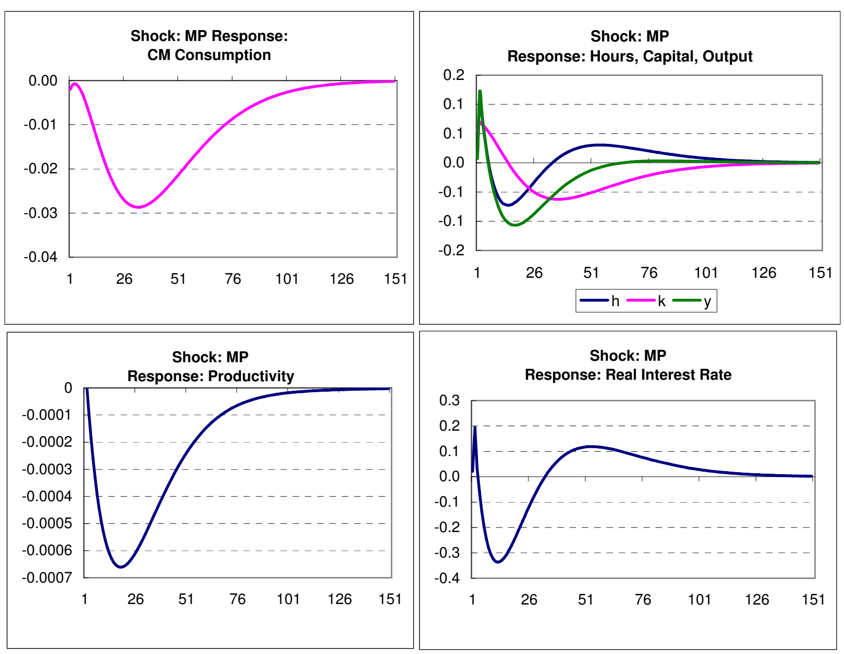

Figure 2: Real Variables' Impulse Responses to a Monetary Policy Shock, % of Steady State

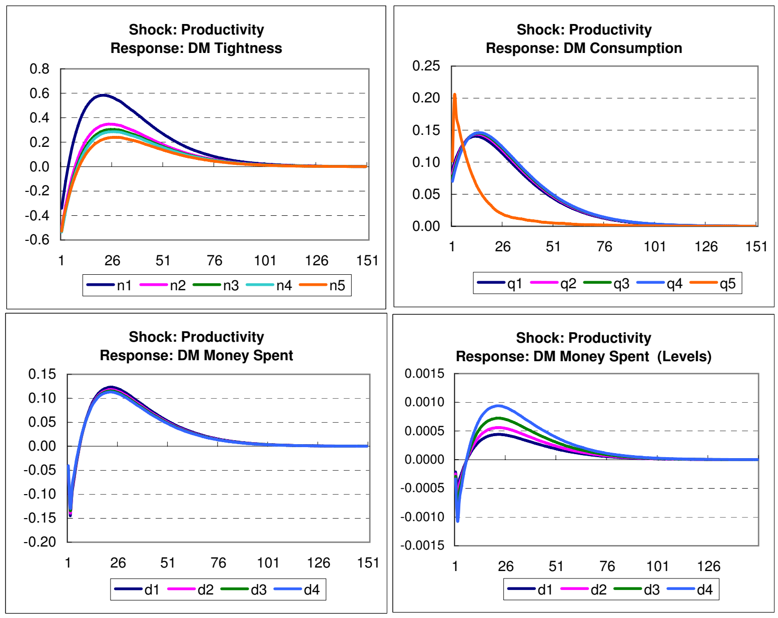

Figure 3: Decentralized Market Impulse Responses to a Productivity Shock, % of Steady State

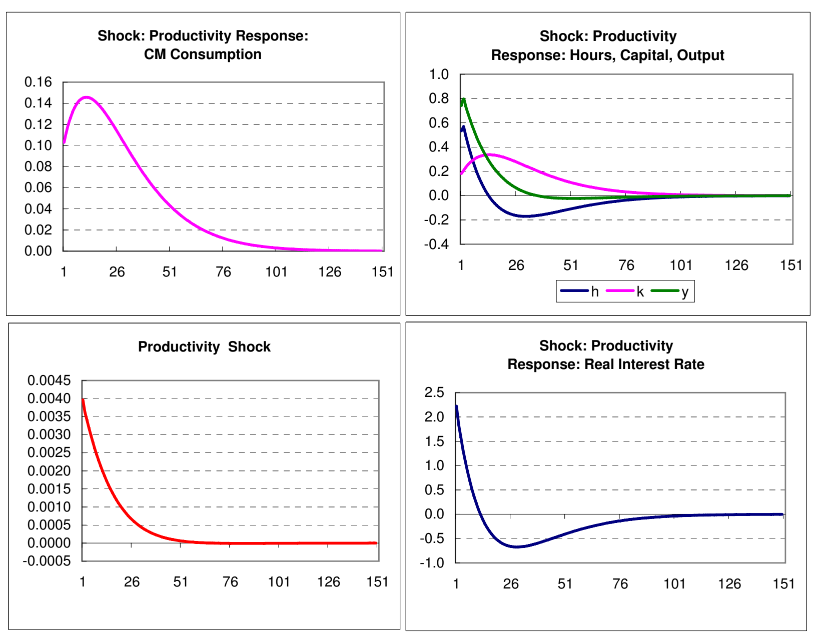

Figure 4: Real Variables' Impulse Responses to a Productivity Shock, % of Steady State

interest rates that would not be there if there were no precautionary demand motive in the model: in that case, everyone's decentralized-market consumption would go down, as q_5 does, with no adjustment in money spent or velocity, upon the impact of the shock.

The dynamics of the decentralized-market variables in the quarters after the shock hits are due to the interaction of productivity and interest rates. Even though the shocks themselves are orthogonalized, the laws of motion for both the productivity shock and the interest rate rule have endogenous components. For example, the nominal interest rate enters the productivity process with a negative coefficient. So once interest rates increase due to a monetary policy shock, the level of productivity drops, which in turn has propagation effects similar to what a negative productivity shock would do. Namely, once the productivity level drops in the quarters following the monetary policy shock, it has negative effects on output, investment and consumption in both centralized and decentralized markets (see figure 2). This is where the subsequent drop in $q_1 - q_4$ comes from in figure 1. As the interest rates decrease and the productivity level starts to recover, prices in the decentralized market decrease, allowing both constrained and unconstrained individuals to increase their consumption, while those who are unconstrained also decrease their spending.

Responses to the productivity shock, upon impact, are of similarly diverse nature for those who are constrained and those who are not. When the productivity shock hits, everyone wants to consume more, as prices of decentralized-market goods have fallen. This is reflected in the fall on impact of $d_1 - d_4$ and the increase on impact of $q_1 - q_5$. Again, notice that this creates a dynamic response of velocity of money that would not occur if all the individuals were equally constrained: in that case, only their consumption would respond. In the quarters following initial impact, the increase in output that has resulted from the high productivity shock (figure 4) feeds into the interest rate rule, causing prices to rise, which in turn means that constrained households decrease their decentralized-market consumption q_5 much more rapidly than the unconstrained households, while the unconstrained can continue to increase their consumption optimally along with their spending. The hump-shaped response in both the centralized- and decentralized-market consumption is standard in the business cycle literature, in response to productivity shocks.

To sum up, the impulse response functions so far discussed highlight the role of preference shocks in producing a dynamic response in velocity that would not be present in the model had everyone's demand for money been deterministic. This is what gives the model the much-improved match to moments listed in table **??**.

5.2 Contributions of Search Frictions

Just as in the above section, we can now discuss the contribution of search frictions to the results in our model. To this end, we compute a version of our model where

Table 5: Dynamic Properties of the Model - Comparison of Model With and Without Search Frictions

Moment	Data	No-Search Model	Full Model
$\mathbb{E}(V_y)$	1.897	1.190	1.156
$\mathbb{E}(V_c)$	1.120	0.907	0.882
$\sigma(V_y)$	0.017	0.013	0.015
$\sigma(V_c)$	0.014	0.011	0.012
$\sigma(y)$	0.009	0.008	0.010
$\sigma(1+i)$	0.003	0.002	0.002
$corr(V_y, y)$	0.638	0.497	0.595
$corr(V_c, y)$	0.447	0.005	0.005
$corr(V_y, g_y)$	0.059	0.105	0.106
$corr(V_c, g_y)$	-0.094	-0.044	-0.068
$corr(V_y, 1+i)$	0.714	0.869	0.821
$corr(V_c, 1+i)$	0.688	0.999	0.998
$corr(1+\pi, 1+i)$	0.529	0.768	0.760

All data logged and BP filtered. Data period: 1984-2007. Velocity moments calculated based on M2. Interest rate is the Fed Funds rate. Inflation measured based on GDP deflator. g_y refers to output growth. "No-Search" model is the version of the model with the search frictions shut down.

we shut down the search frictions. Table 5 compares the moments from this model to the full model. As is apparent, while the search frictions help the model account for the moments in the data (that is, the no-search model does slightly worse on most moments), this effect is not very dramatic quantitatively. It appears that search frictions alone do not add very much to the dynamics of nominal variables in question.

Figure 5 compares the impulse responses of the real variables to a monetary policy shock in the models with and without search frictions. As is apparent from these responses, the search model amplifies somewhat the impact of a monetary policy shock on real variables. For example, the initial response of investment to a monetary shock is six times larger in the model with search than without. Smaller differences are observed in the responses of consumption, labor and output. However, since in either of the models, the real responses to monetary policy shocks are small, this amplification is also small in absolute terms. From this discussion and table 5, these preliminary results seem to suggest that the search wedge discussed in section 2.8.4 does not seem to be very large quantitatively, given our current calibration.

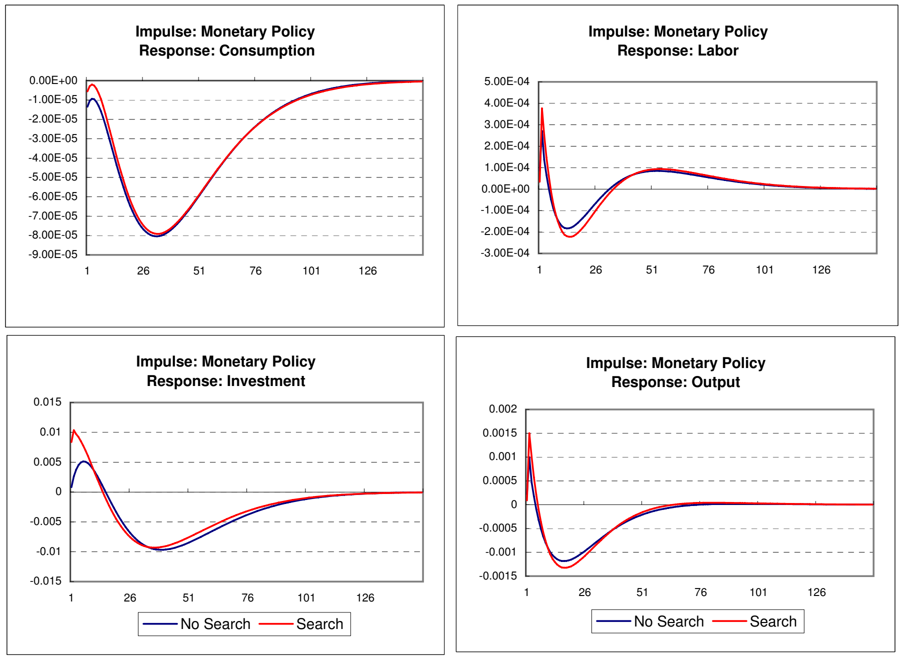

Figure 5: Real Variables' Impulse Responses to a Monetary Policy Shock, No-Search versus Search Models, % of Steady State

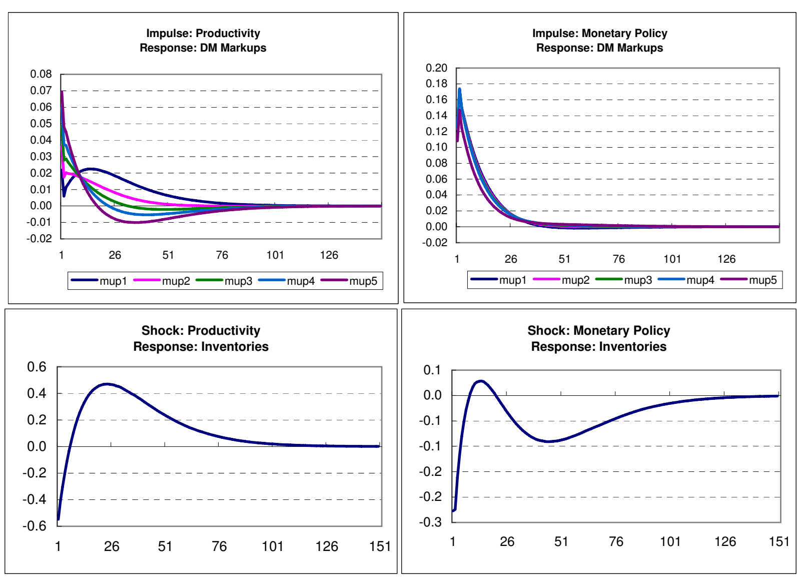

Figure 6: Responses of Markups and Inventories to a Productivity and Monetary Policy Shocks, % of Steady State

The most interesting, and possibly quantitatively significant, potential contribution of the search frictions is that they generate both retail markups (in the prices that retail firms charge to households) and inventories (goods unsold by firms that did not manage to match). Figure 6 shows preliminary results on the impulse responses of these variables, based on both orthogonalized productivity and orthogonalized monetary policy shocks. For example, upon impact of a positive productivity shock, inventories initially drop and then begin rising immediately. In the model, this is explained as follows: as the productivity shock hits, output, investment in capital and consumption all increase (as can be seen in figure 4), but consumption and investment increase by significantly more than output (in the graphs, the impulse responses are shown as percent of each variable's steady state, so the numbers are not comparable across series). This is because real interest rate also increases on impact, making investment profitable, while making the opportunity cost of inventory investment (participation in the decentralized market) more costly. In order to finance the initial increase in investment, households deplete retail inventories initially, but as marginal product of capital drops and investment begins to come down, inventories build up. At the point where the real interest rate drops below steady state level (near period 10), due to feedback effects of the interest rate rule, inventories are built back up to steady state level and then overshoot, as the opportunity cost of inventory investment at this point is very low. The response of inventories to the monetary policy shock can be analyzed similarly, and of markups to either shock, can be analyzed similarly.

In terms of the direction of these impulse responses, a preliminary look at existing research of cyclical behavior of inventories suggests that inventory investment is countercyclical in the short run, and procyclical in the long run. Our model may be confirming this finding, although we have to make decisions still on how to map inventory investment in the data to retail inventories in our model. Our research on markups is incomplete at this stage, and is work in progress. We can study markups explicitly here, and use them in calibration as well (e.g. we plan to use sizes of markups as a calibration target).

The analysis above also underscores a second point that is delivered to us by the search setup: underlying all of the analysis is the fact that the real interest rate in our model plays a role that it would not play in a reduced-form model. The real interest rate here is the opportunity cost of participating in the decentralized market. Namely, firms that decide to invest in goods to take to the decentralized market face the tradeoff between doing so and investing in centralized-market capital instead, thus earning the return r.

Finally, it is worth pointing out that this model, like most monetary models we are aware of, cannot reproduce the short-term liquidity effect observed in the data: a negative relationship between interest rates and money supply growth. In our model this relationship is also positive.

6 Conclusion

We study the aggregate implications of the significant precautionary demand for money in the data, by incorporating it into a stochastic search model of money that combines a real sector with a monetary sector in one economy. The presence of the real sector allows us to incorporate a standard RBC setup with productive capital and stochastic productivity shocks, while the monetary sector, which we model using competitive search and view as a model of retail, allows explicit study of nominal shocks.

Our first goal is to quantify how well precautionary demand for money does in helping to answer a set of standing questions regarding dynamics of nominal variables. We focus in particular on dynamics of velocity of money, inflation and interest rates. We find that relative to a model where demand for money is deterministic, the precautionary motive has a dramatic impact on the ability of the model to replicate facts in the data.

Our second goal is to understand how search models of money add to our understanding of the data relative to models of money that do not model search frictions explicitly. We find that in our model, search frictions have discernible effects in helping the model account for the moments of interest, and in amplifying real effects of monetary policy, but these effects are not very large. The search frictions also give additional implications that would not be present in a reduced-form model of money - namely, implications for the cyclical behavior of retail inventories and markups. Qualitatively, these seem important and interesting distinctions; the quantitative evaluation of these implications is work in progress.

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APPENDIX

A Proofs

A.1 Proof of Household's Value Function Properties, Prop. 1

The value of the problem is bounded by

$$\frac{\sup_x U(x) + \sup_q u(q)}{1-\beta},$$

and hence we can find the finite supremum of the household's problem. For any history the constraint set is nonempty. Then, by Stokey and Lucas, section 4.1, the solution(s) to the sequential formulation of the household's problem and the recursive formulation coincide, and the values of both problems coincide as well. In the next paragraphs we show that the supremum is in fact attained, so we are correct (if not a bit premature) to speak of a maximum, e.g. in (1).

We now want to derive the properties of the value function and characterize the optimal policy function.

Define an operator T by

$$\begin{aligned} T\phi = & \max_{\{c,h,\tilde{m},k',\mathbf{s}_r,\{(q,d,n)(\vartheta)\}\}} \\ & \Bigg\{ U(c) - Ah + \mathbb{E}_\vartheta \Big[a(n_\vartheta) \Big(\vartheta u(q_\vartheta) + \beta \mathbb{E}[\phi(k', m' - d_\vartheta, g', K', G', z', \varpi)] \Big) \\ & + (1 - a(n_\vartheta)) \beta \mathbb{E}[\phi(k', m', g', K', G', z', \varpi)] \Big] \Bigg\} \end{aligned} \tag{75}$$

subject to

$$\phi m + g + (1+r)k + wh \geq c + \phi \tilde{m} + k' + \mathbf{s}_r \mathbf{P}_r^t \tag{76}$$

$$d_\vartheta \leq \tilde{m} \ \forall \ \vartheta \tag{77}$$

$$\begin{aligned} q_\vartheta = & \left(\frac{a(n_\vartheta)}{n_\vartheta} \right) \mathbb{E}\big[\rho(K', G', z', \varpi) \phi'(K', G', z', \varpi)\big] d_\vartheta \\ & + \left(1 - \frac{a(n_\vartheta)}{n_\vartheta} \right) \mathbb{E}\big[\rho(K', G', z', \varpi)\big] q_\vartheta, \quad \forall (q,d,n)(\vartheta), \vartheta, \end{aligned} \tag{78}$$

$$\phi(K, G, z, \varpi), w(K, G, z, \varpi), r(K, G, z, \varpi), \rho(K', G', z' | K, G, z) \tag{79}$$

$$\mathbf{P}_r^t(K, G, z, \varpi), \mathbf{b_q}(K, G, z, \varpi), \mathbf{b_d}(K, G, z, \varpi), \mathbf{n_r}(K, G, z) \tag{80}$$

$$m' = \tilde{m} + \mathbf{s_r} \mathbf{n_r} \mathbf{b_d} \tag{81}$$

$$g' = \mathbf{s_r} \mathbf{n_r} \mathbf{G} \tag{82}$$

$$K'(K, G, z, \varpi), G'(K, G, z, \varpi)$$

$$z' = \xi z + \varepsilon' \tag{83}$$

$$h \in [0, 1] \tag{84}$$

Lemma A.1.1. *T maps the space of continuous and bounded concave functions into itself.*

Proof. Given K, G, z, the fact that K', G' in (19) are deterministic functions, and z is a finite state Markov process, and the preference shock has a finite domain as well, taking expectations with respect to K', G', z', ϑ over a continuous functions $\phi, a(n)$ makes the expectation $\mathbb{E}_{\vartheta}[.]$ a continuous function as well.

The strictly increasing utility functions imply that the budget constraint holds with equality. Moreover, from (78) d is a continuous function in q, n. The solution set given k, m, g is nonempty ($k' = k$, $c = m+g$, $\mathbf{s_r} = \mathbf{0}, \{(q,n)(\vartheta)\}^{\Theta} = \{(0,0)(\vartheta)\}^{\Theta}, h = 0$ is feasible.) Given that we have transformed all constraints into equality constraint, the upper bound on value of the portfolio $\mathcal{P} = \{c, h, \tilde{m}, k', \mathbf{s}_r, \{(q,n)(\vartheta)\}\}$, and the lower bound on debt (take e.g. the natural debt limit), the solution set is closed and bounded in Euclidean space $\mathbb{R}^m$ (the space of $\{c, h, \tilde{m}, k', \mathbf{s}_r, \{(q,n)(\vartheta)\}\}$), and hence compact-valued. Given any sequence $\{k_n, m_n, g_n\}$ converging to $\{k, m, g\}$ we can find

$$\sup\ (1 + r(K,G,z))k_n + \phi(K,G,z)m_n + g_n - D = \varsigma,$$

where D is the natural debt limit. We then know that each variable in the budget constraint is bounded both from above and below, and is an element of a closed and bounded interval (e.g. $k'_n \in [0, \varsigma]$. Hence, any sequence$\{c_n, h_n, \tilde{m}_n, k'_n, \mathbf{s_r}_n, \{(q_n, n_n)(\vartheta)\}\}$ has a convergent subsequence as well. Taking the limits if each variable, we find that the limits of these variables jointly satisfy the budget constraint and the money constraint (by the standard limit rules). Hence the limit of the sequence is feasible, and hence the constraint correspondence is upper hemi-continuous.

To prove lhc, consider a (k, m, g) and a $p(k, m, g)$. Let $\{k_n, m_n, g_n\}_{n=1}^{\infty}$ be a sequence converging to (k, m, g). Construct a feasible sequence by keeping all variables that are at their lower bound at this bound in the sequence, and all other non-state variables except one variable constant. Then, let this variable be determined by all other variables in the budget constraint, (this variable will be feasible for n large enough, i.e. (k_n, m_n, g_n) close enough to (k, m, g)). If all variables are at their lower bound, then $\{k_n, m_n, g_n\}_{n=1}^{\infty}$ can only approach from above, and hence the constructed sequence will still be feasible if we let one of the non-state variables on the lower bound be determined by the budget constraint. Hence, lhc, and hence continuous.

If ϕ is bounded, then given the boundedness of functions $U(c), u(q), a(n)$, and the restriction on $h, h \in [0, 1]$ means that $T\phi$ will be bounded as well. □

Lemma A.1.2. *Mapping T defines a contraction in the space of bounded and continuous functions. T has a fixed point; and its fixed point is the solution to the recursive equation.*

Proof. The last point is immediate; the second point follows from the completeness of the space of bounded and continuous functions, and the application of Banach's fixed point theorem. Blackwell's sufficient conditions are satisfied to show that T defines a contraction. Define the argmax of the mapping T of a function $\phi_1(k, m, g)$, by $a_1^* = \{c_1, h_1, \tilde{m}_1, k_1', \mathbf{s}_{r1}, \{(q_1, n_1)(\vartheta)\}\}$, and the value of

$$\tilde{T}(\phi_1, a_1) \equiv \Big\{ U(c_1) - Ah_1 + \\ \mathbb{E}_\vartheta \Big[a(n_1(\vartheta)) \Big(\vartheta u(q_1(\vartheta)) + \beta \mathbb{E}[\phi(k_1', m' - d_1(\vartheta), g', K', G', z', \varpi)] \Big) \\ + (1 - a(n_1(\vartheta))) \beta \mathbb{E}[\phi_1(k_1', m', g', K', G', z', \varpi)] \Big] \Big\}$$

□

Then if $\phi_2 > \phi_1$,

$$\tilde{T}(\phi_2, a_2^*) \geq \tilde{T}(\phi_2, a_1^*) \geq \tilde{T}(\phi_1, a_1),$$

for some feasible a_2^*, in particular the arg max in (75) with functionϕ_2. Moreover

$$\tilde{T}(\phi_1, a_1^*) + \beta\gamma = \tilde{T}(\phi_1 + \gamma, a_1^*),$$

for each constant function γ, and discount rate $0 < \beta < 1$.

Lemma A.1.3. *T maps increasing functions in the space of continuous, bounded functions into strictly increasing functions*

Standard proof.

A.2 Spelling Out the Two-Subperiod Problem

Then, the maximization problem in the second subperiod, after the preference shock has realized, can be written as

$$W\Big(k', \tilde{m}, \mathbf{s_r}, \vartheta, K, G, z, \{(q, d, n)\}\Big) = \\ \max_{\{q,d,n\}} \Big\{ a(n) \Big(\vartheta u(q_t) + \beta \mathbb{E} V(k', m' - d, g', K', G', z', \varpi) \Big) \\ + (1 - a(n)) \beta \mathbb{E} V(k', m', g', K', G', z', \varpi) \Big\} \quad (85)$$

subject to

$$
\begin{aligned}
&d \leq \tilde{m} \\
&m' = a(n)(\tilde{m} + \mathbf{s_r n_r b_{rd}}) + (1 - a(n))((\tilde{m} - d + \mathbf{s_r n_r b_{rd}})) \\
&g' = \mathbf{s_r n_r b_{rq}} \\
&\mathbf{b_{rd}}(K, G, z), \mathbf{b_{rq}}(K, G, z), \mathbf{n_r}(K, G, z) \\
&G'(K, G, z, \varpi), K'(K, G, z, \varpi), \\
&z' = \xi z + \varepsilon'
\end{aligned}
$$

where we have used that k' is already determined. Agents take the entire vector of submarkets with their equilibrium quantity-price-tightness(n) relation as something that is given: this is why $(q, d, n)(\vartheta)$ appears in the value function. Variables m', g' denote the money and goods holdings after dividends are paid out at the beginning of the next period; $\mathbf{b_{rq}}, \mathbf{b_{rd}}$ the expectations about their quantity given that the aggregate state at the beginning of this period was (K, G, z, ϖ). The remaining functions are the aggregate laws of motion.[13]

In the first subperiod, the value function is

$$
V(k, m, g, K, G, z) = \max_{(q,d,n)(\vartheta), k', \tilde{m}, \mathbf{s_r}, c, h,} \left\{ U(c) - Ah + \mathbb{E}W(k', \tilde{m}, \mathbf{s_r}, \vartheta, K, G, z, \{(q, d, n)(\vartheta)\}) \right\} \tag{86}
$$

subject to

$$
\phi m + g + (1 + r)k + wh = c + \phi \tilde{m} + k' + \mathbf{s}_r' \mathbf{P}_r^t
$$

$$
q_\vartheta = \left(\frac{a(n_\vartheta)}{n_\vartheta} \right) \mathbb{E}\Big[\rho(K', G', z', \varpi) \phi'(K', G', z', \varpi) d_\vartheta \Big] + \left(1 - \frac{a(n_\vartheta)}{n_\vartheta} \right) \mathbb{E}\Big[\rho(K', G', z', \varpi) q_\vartheta \Big], \ \forall \ \vartheta \tag{87}
$$

$$
\begin{aligned}
&\phi(K, G, z, \varpi), w(K, G, z, \varpi), r(K, G, z, \varpi), \rho(K, G, z, \varpi), \mathbf{P}_r^t(K, G, z, \varpi) \\
&G'(K, G, z, \varpi), K'(K, G, z, \varpi) \\
&z' = \xi z + \varepsilon' \\
&h \in [0, 1]
\end{aligned}
$$

where the household chooses multiple q, d, n to post, but without loss of generality we can restrict ourselves to one such vector per realization ϑ; we subsequently indexed these variables by the realization ϑ for which these submarkets will be visited. Putting these two problems together we get the full-period problem in the main text.

[13] Note that we make the assumption, purely for convenience, that households first spread out the share holdings over all firms of the same type, before buying more shares of the same firm. I.e. the household always chooses to invest in $\mathbf{n_r}$ firms, so the only control is the share of these firms that households own, $\mathbf{s_r}$.

A.3 Proofs for the Decentralized Market Maximization Problem

A.3.1 Proposition 2

Proof. Let us solve the case without idiosyncratic uncertainty. The proof for the general case (with adjustment for the fact that money paid, d_ϑ can be nonbinding) is very similar. Moreover, for ease of notation, denote $u(q)/U'(c)$ simply by $u(q)$. From the zero-profit condition of entry, we can derive the following relation between n and (q,d)

$$a(n) = \left(\frac{\beta\widetilde{\mathbb{E}\phi'}d - \beta\widetilde{\mathbb{E}}q}{q - \beta\widetilde{\mathbb{E}}q} \right)^{\frac{\eta}{1-\eta}}.$$

The problem (where we could incorporate the above relation, thus reducing the choice variables) is to maximize

$$-\phi d + a(n)(u(q) - \beta\widetilde{\mathbb{E}\phi'}d) + \beta\widetilde{\mathbb{E}\phi'}d$$

First order conditions are given by

$$\frac{a(n)}{q}\left(-\frac{\eta}{1-\eta}\frac{\beta\widetilde{\mathbb{E}\phi'}d}{\beta\widetilde{\mathbb{E}\phi'}d - \beta\widetilde{\mathbb{E}}q}\left(u(q) - \beta\widetilde{\mathbb{E}\phi'}d\right) + u'(q)q \right) = 0 \tag{88}$$

Notice the familiar

$$\eta(u(q) - \beta\widetilde{\mathbb{E}\phi'}d)\beta\widetilde{\mathbb{E}\phi'}d - (1-\eta)(\beta\widetilde{\mathbb{E}\phi'}d - \beta\widetilde{\mathbb{E}}q)u'(q)q = 0 \tag{89}$$

implied by this first order condition. The first order condition with respect to d is given by

$$-\phi + a(n)\left(\frac{\eta}{1-\eta}\frac{\beta\widetilde{\mathbb{E}\phi'}}{\beta\widetilde{\mathbb{E}\phi'}d - \beta\widetilde{\mathbb{E}}q}\left(u(q) - \beta\widetilde{\mathbb{E}\phi'}d\right) - \beta\widetilde{\mathbb{E}\phi'} \right) + \beta\widetilde{\mathbb{E}\phi'} = 0 \tag{90}$$

Second order conditions, evaluated at the point where both first order conditions equal zero, are then given by

$$\begin{aligned} \frac{a(n)}{q}\Bigg(& -\frac{\eta}{1-\eta}\frac{\beta\widetilde{\mathbb{E}\phi'}d\beta\widetilde{\mathbb{E}}}{(\beta\widetilde{\mathbb{E}\phi'}d - \beta\widetilde{\mathbb{E}}q)^2}\left(u(q) - \widetilde{\mathbb{E}\phi'}d\right) + u''(q)q + u'(q) \\ & - \frac{\eta}{1-\eta}\frac{\beta\widetilde{\mathbb{E}\phi'}d}{\beta\widetilde{\mathbb{E}\phi'}d - \beta\widetilde{\mathbb{E}}q}u'(q)\Bigg) \end{aligned} \tag{91}$$

$$\frac{a(n)}{q}\left(\frac{\eta}{1-\eta}\frac{\beta\widetilde{\mathbb{E}\phi'}\beta\widetilde{\mathbb{E}}q}{(\beta\widetilde{\mathbb{E}\phi'}d - \beta\widetilde{\mathbb{E}}q)^2}\left(u(q) - \beta\widetilde{\mathbb{E}\phi'}d\right) + \frac{\eta}{1-\eta}\frac{\beta\widetilde{\mathbb{E}\phi'}d}{\beta\widetilde{\mathbb{E}\phi'}d - \beta\widetilde{\mathbb{E}}q}\beta\widetilde{\mathbb{E}\phi'} \right) \tag{92}$$

$$\begin{aligned} & a(n)\left(-\frac{\eta}{1-\eta}\frac{(\beta\widetilde{\mathbb{E}\phi'})^2}{(\beta\widetilde{\mathbb{E}\phi'}d - \beta\widetilde{\mathbb{E}}q)^2}\left(u(q) - \beta\widetilde{\mathbb{E}\phi'}d\right) - \frac{(\beta\widetilde{\mathbb{E}\phi'})^2}{\beta\widetilde{\mathbb{E}\phi'}d - \beta\widetilde{\mathbb{E}}q} \right) \\ & \quad + a(n)\frac{\eta}{1-\eta}\frac{\beta\widetilde{\mathbb{E}\phi'}}{\beta\widetilde{\mathbb{E}\phi'}d - \beta\widetilde{\mathbb{E}}q}\left(\frac{\eta}{1-\eta}\frac{\beta\widetilde{\mathbb{E}\phi'}}{\beta\widetilde{\mathbb{E}\phi'}d - \beta\widetilde{\mathbb{E}}q}\left(u(q) - \beta\widetilde{\mathbb{E}\phi'}d\right) - \beta\widetilde{\mathbb{E}\phi'} \right) \end{aligned} \tag{93}$$

Note that from the first order conditions, it must be that $u'(q)q \geq \beta\widetilde{\mathbb{E}\phi'}d$, with equality iff $\phi = \beta\widetilde{\mathbb{E}\phi'}$. Note that we can derive the conditions under which (93) and (91) are negative, using (89) first for (91)

$$-\frac{\beta\widetilde{\mathbb{E}}q}{\beta\widetilde{\mathbb{E}\phi'}d - \beta\widetilde{\mathbb{E}}q}u'(q)q + u''(q)q^2 + u'(q)q - \frac{\eta}{1-\eta}\frac{\beta\widetilde{\mathbb{E}\phi'}d}{\beta\widetilde{\mathbb{E}\phi'}d - \beta\widetilde{\mathbb{E}}q}u'(q)q \tag{94}$$

For coefficient of relative risk aversion larger than 1, it follows. For (93),

$$\frac{a(n)}{d^2}\frac{\beta\widetilde{\mathbb{E}\phi'}d}{\beta\widetilde{\mathbb{E}\phi'}d - \beta\widetilde{\mathbb{E}}q}\left(\left(-u'(q)q - \beta\widetilde{\mathbb{E}\phi'}d\right) + \frac{\eta}{1-\eta}\left(u'(q)q - \beta\widetilde{\mathbb{E}\phi'}d\right)\right). \tag{95}$$

The crossderivatives can be rewritten as

$$\frac{a(n)}{dq}\left(\frac{\beta\widetilde{\mathbb{E}}q}{\beta\widetilde{\mathbb{E}\phi'}d - \beta\widetilde{\mathbb{E}}q}u'(q)q + \frac{\eta}{1-\eta}\frac{\beta\widetilde{\mathbb{E}\phi'}d}{\beta\widetilde{\mathbb{E}\phi'}d - \beta\widetilde{\mathbb{E}}q}\beta\widetilde{\mathbb{E}\phi'}d\right). \tag{96}$$

We can find an upperbound of (93), since $u(q) \geq u'(q)q$ (as long as $u(0) \geq 0$, this is guaranteed)

$$\frac{a(n)}{d^2}\frac{\beta\widetilde{\mathbb{E}\phi'}d}{\beta\widetilde{\mathbb{E}\phi'}d - \beta\widetilde{\mathbb{E}}q}\left(\left(-u'(q)q - \beta\widetilde{\mathbb{E}\phi'}d\right) + \frac{\eta}{1-\eta}\left(u(q) - \beta\widetilde{\mathbb{E}\phi'}d\right)\right). \tag{97}$$

$$= -\frac{a(n)}{d^2}\frac{\beta\widetilde{\mathbb{E}\phi'}d}{\beta\widetilde{\mathbb{E}\phi'}d - \beta\widetilde{\mathbb{E}}q}\left(u'(q)q + \beta\widetilde{\mathbb{E}\phi'}d\right) + \frac{a(n)}{d^2}u'(q)q. \tag{98}$$

$$= -\frac{a(n)}{d^2}\left(\frac{\beta\widetilde{\mathbb{E}}q}{\beta\widetilde{\mathbb{E}\phi'}d - \beta\widetilde{\mathbb{E}}q}u'(q)q + \frac{\beta\widetilde{\mathbb{E}\phi'}s}{\beta\widetilde{\mathbb{E}\phi'}d - \beta\widetilde{\mathbb{E}}q}\beta\widetilde{\mathbb{E}\phi'}d\right) < 0. \tag{99}$$

Moreover, it now can be shown that the Hessian evaluated at the point where both first order conditions are satisfied (99) and (94) are larger in absolute value than (96) (ignoring the $\frac{a(n)}{.}$ term as we will be able to factor them out when evaluating the determinant of the Hessian). Thus the second order conditions for concavity at the points where the first order conditions hold, are satisfied.

It can be shown that there is no utility function that starts at 0, and has a coefficient of relative risk aversion larger than 1 everywhere. Two solutions to the problem of having a utility function with a high relative risk aversion with positive values, simply shifting the standard CRRA function upwards (by adding a constant), or shifting it upwards and to the left, so $u(0) = 0$. In the first case, we have to make sure that $u(q) > u'(q)q$ for the q's picked in the first order conditions. Here follows the proof:

Let us try to prove that in the bargaining solution, whenever $\eta < 0.5$, we must have $u'(q)q < u(q)$. Suppose not. Start from the familiar condition

$$\frac{1-\eta}{\eta}\frac{u'(q)q}{\beta\widetilde{\mathbb{E}\phi'd}} = \frac{u(q)-\beta\widetilde{\mathbb{E}\phi'd}}{\beta\widetilde{\mathbb{E}\phi'd}-\beta\widetilde{\mathbb{E}}q}.$$

We must have $u(q) > \beta\widetilde{\mathbb{E}\phi'd}$ as the outcome of bargaining, and hence

$$\frac{1-\eta}{\eta}\frac{u'(q)q}{u(q)} < \frac{u(q)-\beta\widetilde{\mathbb{E}\phi'd}}{u(q)-\beta\widetilde{\mathbb{E}}q},$$

since the denominator is smaller, positive as $u(q) > \beta\widetilde{\mathbb{E}}q$.

Given $u'(q)q > u(q)$, and $(1-\eta) > \eta$, it must be that the RHS of the above equation is larger than 1. Moreover, since $\beta\widetilde{\mathbb{E}\phi'd} < \beta\widetilde{\mathbb{E}\phi'd}$, the LHS is smaller than 1. Contradiction. Hence, given our assumption about the bargaining power, it must be that the bargaining solution is in the range of q's which exhibit $u(q) \geq u'(q)q$.

For the other solution, the utility function is given by, e.g.

$$u(q) = \frac{(q+b)^{1-\sigma} - b^{1-\sigma}}{1-\sigma}.$$

In this case $u(q) > u'(q)q$ because the function is strictly concave. However, the degree of relative risk aversion approaches σ for q's sufficiently away from 0, but close to $q = 0$, relative risk aversion is lower than 1. Hence, it has to be checked that for the q solving the first order condition, the RRA(q) is indeed larger than 1.

The restriction that $\eta < 0.5$ is not unintuitive: we see that with an ex ante constant cost of bringing each unit to the decentralized market, the matching rate depends only on the ratio d/q. If $\eta/(1-\eta) > 1$, there are increasing returns to d/q: a doubling of the ratio more than doubles the probability of matching. If we choose η sufficiently larger than 0.5, utility becomes unbounded, by letting d/q approach infinitely, while both d and q become very small (but q at a faster rate), using that the matching rate of buyers approaches infinity at the fastest rate. There are two assumptions behind this: first that $\lim_{q\downarrow 0} u'(q) = \infty$, and secondly that with the usual Cobb-Douglas constant elasticity matching function, the matching probability within one period is not bounded by one (on the contrary, it could go to infinity). This has the effect of allowing multiple small matches to substitute for one large one. If we rewrite the maximization problem as a function of d/q and q, it can be derived that for any utility level, we can find a larger d/q and a smaller q that will give higher utility.

On the other hand, if we had a fixed cost of retail firms to enter the decentralized market, and production on the spot (as in RW), we would not have to worry about the returns to scale of the *buyer* matching function, as the firms with high d/q, but low d would not be able to recoup their fixed cost needed to enter. $\square$

A.3.2 Lemma 1

Proof. Take a ϑ; consider the nonbinding case: q_ϑ is fixed and a unique solution to[14]

$$\eta\vartheta u(q_\vartheta) - \vartheta u'(q_\vartheta)q_\vartheta = -(1-\eta)\beta\widetilde{\mathbb{E}}q_\vartheta < 0, \tag{100}$$

as the RHS is decreasing and the LHS is increasing in q_ϑ. Money spent d_ϑ is given by $\vartheta u'(q_\vartheta)q_\vartheta = \beta\widetilde{\mathbb{E}\phi'}d_\vartheta$, which is a decreasing continuous function of $\beta\widetilde{\mathbb{E}\phi'}$. For the binding cash constraint, $d_\vartheta = 1$, and q_ϑ is given by

$$\eta(\vartheta u(q_\vartheta) - \beta\widetilde{\mathbb{E}\phi'})\beta\widetilde{\mathbb{E}\phi'} - (1-\eta)(\beta\widetilde{\mathbb{E}\phi'} - \beta\widetilde{\mathbb{E}}q_\vartheta)\vartheta u'(q_\vartheta)q_\vartheta$$

$$\eta\vartheta u(q_\vartheta) - (1-\eta)\vartheta u'(q_\vartheta)q_\vartheta(1 - \frac{\widetilde{\mathbb{E}}q_\vartheta}{\widetilde{\mathbb{E}\phi'}}) = \eta(\beta\widetilde{\mathbb{E}\phi'}) > 0 \tag{101}$$

Again, the LHS is an increasing function of q_ϑ, while the RHS is constant, so the q_ϑ is unique. Moreover, it can be shown that q_ϑ is increasing with $\beta\widetilde{\mathbb{E}\phi'}$. To establish continuity of the q_ϑ, d_ϑ choice, it is now sufficient to establish that for a $\beta\widetilde{\mathbb{E}\phi'}$ for which both the unconstrained and constrained case give $d_\vartheta = 1$, it is the case that q_ϑ is the same in both cases (or equivalently, in the case that q_ϑ in the constrained case equals the unconstrained q_ϑ, $d_\vartheta = 1$). This follows from the fact that, at this $\beta\widetilde{\mathbb{E}\phi'}$, it must be that $\vartheta u'(q_\vartheta)q_\vartheta = \beta\widetilde{\mathbb{E}\phi'}$ substitution then finds that both 'bargaining conditions' (100) and (101) are equivalent. Hence, q_ϑ, d_ϑ are continuous functions of $\beta\widetilde{\mathbb{E}\phi'}$; it follows straightforwardly from (16) and the continuous matching rate $a(n_\vartheta)$ that n_ϑ is a continuous function of $\beta\widetilde{\mathbb{E}\phi'}$. Finally, since μ_ϑ is a continuous function $q_\vartheta, d_\vartheta, n_\vartheta$ which in turn are continuous functions of $\beta\widetilde{\mathbb{E}\phi'}$, ϕ is a continuous function of $\beta\widetilde{\mathbb{E}\phi'}$ as well. □

A.3.3 Lemma 2

Proof. In the unconstrained case, q_ϑ can be determined by solving[15]

$$\eta(\vartheta u(q) - \vartheta u'(q)q) - (1-\eta)(\vartheta u'(q)q - \beta q)$$

We can derive the response $\frac{dq}{d\vartheta}$ from the above function as

$$\frac{dq}{d\vartheta} = -\frac{\eta u(q) - u'(q)q}{\eta\vartheta u'(q) - \vartheta u'(q) - \vartheta u''(q)q + (1-\eta)\beta} \tag{102}$$

which can be rewritten as

$$\frac{dq/q}{d\vartheta/\vartheta} = \frac{(1-\eta)\beta q}{\eta\vartheta u'(q)q - \vartheta u'(q)q - \vartheta u''(q)q^2 + (1-\eta)\beta q} \tag{103}$$

[14]For ease of notation, we have normalized the utility function by $U'(c)$.

[15]We will not write $u'(c)$ in the general case, but normalize the utility function by it. I.e. $u(q) = u^{old}(q)/u'(c)$. Furthermore, we redefined β to mean $\beta\widetilde{\mathbb{E}}$.

Then

$$\frac{dq/q}{d\vartheta/\vartheta} = \frac{1}{\frac{\eta u'(q)q - u'(q)q - u''(q)q^2}{u'(q)q - \eta u(q)} + 1} \tag{104}$$

If

$$\frac{\eta u'(q)q - u'(q)q - u''(q)q^2}{u'(q)q - \eta u(q)} > -1,$$

q will indeed go up as ϑ goes up; or equivalently,

$$\eta u'(q)q - u'(q)q - u''(q)q^2 > \eta u(q) - u'(q)q$$

This means that

$$\eta u'(q)q - \eta u(q) - u''(q)q^2 > 0.$$

Next we substitute $-(1-\eta)\frac{(\vartheta u'(q)q - \beta q)}{\vartheta}$ for $\eta u'(q)q - \eta u(q)$. In the optimum, it must be that $\vartheta u'(q) \geq 1$, as 1 is the marginal cost of buying 1 unit for retailers (and $\vartheta u'(q)/u'(c) \geq 1$ in the non-rescaled version of the utility) in the optimum. Hence, we can find that

$$-(1-\eta)(\vartheta u'(q)q - \beta q) - \vartheta u''(q)q^2 > 0$$

which is satisfied if the following bound holds:

$$-\frac{qu''(q)}{u'(q)} > (1-\eta) > (1-\eta)\left(1 - \frac{\beta}{\vartheta u'(q)}\right),$$

which means that the coefficient of relative risk aversion larger than 1 should suffice for an increase in q in response to a change in ϑ.

The response of n depends on the response of d/q. From the optimization, we have $(\beta\phi' d)/q = \vartheta u'(q)$. Using that $\beta\widetilde{\mathbb{E}\phi'}d = \vartheta u'(q_\vartheta)q$, we find that

$$\eta(\vartheta u(q_\vartheta) - \vartheta u'(q_\vartheta)q_\vartheta) - (1-\eta)(\vartheta u'(q_\vartheta)q_\vartheta - \beta u'(c')q_\vartheta) = 0, \tag{105}$$

and rewriting, we get

$$\eta(\vartheta u(q_\vartheta)) + (1-\eta)\beta u'(c')q_\vartheta = \vartheta u'(q_\vartheta)q_\vartheta$$

From this equation, if ϑ increases, and q_ϑ increases in response, then it must be that the two terms on the LHS increase as well. Hence, $\vartheta u'(q_\vartheta)q_\vartheta$ also goes up. But then $\beta\widetilde{\mathbb{E}\phi'}d$ goes up, which implies d goes up.

Let us now look at d/q: this goes up in the optimum if and only if $\vartheta u'(q)$ goes up. Again, calculate the elasticity

$$\frac{d\ln\vartheta u'(q)}{d\ln\vartheta} = 1 + \frac{d\ln u'(q)}{d\ln q}\frac{d\ln q}{d\ln\vartheta} \tag{106}$$

Above, we have calculated $\frac{d\ln q}{d\ln\vartheta}$ already, and we will use it here. Continuing the above,

$$\frac{d\ln\vartheta u'(q)}{d\ln\vartheta} = 1 + \frac{qu''(q)}{u'(q)}\frac{d\ln q}{d\ln\vartheta},$$

$$\frac{d\ln\vartheta u'(q)}{d\ln\vartheta} = 1 - RRA(q)\frac{d\ln q}{d\ln\vartheta},$$

has the same sign, as long as quantity increases with ϑ (for which a sufficient condition is $RRA > 1-\eta$).

$$\begin{aligned} &\frac{d\ln\vartheta}{d\ln q} - RRA(q) \\ &= \frac{\eta u'(q)q - u'(q)q - u''(q)q^2}{u'(q)q - \eta u(q)} + 1 - RRA(q) \end{aligned} \tag{107}$$

$$\begin{aligned} &= \frac{u'(q)q}{u'(q)q - \eta u(q)}\left(\eta - 1 + RRA(q)\right) + 1 - RRA(q) \\ &= \frac{\eta}{u'(q)q - \eta u(q)}\left(u'(q)q - u(q) + u(q)RRA(q)\right) > 0 \end{aligned}$$

Thus, given $RRA(q) > (1-\eta)$, we can show that this term is positive for any concave function that starts at $u(0) = 0$ or $u(0) > 0$, since then $u(q) > u'(q)q$. Moreover, in the proof of proposition 2 we establish that in fact $u(q) > u'(q)q$ holds in different utility functions as well.

Thus, we have derived relatively mild conditions under which a multiplicative preference shock leads to more money being spend, more q being bought, and a higher buyer matching rate. □

A.3.4 Lemma 3

Proof. Rewriting the 'bargaining' condition

$$\eta\left(\vartheta u(q_\vartheta) - \beta\phi'\right)\beta\phi' - (1-\eta)\left(\beta\phi' - \beta q_\vartheta\right)\vartheta u'(q_\vartheta)q_\vartheta = 0$$

as

$$\frac{\eta(u(q_\vartheta) - \frac{\beta\phi'}{\vartheta}}{u'(q_\vartheta)q_\vartheta} + \frac{(1-\eta)\beta q_\vartheta}{\beta\phi'} = (1-\eta).$$

Now, raising ϑ can only be offset by changing q to keep the RHS constant. If $RRA(q) > 1$, lowering q will achieve this ($u(q)$ and q will decrease while denominator $u'(q)q$ will increase), while raising q will not. Hence, at a higher ϑ we have lower q_ϑ. Since $d_\vartheta/q_\vartheta = 1/q_\vartheta$ is increasing in ϑ among the binding ϑ's, n_ϑ is increasing. From (49) it follows that $\mu_\vartheta/\mathbb{P}(\vartheta)$ is increasing, as $\vartheta u'(q_\vartheta)q_\vartheta$ increases with ϑ among binding shocks. □

A.3.5 Lemma 4

Proof. Let us turn our attention to the effect of a change in the value of money tomorrow on the value of money today, given that the money market clears. To save notation, denote $\widetilde{\mathbb{E}\phi'}$ by ϕ'. To begin, note that the system of equations (41)-(43)

is solved sequentially. First, (41) directly links ϕ' with q_ϑ, then (42)-(43) link the q_ϑ's for which the cash constraint is *binding* with ϕ, given market clearing $\tilde{m}$. Since shocks for which the cash constraint does not bind, do not influence the value of money for marginal changes (and for larger changes the larger-order effects move to amplify our conclusions) we can focus only on those shocks for which the cash constraint is currently binding. First, we can directly calculate the impact of changes in $\beta\phi'$ on q_ϑ, by looking at the 'bargaining condition' with a binding cash constraint,

$$\eta(\vartheta u(q_\vartheta) - \beta\phi')\beta\phi' - (1-\eta)(\beta\phi' - \beta\widetilde{\mathbb{E}}q_\vartheta)\vartheta u'(q_\vartheta)q_\vartheta = 0 \tag{108}$$

Then, $\frac{dq_\vartheta}{d\beta\phi'}$ is given by

$$\frac{-\eta(\vartheta u(q_\vartheta) - \beta\phi') + \eta\beta\phi' + (1-\eta)\vartheta u'(q_\vartheta)q_\vartheta}{\eta\vartheta u'(q_\vartheta)\beta\phi' + (1-\eta)\beta\widetilde{\mathbb{E}}\vartheta u'(q_\vartheta)q_\vartheta - (1-\eta)(\beta\phi' - \beta\widetilde{\mathbb{E}}q_\vartheta)(\vartheta u''(q_\vartheta)q_\vartheta + \vartheta u'(q_\vartheta))} \tag{109}$$

The numerator is larger than zero, as $\eta(\vartheta u(q_\vartheta) - \beta\phi')\beta\phi' < (1-\eta)\beta\phi'\vartheta u'(q_\vartheta)q_\vartheta$; the denominator is positive for a coefficient of relative risk aversion close to one or larger than 1. Hence, not very surprisingly, an increase in ϕ' will lead to more units being transacted in the decentralized market (given that at market clearing and binding $\tilde{m} = d_\vartheta = 1$). We can also calculate what happens to the ratio $\beta\widetilde{\mathbb{E}\phi'}/q_\vartheta$.

$$\frac{dq/q}{d\beta\phi'/\beta\phi'} = \frac{-\eta\vartheta(u(q_\vartheta) - \beta\phi')\beta\phi' + \eta(\beta\phi')^2 + (1-\eta)\vartheta u'(q_\vartheta)q_\vartheta\beta\phi'}{\Big(\eta\beta\phi' + (1-\eta)\beta\widetilde{\mathbb{E}}q_\vartheta + (1-\eta)(\beta\phi' - \beta\widetilde{\mathbb{E}}q_\vartheta)(RRA(q_\vartheta) - 1)\Big)\,\vartheta u'(q_\vartheta)q_\vartheta} \tag{110}$$

It follows that this term is

$$\varepsilon_{q_\vartheta,\beta\phi'} = \frac{\eta(\beta\phi')^2 + (1-\eta)\vartheta u'(q_\vartheta)q_\vartheta\beta\widetilde{\mathbb{E}}q_\vartheta}{\Big(\eta\beta\phi' + (1-\eta)\beta\widetilde{\mathbb{E}}q_\vartheta + (1-\eta)(\beta\phi' - \beta\widetilde{\mathbb{E}}q_\vartheta)(RRA(q_\vartheta) - 1)\Big)\,\vartheta u'(q_\vartheta)q_\vartheta} < 1 \tag{111}$$

since $\beta\phi' < \vartheta u'(q_\vartheta)q_\vartheta$ and $RRA(q_\vartheta) - 1 > 0$. Thus, we get the remarkably clean-cut result that if prices go up, q goes up, but less than proportionally so: $0 < \varepsilon_{q_\vartheta,\beta\phi'} < 1$. Since the elasticity of q with respect to ϕ' is less than 1, it follows immediately that $\frac{\beta\phi' - \beta\widetilde{\mathbb{E}}q_\vartheta}{q_\vartheta - \beta\widetilde{\mathbb{E}}q_\vartheta}$ goes up, and hence n_ϑ must go up as well.

From

$$\phi = \sum_{\{\vartheta\}^b} \Big(\mathbb{P}(\vartheta)a(n_\vartheta)\left(\vartheta u'(q_\vartheta)q_\vartheta - \beta\phi'\right)\Big) + \beta\phi', \tag{112}$$

we can calculate the elasticity of ϕ with respect to $\beta\phi'$ as

To relate the equilibrium changes in ϕ to variation in ϕ', we can calculate $\frac{d\phi}{d\beta\phi'}$ as

$$a(n)\frac{du'(q)q}{d\phi'} + (1 - a(n)) + \frac{da(n)}{d\beta\phi'}(u'(q)q - \beta\phi) \tag{113}$$

□

B Computational Algorithm

The full PEA algorithm is as follows:

1. Solve a linearized version of the model.

2. Use the decision rules of the linearized problem to find the initial set of parameters $\gamma^0 = \{\gamma^{10}, \gamma^{20}, \gamma^{30}\}$. This is done by solving nonlinear projections of expectation terms computed from simulations on the polynomial approximants of the expectation terms; denote the resulting parameters as vector ξ.

3. Nonlinear Model: set $\gamma^0 = \xi$. Using γ^0, simulate the economy, solving the system of Euler equations. Assume first that the cash constraint does not bind. If it does, re-solve the system under binding constraints.

4. Run the nonlinear projections to get a new estimate ξ.

5. Compare ξ and γ^0. If they are close according to a chosen tolerance parameter, we have convergence. If not, update $\gamma^1 = \lambda\xi + (1 - \lambda)\gamma^0$. Repeat from step 3.

6. Once we have the final estimates γ, we have the decision rules of the problem.