The $(\mathbf{Q}, \mathbf{S}, \mathbf{s})$ Pricing Rule*

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Abstract

We study the effect of menu costs on the pricing behavior of sellers and on the cross-sectional distribution of prices in the search-theoretic model of imperfect competition of Burdett and Judd (1983). We find that, when menu costs are not too large, the equilibrium is such that sellers follow a (Q, S, s) pricing rule. According to a (Q, S, s) rule, a seller lets inflation erode the real value of its nominal price until it reaches some point s. Then, the seller pays the menu cost and changes its nominal price so that the real value of the new price is randomly drawn from a distribution with support $[S, Q]$, where Q is the buyer's reservation price and S is some price between s and m. In a (Q, S, s) equilibrium, both menu costs and search frictions contribute to price stickiness. A calibrated version of the model reveals that search frictions are more important than menu costs in explaining the observed duration of nominal prices.

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1 Introduction

It is a well-documented fact that sellers leave the nominal price of their goods unchanged for months and months in the face of a continuously increasing aggregate price level (see, e.g., Klenow and Kryvtsov 2008 or Nakamura and Steinsson 2008). The standard explanation for nominal price stickiness is that sellers have to pay a menu cost in order to change their nominal price (see, e.g., Sheshinski and Weiss 1982 or Caplin and Leahy 1997). In the presence of such a menu cost, sellers find it optimal to follow an (S, s) pricing rule: they let inflation erode the real value of their nominal price until it reaches some point s and, then, they pay the menu cost and reset their nominal price so that its real value is some S, with $S > s$.

Head, Liu, Menzio and Wright (2012) have recently advanced an alternative explanation for nominal price stickiness. In the presence of search frictions, the equilibrium distribution of real prices posted by sellers is non-degenerate. Sellers are indifferent between posting any two real prices on the support of the equilibrium distribution. If the seller posts a relatively high price, it enjoys a higher profit per trade but it faces a lower probability of trade. If the seller posts a relatively low price, it enjoys a lower profit per trade but it faces a higher probability of trade. For this reason, sellers are also willing to keep their nominal price unchanged until inflation pushes its real value outside of the support of the equilibrium price distribution. Only then sellers find it necessary to change their nominal price.

The menu cost and the search friction theories of nominal price stickiness have very different implications with respect to the real effects of monetary policy. If nominal price stickiness is due to menu costs, an unexpected change in the quantity of money will in general affect the distribution of real prices and, in turn, consumption and production. In contrast, if nominal price stickiness is due to search frictions, an unexpected change in the quantity of money will have no effect on the distribution of real prices, on consumption and on production because sellers do not face any cost to adjusting their nominal price. For this reason, it is important to understand to what extent price stickiness is caused by search frictions and to what extent it is caused by menu costs. The answer to this question requires developing and calibrating a monetary model that incorporates both menu costs and search frictions, a task that we carry out in this paper.

We introduce menu costs in a simple search-theoretic model of imperfect competition in the spirit of Butters (1977), Varian (1980) and Burdett and Judd (1983). In the model, sellers post nominal prices whose real value is eroded by inflation and that can only be changed by paying a menu cost. Buyers search for sellers. In some meetings buyers are captive, in the sense that they are in contact with only one seller, while in other meetings they are not, in the sense that they are in contact with multiple sellers.

It is well known that, absent menu costs, the unique equilibrium of this model involves price dispersion (see, e.g, Burdett and Judd 1983). When menu costs are positive but sufficiently small, we show that the unique equilibrium of this model is such that sellers follow a (Q, S, s) pricing rule. According to this rule, the seller lets inflation erode the real value of its nominal price until it reaches some point s. Then, the seller pays the menu cost and changes its nominal price so that the real value of the *new price is a random draw* from a distribution with some support $[S, Q]$, with $Q > S > s$. When menu costs are sufficiently large, we show that all equilibria are such that seller follow a standard (S, s) rule. And when menu costs take on intermediate values, we find that there may be equilibria in which sellers follow a (Q, S, s) pricing rule and equilibria in which sellers follow an (S, s) pricing rule. Finally, we show that the forces that rule out the existence of an (S, s) equilibrium when menu costs are small are exactly the same forces that rule out a one-price equilibrium when there are no menu costs.

A (Q, S, s) equilibrium looks like a hybrid between a menu cost model (e.g., Caplin and Spulber 1986 and Benabou 1988) and a standard search-theoretic model of price dispersion (e.g., Butters 1977, Varian 1980 and Burdett and Judd 1983). As in a standard search-theoretic model, the seller's present discounted value of profits remains constant as the real value of the seller's nominal prices falls from Q to S. As in a standard menu cost model, the seller's present discounted value of profits declines monotonically as the real value of the nominal price falls from S to s. Similarly, over the interval $[S, Q]$, the stationary distribution of prices is such that the seller's flow profit is constant, as in a standard search-theoretic model. Over the interval $[s, S]$, the stationary distribution of prices is log-uniform, as in a standard menu costs model. These natural equilibrium outcomes emerge from a rather surprising behavior of sellers: Sellers draw their reset price from a distribution that has support $[S, Q]$ and mass points at both S and Q.

In a (Q, S, s) equilibrium, both search frictions and menu costs contribute to price stickiness. In a (Q, S, s) equilibrium, a seller maintains the same nominal price until its real value reaches some point s. While the real value of the nominal price falls from Q to

S, the seller would not want to change the price even if it could do it for free. While the real value of the nominal price falls from S to s, the seller would like to change its price but chooses not to in order to avoid paying the menu cost. In this sense, only part of the stickiness of nominal prices is due to menu costs. The other part is due to the fact that search frictions create a region of prices over which the sellers' profits are constant and maximized.

We calibrate the model to match data on the average duration of nominal prices for individual sellers and on the dispersion of prices at which different sellers trade the same good. For goods characterized by a below-average duration of prices or by an above-average cross-sectional dispersion of prices, we find no menu costs. For goods characterized by an above-average duration of prices and by a below-average price dispersion, we find menu costs that are positive, albeit small. For example, for goods with a price duration of prices of 5 quarters and a standard deviation of prices of 4%, menu costs account for somewhere between 10 and 35% of price stickiness, depending on how the decomposition is carried out. These findings suggest that—as in Head, Liu, Menzio and Wright (2012)—price stickiness may play a limited role in transmitting monetary policy shocks to the real side of the economy.

Our paper contributes to two strands of literature. First, our paper contributes to the literature on menu costs. In particular, ours is the first paper to show that the optimal pricing strategy of a seller facing menu costs need not be an (S, s) rule or some analogous rule where the reset price is deterministic. Sheshinski and Weiss (1977) prove the optimality of an (S, s) rule for a monopolist facing a well-behaved, exogenous downward sloping demand curve. Caplin and Spulber (1987) and Caplin and Leahy (1997, 2010) show that an (S, s) rule is optimal for a monopolist facing an exogenously given demand curve which depends on its own price, on the average price and on the quantity of money. Dotsey, King and Wolman (1999) and Golosov and Lucas (2007) establish the optimality of an (S, s) rule in an equilibrium model where a continuum of monopolistic competitors faces an endogenous demand curve derived from the buyers' Dixit-Stiglitz utility function. Benabou (1988, 1992) shows that (S, s) rules are optimal in the search model of Diamond (1971). Midrigan (2011) and Alvarez and Lippi (2014) generalize the notion of an (S, s) rule for sellers of multiple goods. All of the papers listed above consider models of imperfect competition where, absent menu costs, identical sellers would set the same price. Our paper considers a model of imperfect competition where, absent menu costs, identical sellers would set different prices. This is the reason why we find that the optimal pricing strategy of a seller may be a (Q, S, s) rule rather than an (S, s) rule.

Second, our paper contributes to the literature on search-theoretic models of price dispersion. In particular, ours is the first paper to introduce menu costs in this class of models. Butters (1977) considers an environment in which some buyers contact only one seller and some contact two sellers. Since they cannot discriminate between the two types of buyers, sellers find it optimal to randomize with respect to their price. Varian (1980) considers an environment in which some buyers contact one seller and some buyers contact all the sellers. Again, sellers find it optimal to randomize with respect to their price. Burdett and Judd (1983) consider an environment in which buyers choose how many sellers to contact. They show that there exists an equilibrium in which some buyers choose to contact one seller, some buyers choose to contact two sellers and sellers randomize over their price. Head, Liu, Menzio and Wright (2012) embed Burdett and Judd (1983) into a monetary model in the spirit of Lagos and Wright (2005). In all of the papers mentioned above, the seller's problem is static because sellers are free to change their price in every period. In our paper, the seller's problem is dynamic because of menu costs.

The remainder of the paper is organized as follows. In Section 2, we describe the environment and compare it with Burdett and Judd (1983) and Benabou (1988). In Section 3, we derive a necessary and sufficient condition for the existence of a (Q, S, s) equilibrium, we characterize the salient features of the equilibrium and we carry out comparative statics with respect to the menu cost and the inflation rate. In Section 4, we derive a necessary and sufficient condition for the existence of an (S, s) equilibrium and we show that this type of equilibrium does not exist when the menu cost is sufficiently small. In Section 5, we calibrate the model and we break down price stickiness and price dispersion into a component due to menu costs and a component due to search frictions. In Section 6, we briefly conclude.

2 Environment

We study a dynamic and monetary version of a model of imperfect competition in the spirit of Butters (1977), Varian (1980) and Burdett and Judd (1983). The market for an indivisible good is populated by a continuum of identical sellers with measure 1. Each seller maximizes the present value of real profits, discounted at the rate $r > 0$. Each seller produces the good at a constant marginal cost, which, for the sake of simplicity, we assume to be zero. Each seller posts a nominal price d for the good, which can only be changed by paying the real cost c, with $c > 0$.

The market is also populated by a continuum of identical buyers. In particular, during each interval of time of length dt, a measure bdt of buyers enters the market. A buyer comes into contact with one seller with probability α and with two sellers with probability $1 - \alpha$, where $\alpha \in (0, 1)$. We refer to a buyer who contacts only one seller as captive, and to a buyer who contacts two sellers as non-captive. Then, the buyer observes the nominal prices posted by the contacted sellers and decides whether and where to purchase a unit of the good. If the buyer purchases the good at the nominal price d, he obtains a utility of $Q - \mu(t)d$, where $\mu(t)$ is the utility value of a dollar at date t and $Q > 0$ is the buyer's valuation of the good. If the buyer does not purchase the good, he obtains a reservation utility, which we normalize to zero. Whether the buyer purchases the good or not, he exits the market.

The utility value of a dollar declines at the constant rate $\pi > 0$. Therefore, if a nominal price remains unchanged during an interval of time of length dt, the real value of the price falls by $\exp(-\pi dt)$. In this paper, we do not describe the demand and supply of dollars. It would, however, be straightforward to embed our model into either a standard cash-in-advance framework (see, e.g., Lucas and Stokey 1987) or in a standard money-search framework (see, e.g., Lagos and Wright 2005) and show that, in a stationary equilibrium, the depreciation rate π would be equal to the growth rate of the money supply.

Even without inflation and menu costs, the equilibrium of the model features a non-degenerate distribution of prices. The logic behind this result is clear. If all sellers post the same price, an individual seller can increase its profits by charging a slightly lower price and sell not only to the contacted buyers who are captive, but also to the contacted buyers who are not captive. This Bertrand-like process of undercutting cannot push all prices down to the marginal cost. In fact, if all sellers post a price equal to the marginal cost, an individual seller can increase its profits by charging the reservation price Q and sell only to the contacted buyers who are captive. Thus, in equilibrium, there must be price dispersion.

There are two differences between our model and Burdett and Judd (1983) (henceforth, BJ83). First, in our model sellers post nominal prices that can only be changed by paying a menu cost, while in BJ83 sellers post real prices that can be freely changed in every period. This difference is important because it implies that in our model the problem of the seller is dynamic, while in BJ83 it is static. Second, in our model the fraction of buyers meeting one and two sellers is exogenous, while in BJ83 it is an endogenous outcome of buyers' optimization. We believe that our results would generalize to an environment where buyers' search intensity is endogenous.[1]

There are also two differences between our model and Benabou (1988) (henceforth, B88). First, in our model there are some buyers who are in contact with one seller and some who are in contact with multiple sellers, while in B88 all buyers are temporarily captive. This difference is important because it implies that, even without menu costs, the equilibrium of our model features price dispersion, while in B88 every seller would charge the monopoly price (as in Diamond 1971). Second, in our model buyers have to leave the market after they search today, while in B88 they can choose to stay in the market and search again tomorrow. We believe that our results would generalize to an environment where buyers are allowed to search repeatedly.[2]

3 (Q, S, s) Equilibrium

A (Q, S, s) equilibrium is an equilibrium where every seller lets inflation erode the real value of its price until it reaches some point $s \in (0, Q)$, then it pays the menu cost and it resets the nominal price so that its real value is a random draw from some distribution with support $[S, Q]$, where $S \in (s, Q)$. In a (Q, S, s) equilibrium, the stationary distribution of real prices is some continuous distribution F with support $[s, Q]$ and the distribution of new prices is some distribution G with support $[S, Q]$. In subsection 3.1, we formally define a (Q, S, s) equilibrium. In subsection 3.2, we derive a necessary and sufficient condition for the existence of a (Q, S, s) equilibrium. In subsection 3.3, we describe several novel features of a (Q, S, s) equilibrium, as well as their implications for the estimation of menu costs, seller-specific shocks and search costs. In subsection 3.4, we analyze the effect on equilibrium outcomes of changes in the menu cost and in the inflation rate. The main

[1] Consider a model where buyers can choose whether to search once or twice. Clearly, given the appropriate choice of the distribution of search costs across buyers, the equilibrium of our model is also an equilibrium of the model with endogenous search intensity.

[2] Consider a model where buyers can search repeatedly and have a valuation of the good $Z \geq Q$ and a discount factor ρ. Given the appropriate choice of Z and ρ, the equilibrium of our model is also an equilibrium of the model with repeated search. In the model with repeated search, Q does not represent the buyer's valuation of the good, but the buyer's reservation price.

result of this section is that, as long as menu costs are not too large, a (Q, S, s) equilibrium exists and is unique.[3]

3.1 (Q, S, s) Equilibrium: Definition

Consider a seller whose nominal price has a real value of $Q\exp(-\pi t)$. Let $V(t)$ denote the present discounted value of the real profits of this seller, which is given by

$$V(t) = \max_{T} \left\{ \int_t^T e^{-r(x-t)} R\left(Qe^{-\pi x}\right) dx + e^{-r(T-t)}\left(V^* - c\right) \right\}, \tag{1}$$

where

$$R(Qe^{-\pi x}) = b\alpha Qe^{-\pi x} + 2b\left(1-\alpha\right)\left(1 - F(Qe^{-\pi x})\right)Qe^{-\pi x}. \tag{2}$$

The above expressions are easy to understand. After $x - t$ units of time, the real value of the seller's nominal price is $Q\exp(-\pi x)$. When the real value of the price is $Q\exp(-\pi x)$, the seller's profit is $R(Q\exp(-\pi x))$, which is given by the sum of two terms. The first term is the rate at which the seller meets a captive buyer, $b\alpha$, times the probability that the seller trades with the captive buyer, 1, times the profit enjoyed by the seller if trade takes place, $Q\exp(-\pi x)$. The second term is rate at which the seller meets a non-captive buyer, $2b(1-\alpha)$, times the probability that the seller trades with the non-captive buyer, $1-F(Q\exp(-\pi x))$, times the profit enjoyed by the seller if trade takes place, $Q\exp(-\pi x)$. After $T - t$ units of time, the seller pays the menu cost c, resets the nominal price and attains the maximized value V^*.

The seller finds it optimal to change the nominal price after $T^* - t$ units of time, where T^* is such that

$$R\left(Qe^{-\pi T^*}\right) = r\left(V^* - c\right). \tag{3}$$

The left-hand side of (3) is the marginal benefit of deferring the price change, which is given by the expected profit associated with the real price $Q\exp(-\pi T^*)$. The right-hand side of (3) is the marginal cost of deferring the price change, which is given by the annuitized value of paying the menu cost c and attaining the maximized profit V^*. Thus, T^* equates the marginal benefit and the marginal cost of deferring the price change. From

[3] We do not need to look for equilibria in which a seller resets the real value of its nominal price to an interval $[S, P]$, where $P < Q$. Indeed, it is easy to verify that this class of equilibria does not exist because, if $F(P)$ were equal to 1, an individual seller would be strictly better off resetting the real value of its nominal price to Q rather than to P. Similarly, we can rule out equilibria in which a seller randomizes over the real value p of the nominal price at which it pays the menu cost. All details are available upon request.

(3), it follows that the seller finds it optimal to change the nominal price when its real value reaches s, where s is such that

$$R(s) = r(V^* - c). \tag{4}$$

The seller finds it optimal to reset the nominal price to any real value between S and Q if and only if $V(t) = V^*$ for all $t \in [0, T_1]$, and $V(t) \leq V^*$ for all $t \in [T_1, T_1 + T_2]$, where $T_1 = \log(Q/S)/\pi$ denotes the time it takes for the real value of a nominal price to travel from Q to S and $T_2 = \log(S/s)/\pi$ denotes the time it takes for the real value of a nominal price to travel from S to s.

The condition $V(t) = V^*$ for all $t \in [0, T_1]$ is satisfied if and only if

$$\int_{T_1}^{T_1+T_2} e^{-r(x-T_1)} R\left(Qe^{-\pi x}\right) dx + e^{-rT_2}\left(V^* - c\right) = V^*, \tag{5}$$

$$V'(t) = rV^* - R(Qe^{-\pi t}) = 0, \ \forall t \in [0, T_1]. \tag{6}$$

Equation (5) guarantees that the present discounted value of profits for a seller with a real price of $Q\exp(-\pi T_1)$ is equal to the maximum value V^*. Equation (6) guarantees that the derivative with respect to t of the present discounted value of profits for a seller with a real price of $Q\exp(-\pi t)$ is equal to zero for all $t \in [0, T_1]$. Similarly, the condition $V(t) \leq V^*$ for all $t \in [T_1, T_1 + T_2]$ is satisfied if and only if

$$\int_{t}^{T_1+T_2} e^{-r(x-t)} R\left(Qe^{-\pi x}\right) dx + e^{-r(T_1+T_2-t)}\left(V^* - c\right) \leq V^*. \tag{7}$$

The distribution of prices, F, is stationary if the measure of sellers whose real price enters the interval $[s, p]$ is equal to the measure of sellers whose real price exits the interval $[s, p]$ during any arbitrarily small period of time dt. For any $p \in (s, S)$, the inflow-outflow equation is given by

$$F(e^{\pi dt}p) - F(p) = F(e^{\pi dt}s) - F(s). \tag{8}$$

The flow of sellers into the interval $[s, p]$ is equal to the measure of sellers who, at the beginning of the period, have a real price between p and $p\exp(\pi dt)$. Each one of these sellers sees the real value of its price fall below p during a period of time of length dt. The flow of sellers out of the interval $[s, p]$ is equal to the measure of sellers who, at the beginning of the period, have a real price between s and $s\exp(\pi dt)$. Each one of these sellers pays the menu cost and resets the nominal price so that its real value is somewhere in the interval $[S, Q]$, $S > p$. Dividing both sides of (8) by dt and taking the limit $dt \to 0$, we obtain

$$F'(p)p = F'(s)s, \; \forall p \in (s, S). \tag{9}$$

For any $p \in (S, Q)$, the inflow-outflow equation is given by

$$F(e^{\pi dt}p) - F(p) = \left[F(e^{\pi dt}s) - F(s)\right][1 - G(p)]. \tag{10}$$

The flow of sellers into the interval $[s, p]$ is equal to the measure of sellers who, at the beginning of the period, have a real price between p and $p\exp(\pi dt)$. Each one of these sellers sees the real value of its price fall below p during a period of time of length dt. The flow of sellers out of the interval $[s, p]$ is equal to a fraction $1 - G(p)$ of the measure of sellers who, at the beginning of the period, have a real price between s and $s\exp(\pi dt)$. Every seller with a real price between s and $s\exp(\pi dt)$ pays the menu cost and resets its nominal price. A fraction $1 - G(p)$ of them resets the nominal price so that its real value is greater than p. Dividing both sides of (10) by dt and taking the limit $dt \to 0$, we obtain

$$F'(p)p = F'(s)s[1 - G(p)], \; \forall p \in (S, Q). \tag{11}$$

The stationary price distribution F must satisfy the differential equation (9) over the interval (s, S) and the differential equation (11) over the interval (S, Q). In addition, the stationary price distribution F must satisfy the boundary conditions

$$F(s) = 0 \text{ and } F(Q) = 1. \tag{12}$$

The above observations motivate the following definition of equilibrium.

Definition 1. *A stationary (Q, S, s) equilibrium is a CDF of prices $F : [s, Q] \to [0, 1]$, a CDF of new prices $G : [S, Q] \to [0, 1]$, a lower bound on the price distribution $s \in (0, Q)$, a lower bound on the new price distribution $S \in (s, Q)$, and a seller's maximum value V^* that jointly satisfy the optimality conditions (4)-(7) and the stationarity conditions (9), (11) and (12).*

3.2 (Q, S, s) Equilibrium: Existence

The equilibrium condition (6) implies that $R(Q)$, the seller's profit when the real value of its nominal price is Q, must be equal to rV^*, the annuitized maximum value of the seller. Since $R(Q)$ is given by $b\left[\alpha+2(1-\alpha)\left(1-F(Q)\right)\right]Q$ and $F(Q)$ is equal to 1, the maximum value of the seller is equal to

$$V^{*}=\frac{b\alpha Q}{r}. \tag{13}$$

The seller's maximum value is equal to the present discounted value of profits for a hypothetical seller that always charges the buyer's reservation price Q and never has to incur the menu cost. The fact that the seller's maximum value is independent of the menu cost is a surprising property of the equilibrium. Intuitively, the property obtains because any increase in the menu cost is entirely passed through to the buyers through the response of the equilibrium distribution of prices.

The equilibrium condition (6) states that $R(p)$ must be equal to rV^{*} for all $p\in[S,Q]$. As $R(p)$ is given by $b\left[\alpha+2(1-\alpha)(1-F(p))\right]p$ and rV^{*} is equal to $b\alpha Q$, it follows that the stationary price distribution is equal to

$$F(p)=1-\frac{\alpha}{2\left(1-\alpha\right)}\frac{Q-p}{p},\ \forall p\in[S,Q]. \tag{14}$$

Over the interval $[S,Q]$, the stationary price distribution is exactly the same as in BJ83. This property of the equilibrium follows from the fact that, as in BJ83, the stationary price distribution is such that the seller's profits are constant over the interval $[S,Q]$.

The equilibrium condition (9) states that the derivative of the price distribution $F'(p)$ must be equal to $F'(s)s/p$ for all $p\in(s,S)$. Integrating this equilibrium condition on $F'(p)$ and using the fact that $F(s)$ is equal to 0, we find that the stationary price distribution is equal to

$$\begin{aligned} F(p) &= F'(s)s\log(p/s), && \forall p\in[s,S],\\ &= \left[1-\frac{\alpha}{2\left(1-\alpha\right)}\frac{Q-S}{S}\right]\frac{\log(p/s)}{\log(S/s)}, && \forall p\in[s,S], \end{aligned} \tag{15}$$

where the second line makes use of the continuity condition $F(S-)=F(S+)$. Over the interval $[s,S]$, the stationary price distribution is log-uniform, exactly as in B88 and in other models where sellers follow an (S,s) pricing rule. This property of the equilibrium follows from the fact that, as in B88, sellers enter the interval $[s,S]$ from the top, they travel through the interval $[s,S]$ at the constant rate π, and they exit the interval $[s,S]$ from the bottom. And, as first pointed out by Caplin and Spulber (1987), the log-uniform distribution is the only stationary distribution consistent with this type of behavior.

The equilibrium condition (11) states that the derivative of the price distribution $F'(p)$

must be equal to $F'(s)s(1-G(p))/p$ for all $p \in (S,Q)$. Since we know that $F'(p)$ is equal to the derivative of the right-hand side of (14), we can solve the equilibrium condition (11) with respect to the distribution of new prices G and obtain

$$G(p) = 1 - \left(1 - \frac{\alpha}{2(1-\alpha)} \frac{Q-S}{S}\right)^{-1} \frac{\alpha \log(S/s) Q}{2(1-\alpha)p}, \ \forall p \in (S,Q). \tag{16}$$

The role of the distribution of new prices G is to generate the cross-sectional distribution of prices F that makes the seller's profit and value constant over the interval $[S,Q]$ and, hence, makes sellers indifferent between resetting their price anywhere in the interval $[S,Q]$. Interestingly, in order to fulfill its role, the distribution of new prices must have mass points at S and Q with measure

$$\chi(S) = 1 - \left(1 - \frac{\alpha}{2(1-\alpha)} \frac{Q-S}{S}\right)^{-1} \frac{\alpha \log(S/s) Q}{2(1-\alpha)S}, \tag{17}$$

$$\chi(Q) = \left(1 - \frac{\alpha}{2(1-\alpha)} \frac{Q-S}{S}\right)^{-1} \frac{\alpha \log(S/s)}{2(1-\alpha)}. \tag{18}$$

Notice that, although the distribution of new prices has two mass points, the stationary price distribution is everywhere continuous. Intuitively, F is continuous because the fraction of sellers who reset their price to S and Q is strictly positive, but the measure of sellers who reset their price in a particular instant is zero.

Finally, we need to solve for the equilibrium prices s and S. The equilibrium condition (4) states that $R(s)$ must be equal to $r(V^*-c)$. From $F(s)=0$, it follows that $R(s)$ is equal to $b\left[\alpha + 2(1-\alpha)\right]s$. From (13), it follows that $r(V^*-c)$ is equal to $b\alpha Q - rc$. Hence, the lowest price s is given by

$$s = \frac{\alpha Q - rc/b}{2-\alpha}. \tag{19}$$

The equilibrium condition (6) states that the present discounted value of profits for a seller with a real price S must equal the maximum value V^*. After substituting out F, G and V^* and solving the integral, we can rewrite the equilibrium condition (6) as

$$\begin{aligned} \varphi(S) \equiv \quad & \left[\frac{1 - e^{-(r+\pi)T_2(S)}\left(1+(r+\pi)T_2(S)\right)}{(r+\pi)^2}\right] \frac{(2-\alpha)S - \alpha Q}{T_2(S)} \\ & + \left[\frac{1-e^{-(r+\pi)T_2(S)}}{r+\pi}\right] \alpha Q + e^{-rT_2(S)} \left(\frac{\alpha Q}{r} - \frac{c}{b}\right) - \frac{\alpha Q}{r} = 0. \end{aligned} \tag{20}$$

where

$$T_2(S) \equiv \frac{\log(S/s)}{\pi}, \quad s = \frac{\alpha Q - rc/b}{2-\alpha}.$$

A necessary condition for the existence of a (Q, S, s) equilibrium is that the equation $\varphi(S) = 0$ admits a solution for some S in the interval (s, Q). Notice that $\varphi(S)$ is strictly negative for all $S \in [s, \alpha Q/(2-\alpha)]$. Also, $\varphi(S)$ strictly increasing for all $S \in [\alpha Q/(2-\alpha), Q]$. Therefore, the equation the equation $\varphi(S) = 0$ admits a solution for some S in the interval (s, Q) if and only if $\varphi(Q)$ is strictly positive. As stated in Theorem 1, the condition $\varphi(Q) > 0$ is not only necessary, but also sufficient for the existence of a (Q, S, s) equilibrium.

Theorem 1: (Existence of a (Q, S, s) equilibrium). *A (Q, S, s) equilibrium exists iff $\varphi(Q) > 0$. Moreover, if a (Q, S, s) equilibrium exists, it is unique and: (a) $V'(t) = 0$ for all $t \in (0, T_1)$ and $V'(t) < 0$ for all $t \in (T_1, T_1{+}T_2)$; (b) $\hat{R}'(t) = 0$ for all $t \in (0, T_1)$, $\hat{R}'(t) > 0$ for all $t \in (T_1, \hat{T})$ and $\hat{R}'(t) < 0$ for all $t \in (T, T_1 + T_2)$, where $\hat{R}(t) \equiv R(Q\exp(-\pi t))$ and $\hat{T} \in (T_1, T_1 + T_2)$.*

Proof: In Appendix A.

It is useful to note that the condition $\varphi(Q) > 0$ does not define an empty set of parameter values. For instance, it is straightforward to verify that the condition is satisfied when the inflation rate π is sufficiently low or when the menu cost c is sufficiently small.

It is also useful to sketch a proof for the sufficiency of condition $\varphi(Q) > 0$. If $\varphi(Q) > 0$, the equation $\varphi(S) = 0$ admits a solution S^* in the interval $(\alpha Q/(2-\alpha), Q)$. Given S^*, let s^* be given by (19). Given S^* and s^*, let F^* be given by (14) and (15), let G^* be given by (16)-(18) and let V^* be given by (13). The tuple $(F^*, G^*, s^*, S^*, V^*)$ is the candidate equilibrium. First, notice that the candidate equilibrium satisfies the regularity conditions in Definition 1. In fact, it is easy to verify that s^* belongs to the interval $(0, S^*)$ and S^* belongs to the interval (s^*, Q). Moreover, it is easy to verify that F^* is a proper cumulative distribution function with support $[s^*, Q]$ and G^* is a proper cumulative distribution function with support $[S^*, Q]$. Second, notice that the candidate equilibrium satisfies the stationarity conditions (9), (11) and (12) and the optimality conditions (4)-(7). In fact, by construction of F^* and G^*, the candidate equilibrium satisfies (9), (11) and (12). By construction of F^*, S^* and s^*, the candidate equilibrium satisfies the optimality conditions (4)-(6). That is, by construction, the candidate equilibrium is such that the seller attains the present value of profits V^* for all prices in the interval $[S^*, Q]$ and such that the seller

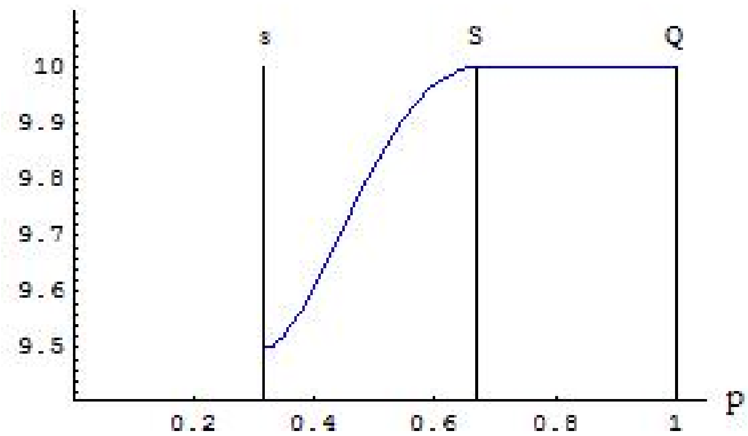


Figure 1: Present value of profits

finds it optimal to pay the menu cost when its price is equal to s^*. Finally, we need to verify that the candidate equilibrium satisfies the optimality condition (7), i.e. the seller's present value of profits is non-greater than V^* for all prices in the interval $[s^*, S^*]$. To this aim, we first show that the seller's flow profit $R(p)$ increases, peaks and then decreases as the real price falls from S^* to s^*. Then, we show that these properties of $R(p)$ together with the fact that $V^* = b\alpha Q/r$ guarantee that the seller's present discounted value of profits declines as the real price falls from S^* to s^*.

3.3 Properties of a (Q, S, s) Equilibrium

Figures 1 through 4 illustrate the main qualitative features of a (Q, S, s) equilibrium. Figure 1 plots the seller's present value of profits as a function of the real value of the seller's price. Like in a standard search-theoretic model of price dispersion (e.g. BJ83), the seller's present value of profits remains equal to its maximum V^* as the real value of the nominal price goes from Q to S. Like in a standard menu cost model (e.g., B88), the seller's present value of profits declines monotonically from V^* to $V^* - c$ as the real value of the nominal price falls from S to s. When the real value of the nominal price reaches s, the seller finds it optimal to pay the menu cost and reset its price.

When the seller pays the menu cost, it is indifferent between resetting its nominal price to any real value between S and Q. At first blush, this property of the equilibrium may seem puzzling as the seller would have to pay the menu cost less frequently if it were

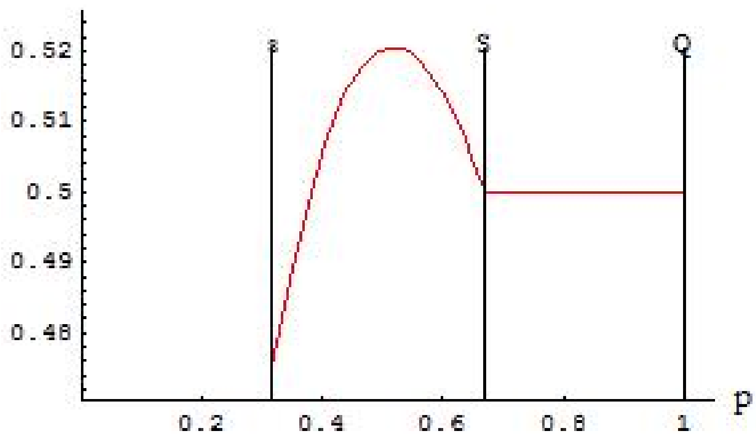
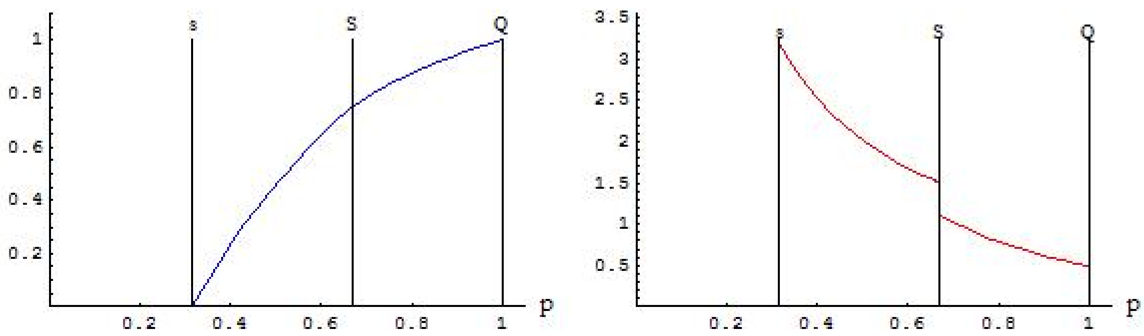


Figure 2: Flow profit

to reset the real price to Q rather than to, say, S. The solution to the puzzle is contained in Figure 2, which plots the seller's flow profit as a function of the real price. As the real price falls from Q to S, the seller's flow profit is constant and equal to rV^*. As the real price falls below S, the seller's flow profit begins to increase, it reaches a maximum, it begins to fall and, eventually, it attains the value $r(V^* - c)$. Thus, if the seller resets its real price to Q rather than to some lower value, it will pay the menu cost less frequently but it will also enjoy the highest flow profit less frequently. The two effects exactly balance each other and, for this reason, the seller is indifferent between resetting its real price to any value between S and Q.

Figure 3 plots the stationary distribution of prices. Over the interval $[S, Q]$, the distribution of prices is such that the seller's flow profit is constant, just as in a standard search-theoretic model of price dispersion. Over the interval $[s, S]$, the distribution is log-uniform, just as in a standard menu cost model. At the border between the two intervals (i.e. at the price S), there is a kink in the distribution and, consequently, a discontinuity in the density. In particular, the density of prices to the right of S is discontinuously lower than the density of prices to the left of S. This discontinuity explains why the seller's flow profit increases when the real price falls below S. In fact, when its real price falls, the seller experiences an increase in the volume of trade that is proportional to the increase in the number of firms charging a price higher than the seller's. That is, when its real price falls, the seller experiences an increase in the volume of trade that is proportional to the density of the price distribution. When the real price falls from Q to S, the density



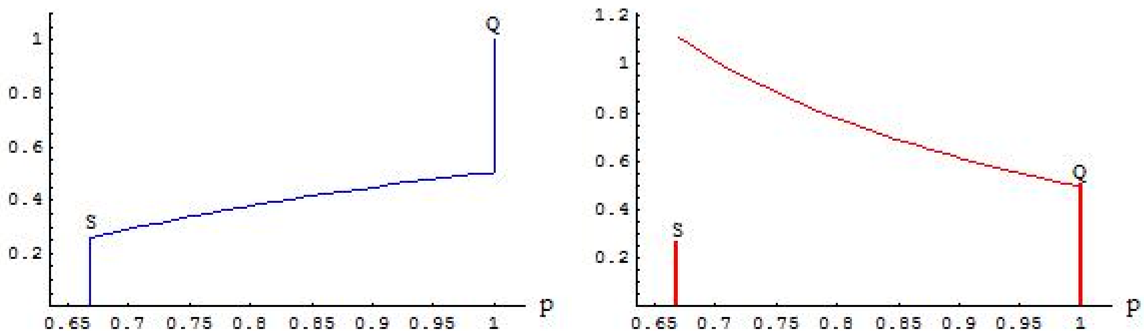
(a) CDF (b) PDF

Figure 3: Price distribution F

of the price distribution is such that the increase in the seller's volume of trade exactly offsets the decline in the seller's price. As the real price falls below S, the density of the price distribution jumps up and, hence, the increase in the seller's volume of trade more than offsets the decline in the seller's price. Hence, the seller's flow profit increases.

Figure 4 plots the distribution of new prices. The support of the distribution of new prices is the interval $[S, Q]$. The distribution has mass points at S and Q, and it is continuous everywhere else. The fact that the distribution of new prices has a mass point at S explains why the stationary price distribution has a kink at S and, in turn, why the density of the stationary price distribution has a discontinuity at S, why the seller's flow profit increases when its real price falls below S and, ultimately, why the seller is indifferent between resetting its price anywhere between S and Q. The fact that the distribution of new prices has a mass point at Q explains why the density of the stationary price distribution is strictly positive at S, which is necessary for the seller's flow profit to remain constant as the real value of its price falls below Q.

In a (Q, S, s) equilibrium, both search frictions and menu costs contribute to the stickiness of nominal prices. Consider a seller that just reset the real value of its nominal price to some $p \in [S, Q]$. The seller will keep this nominal price unchanged for $\log(p/s)/\pi$ units of time. During the first $\log(p/S)/\pi$ units of time in the life of its nominal price, the seller enjoys the maximized present value of profits V^*. Hence, the seller would not want to change its nominal price even if it could do it for free. During the last $\log(S/s)/\pi$ units of time in the life of its nominal price, the seller sees the present value of profits decline



(a) CDF

(b) PDF

Figure 4: New price distribution G

monotonically from V^* to $V^* - c$. Hence, the seller would like to change its nominal price but it chooses not to in order to avoid paying the menu cost. Thus, the first part of the duration of a nominal price is caused by search frictions—which make the seller's present value of profits constant over the region $[S, Q]$—and the second part is caused by menu costs—which prevent the seller from changing its price even though the present value of profits is declining.

Similarly, in a (Q, S, s) equilibrium, both search frictions and menu costs contribute to the dispersion of prices. The support of the price distribution is the interval $[s, Q]$. The presence of search frictions induces identical sellers to post different prices over the interval $[S, Q]$. The presence of menu costs and inflation pushes some of these prices into the interval $[s, S]$. In this sense, both search frictions and menu costs contribute to price dispersion.

3.4 (Q, S, s) Equilibrium: Comparative Statics

Now we want to understand the effect on the (Q, S, s) equilibrium of changes in some key parameters of the model. In particular, we want to understand the effect of changes in the menu cost c and in the inflation rate π on the cutoff prices s and S, on the traveling times T_1 and T_2, and on the price distributions F and G.

In Appendix B, we prove that a (Q, S, s) equilibrium exists if and only if the menu cost c belongs to the interval $(0, \bar{c})$, where $\bar{c}$ is a strictly positive number that depends on the value of the other parameters. Over the interval $(0, \bar{c})$, an increase in the menu cost

c leads to a decrease in the lowest price s in the cross-sectional distribution of prices, as well as to an increase in the lowest price S in the distribution of new prices. Intuitively, s decreases because the marginal cost of deferring a nominal price adjustment increases with c. Similarly, S increases because the marginal benefit of setting a higher nominal price increases with c. Given the behavior of s and S, it follows that an increase in c leads to: (a) an increase in price stickiness, as measured by the maximum duration $T_1 + T_2$ of a nominal price; (b) an increase in the contribution of menu costs to price stickiness, as measured by the ratio $T_2/(T_1 + T_2)$; (c) an increase in price dispersion, as measured by the range of prices $Q - s$ in the cross-section; (d) an increase in the contribution of menu costs to price stickiness, as measured by the ratio $(S - s)/(Q - s)$.

As the menu cost c approaches $\overline{c}$, the (Q, S, s) equilibrium converges to an (S, s) equilibrium, as in a standard menu cost model (see, e.g., B88). In fact, for $c \to \overline{c}$, the lowest new price S converges to the highest new price Q. Hence, for $c \to \overline{c}$, the distribution of new prices becomes degenerate at Q, the cross-sectional distribution of prices becomes log-uniform and the profits of the seller become decreasing everywhere on the support $[s, Q]$. In this limit, all of the stickiness of prices (and all of the dispersion of prices) is caused by menu costs. Conversely, as the menu cost c approaches 0, the (Q, S, s) equilibrium converges to the equilibrium of a standard search-theoretic model of price dispersion (see, e.g., BJ83). In fact, for $c \to 0$, the lowest price s converges to the lowest new price S. Hence, for $c \to 0$, the cross-sectional distribution of prices is such that sellers are indifferent between posting any price on the support $[s, Q]$. In this limit, all of the stickiness of prices (and all of the dispersion of prices) is caused by search frictions. [4]

In Appendix B, we also prove that a (Q, S, s) equilibrium exists if and only if the inflation rate π belongs to the interval $(0, \overline{\pi})$, where $\overline{\pi}$ is a strictly positive number that depends on the value of the other parameters. Over the interval $(0, \overline{\pi})$, an increase in the inflation rate π leads to an increase in the lowest new price S, but has no effect on the lowest price s. Intuitively, s does not change because neither the marginal cost nor the marginal benefit of deferring a nominal price adjustment depend on π. In contrast,

[4] Head, Liu, Menzio and Wright (2012) analyze a version of our model without menu costs. They show that the equilibrium uniquely pins down the distribution of real prices F, but it does not uniquely pin down the distribution of new real prices G or the pricing strategy of the individual sellers. In the limit as the menu cost converges to zero, the equilibrium of our model is exactly the same as in theirs. However, the limit of our model as the menu cost converges to zero uniquely pins down the distribution of new prices G and the pricing strategy of the individual sellers. In this sense, the limit of our model for c going to zero provides a natural refinement of the indeterminate equilibrium objects in Head, Liu, Menzio and Wright (2012).

S increases because the marginal benefit of setting a higher nominal price increases with π. Moreover, we show that an increase in the inflation rate leads to: (a) a decline in price stickiness, as measured by $T_1 + T_2$; (b) an increase in the contribution of menu costs to price stickiness, as measured by $T_2/(T_1 + T_2)$; (c) no change in price dispersion, as measured by $Q - s$; (d) an increase in the contribution of menu costs to price dispersion, as measured by $(S - s)/(Q - s)$. Finally, we show that, as the inflation rate π approaches $\overline{\pi}$, the (Q, S, s) equilibrium converges to the equilibrium of a standard menu cost model. As the inflation rate π approaches 0, the (Q, S, s) equilibrium does not have any special properties, except that the travelling times T_1 and T_2 go to infinity.

The comparative statics results are collected in the following theorem.

Theorem 2: (Comparative statics for (Q, S, s) equilibrium). (i) *A (Q, S, s) equilibrium exists iff $c \in (0, \overline{c})$, where $\overline{c} > 0$ depends on other parameters. As c increases in $(0, \overline{c})$, s falls and S increases. Price stickiness $T_1 + T_2$ increases and so does the fraction $T_2/(T_1 + T_2)$ due to menu costs. Price dispersion $Q - s$ increases and so does the fraction $(S - s)/(Q - s)$ due to menu costs.* (ii) *A (Q,S,s) equilibrium exists iff $\pi \in (0, \overline{\pi})$, where $\overline{\pi} > 0$ depends on other parameters. As π increases in $(0, \overline{\pi})$, s does not change and S increases. Price stickiness $T_1 + T_2$ falls, but the fraction $T_1/(T_1 + T_2)$ due to menu costs increases. Price dispersion $Q - s$ does not change, but the fraction $(S - s)/(Q - s)$ due to menu costs increases.*

Proof: In Appendix B.

4 (S, s) Equilibrium

An (S, s) equilibrium is an equilibrium where every seller lets inflation erode the real value of its nominal price until it reaches some point $s \in (0, Q)$, then the menu cost is paid and the nominal price changed so that its real value is S, where $S = Q$. In subsection 4.1, we formally define an (S, s) equilibrium. In subsection 4.2, we derive a necessary and sufficient condition for the existence of an (S, s) equilibrium. Finally, in subsection 4.3, we characterize the set of parameter values for which an (S, s) equilibrium exists. The main result of this section is that, as long as menu costs are sufficiently small, an (S, s) equilibrium does not exist, but a (Q, S, s) equilibrium does. The result implies that a (Q, S, s) equilibrium is not some odd outcome that exists alongside the more natural

(S, s) equilibrium, but that, in fact, a (Q, S, s) equilibrium it is the only possible outcome for some parameter values.[5]

4.1 (S, s) Equilibrium: Definition

The problem of an individual seller is the same as in Section 3. That is, the present discounted value of profits for a seller whose nominal price has a real value of $Q \exp(-\pi t)$ is given by

$$V(t) = \max_T \left\{ \int_t^T e^{-r(x-t)} R\left(Qe^{-\pi x}\right) dx + e^{-r(T-t)} \left(V^* - c\right) \right\}, \tag{21}$$

where

$$R(Qe^{-\pi x}) = b\alpha Qe^{-\pi x} + 2b\left(1-\alpha\right)\left(1 - F(Qe^{-\pi x})\right) Qe^{-\pi x}. \tag{22}$$

The seller finds it optimal to change the nominal price after $T^* - t$ units of time, where T^* is such that the marginal benefit of deferring a price change, $R(Q \exp(-\pi T^*))$, equals the marginal cost of deferring a price change, $r(V^* - c)$. This implies that the seller finds it optimal to change the nominal price when the price's real value is s, where s is given by

$$R(s) = r\left(V^* - c\right). \tag{23}$$

The seller finds it optimal to reset the real value of the nominal price to $S = Q$ if and only if $V(0) = V^*$ and $V(t) \leq V^*$ for all $t \in [0, T]$, where $T = \log(S/s)/\pi$ denotes the time it takes for the real value of a nominal price to fall from Q to s. Using (21), we can write the condition $V(0) = V^*$ as

$$\int_0^T e^{-rx} R\left(Qe^{-\pi x}\right) dx + e^{-rT}\left(V^* - c\right) = V^*. \tag{24}$$

Similarly, we can write the condition $V(t) \leq V^*$ for all $t \in [0, T]$ as

$$\int_t^T e^{-r(x-t)} R\left(Qe^{-\pi x}\right) dx + e^{-r(T-t)}\left(V^* - c\right) \leq V^*. \tag{25}$$

In an (S, s) equilibrium, a seller only changes its nominal price when the real value

[5] We do not need to look for equilibria in which a seller resets the real value of its nominal price to S, where $S < Q$. Indeed, it is easy to verify that this class of equilibria does not exist because, if $F(S)$ were equal to 1, an individual seller would be strictly better off resetting the real value of its nominal price to Q rather than to P. All details are available upon request.

of the price falls down to s. Moreover, when a seller changes its nominal price, it always resets it so that the real value of the new price is $S = Q$. As all sellers follow this (S, s) pricing rule, the stationary distribution of real prices is given by

$$F(p) = \frac{\log(p) - \log(s)}{\log(S) - \log(s)}, \forall p \in [s, S]. \tag{26}$$

The above observations motivate the following definition of equilibrium.

Definition 2: *A stationary (S, s) equilibrium is a CDF of prices $F : [s, Q] \to [0, 1]$, a lower bound on the price distribution $s \in (0, Q)$, an upper bound on the price distribution $S = Q$, and a seller's maximum value V^* that jointly satisfy the optimality conditions (23)-(25) and the stationarity condition (26).*

4.2 (S, s) Equilibrium: Existence

The equilibrium condition (23) states that the profit of a seller with a real price of s must be equal to the annuitized value of paying the menu cost and resetting the nominal price optimally. Since $R(s)$ is given by $b\,[\alpha + 2(1-\alpha)(1-F(s))]\,s$ and $F(s)$ is equal to 0, the equilibrium condition (23) implies that the lower bound on the price distribution must be equal to

$$s = \frac{r\,(V^* - c)}{b\,(2-\alpha)}. \tag{27}$$

The equilibrium condition (24) states that the present discounted value of profits for a seller with a real price of $S = Q$ must be equal to the maximum value V^*. After substituting out F and s and after solving the integral, we can rewrite the equilibrium condition (24) as one equation in the one unknown V^*, i.e.,

$$\begin{aligned} &\left[\frac{1 - e^{-(r+\pi)T(V^*)}\left(1 + (r+\pi)T(V^*)\right)}{(r+\pi)^2}\right]\frac{2b(1-\alpha)Q}{T(V^*)} \\ &+ \left[\frac{1 - e^{-(r+\pi)T(V^*)}}{r+\pi}\right] b\alpha Q + e^{-rT(V^*)}\left(V^* - c\right) - V^* = 0, \end{aligned} \tag{28}$$

where

$$T(V^*) \equiv \frac{1}{\pi}\log\left(\frac{b(2-\alpha)Q}{r(V^*-c)}\right).$$

If (F^*, S^*, s^*, V^*) is an (S, s) equilibrium, V^* must be a solution to (28). Conversely, if $V^* \in (c, b\alpha Q/r]$ is a solution to (28), then an (S, s) equilibrium exists. Let us briefly show why this is the case. Given V^*, let S^* be given by Q, let s^* be given by (27) and let F^* be given by (26). The tuple (F^*, S^*, s^*, V^*) is our candidate equilibrium. First, notice that the candidate equilibrium satisfies the regularity conditions in Definition 2. In fact, it is easy to verify that s^* belongs to the interval $(0, S^*)$ and that F^* is a proper cumulative distribution function with support $[s^*, S^*]$. Second, notice that the candidate equilibrium satisfies the stationarity condition (26) and the optimality conditions (23)-(25) and, hence, it is an actual equilibrium. In fact, by construction of F^*, the candidate equilibrium satisfies (26). By construction of s^* and V^*, the candidate equilibrium satisfies (23) and (24). That is, by construction, the candidate equilibrium is such that the seller finds it optimal to pay the menu cost when its price is equal to s^* and such that the seller attains the present value of profits V^* when its price is equal to S^*. Finally, in Appendix C we prove that the candidate equilibrium satisfies the optimality condition (25), i.e. the seller's present value of profits is non-greater than V^* for all prices in the interval $[s^*, S^*]$.

If there is no solution to equation (28) for V^* in the interval $(c, b\alpha Q/r]$, an (S, s) equilibrium does not exist. In fact, there is no (S, s) equilibrium associated with a solution $V^* \leq c$ to equation (28). This follows from the fact that $V^* \leq c$ implies $s \leq 0$, i.e. the seller finds it optimal to pay the menu cost when the real value of its nominal price becomes negative, an event that never occurs. Similarly, there is no (S, s) equilibrium associated with a solution $V^* > b\alpha Q/r$ to equation (28). This follows from the fact that $V^* > b\alpha Q/r$ implies $V'(0) > 0$, i.e. the seller finds it optimal to reset the real value of its nominal price strictly smaller than S^*.

The above observations are formalized in the following theorem.

Theorem 3: (Existence of an (S, s) equilibrium). *An (S, s) equilibrium exists if and only if equation (27) admits a solution $V^* \in (c, b\alpha Q/r]$.*

Proof: In Appendix C.

4.3 (S, s) Equilibrium: Comparative Statics

Theorem 3 states that an (S, s) equilibrium exists if and only if there exists a V^* such that: (a) V^* is a solution to (28); (b) V^* is greater than c and smaller than $b\alpha Q/r$. It is convenient to rewrite these two conditions in terms of $x \equiv V^* - c$ rather than V^*.

Condition (a) is equivalent to x being a solution to

$$\begin{aligned}&\left[\frac{1-e^{-(r+\pi)T(x)}\left(1+(r+\pi)T(x)\right)}{(r+\pi)^2}\right]\frac{2b(1-\alpha)Q}{T(x)}\\&+\left[\frac{1-e^{-(r+\pi)T(x)}}{r+\pi}\right]b\alpha Q+e^{-rT(x)}x-x=c,\end{aligned}\tag{29}$$

where

$$T(x)\equiv\frac{1}{\pi}\log\left(\frac{b(2-\alpha)Q}{rx}\right).$$

Condition (b) is equivalent to $x>0$ and such that

$$\frac{b\alpha Q}{r}-x\geq c.\tag{30}$$

Let $\psi(x)$ denote the left-hand side of (30). Clearly, the function $\psi(x)$ is such that $\psi(0)=b\alpha Q/r$, $\psi(b\alpha Q/r)=0$ and $\psi'(x)=-1$. Let $\omega(x)$ denote the left-hand side of (29). It is straightforward to verify that the function $\omega(x)$ is such that $\omega(0)=b\alpha Q/(r+\pi)$, $\omega'(x)>-1$ and $\omega(x)>0$ for all $x\in[0,b\alpha Q/r]$. Figure 5 illustrates the properties of $\psi(x)$ and $\omega(x)$ in the (x,c) space. The properties of $\omega(x)$ and $\psi(x)$ guarantee the existence of a unique x^* in the interval $(0,b\alpha Q/r)$ such that $\psi(x^*)=\omega(x^*)$, $\psi(x)>\omega(x)$ for all $x\in[0,x^*)$ and $\psi(x)<\omega(x)$ for all $x\in(x^*,b\alpha Q/r]$. Let c_h denote the maximum of $\omega(x)$ over the interval $[0,x^*]$. Notice that, since $\omega(x)<\psi(x)$ for all $x\in[0,x^*)$ and $\psi(x)\leq b\alpha Q/r$, c_h is strictly smaller than $b\alpha Q/r$. Similarly, let c_ℓ denote the minimum of $\omega(x)$ over the interval $[0,x^*]$. Notice that, since $\omega(x)>0$ for all $x\in[0,x^*]$, c_ℓ is strictly greater than zero. Figure 5 illustrates the definition of x^*, c_ℓ and c_h.

Given the menu cost c, an (S,s) equilibrium is an x such that $\omega(x)=c$ and $\psi(x)\leq c$. If $c<c_\ell$, an (S,s) equilibrium cannot exist because, as one can see from Figure 5, there is no x such that $\omega(x)=c$ and $\psi(x)\leq c$. Similarly, if $c>c_h$, an (S,s) equilibrium cannot exist because, as one can see from Figure 5, there is no x such that $\omega(x)=c$ and $\psi(x)\leq c$. In contrast, if $c\in[c_\ell,c_h]$, there exists at least one (S,s) equilibrium. There may exist multiple (S,s) equilibria because the function $\omega(x)$ may not be monotonic. This case is illustrated in panel (b) of Figure 5. Intuitively, multiplicity may arise because of a feedback effect between the stationary distribution of prices, F, and the seller's maximum value, V^*. The higher is the seller's maximum value, the lower is the point s at which a seller finds it optimal to reset its nominal price and, hence, the lower is the stationary

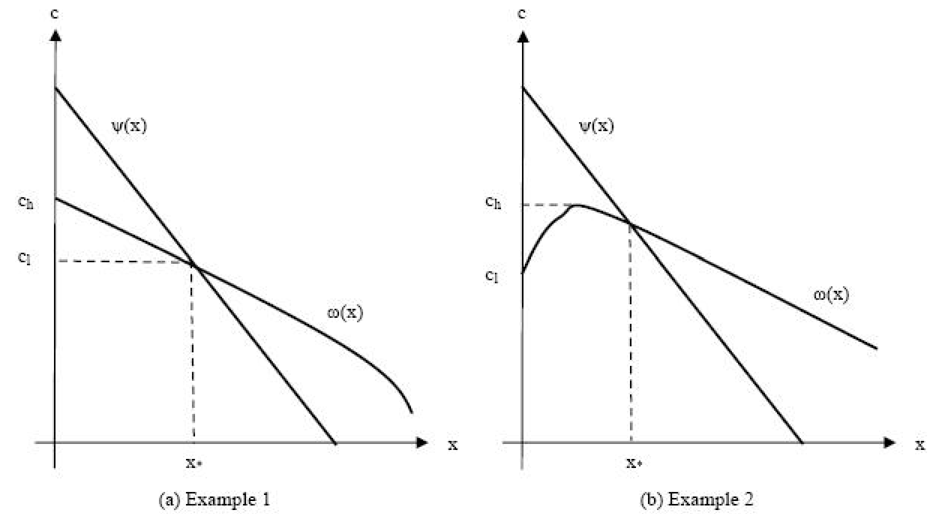


Figure 5: Existence of an (S, s) equilibrium

distribution of prices F. Conversely, the lower is the stationary distribution of prices F, the stronger is the competition faced by a seller offering any particular price and, hence, the lower is the seller's maximum value V^*. For some parameter values, this feed-back effect is so strong as to generate multiple (S, s) equilibria.

Finally, note that the interval of menu costs $[c_\ell, c_h]$ for which an (S, s) equilibrium exists always includes the upper bound $\overline{c}$ of the interval of menu costs for which a (Q, S, s) equilibrium exists. In fact, for $c = \omega(x^*)$, there exists an (S, s) equilibrium in which the maximum value of the seller is $b\alpha Q/r$. For $c \to \overline{c}$, the (Q, S, s) equilibrium converges to an (S, s) equilibrium in which the maximum value of the seller is $b\alpha Q/r$. Therefore, $\overline{c} = \omega(x^*)$.

The above results are summarized in the following theorem.

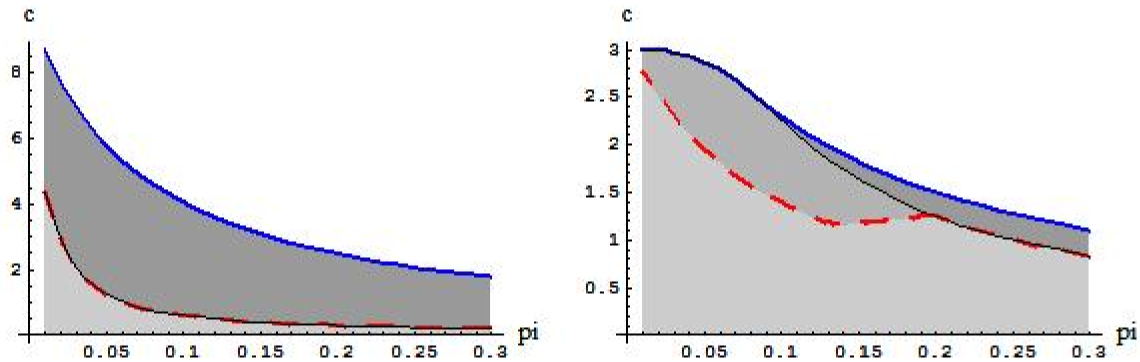
Theorem 4: (Comparative statics for (S, s) equilibrium). *For any menu cost c in the interval $[c_\ell, c_h]$, there exists at least one (S, s) equilibrium. The bounds c_ℓ and c_h are such that $c_\ell > 0$, $c_h < b\alpha Q/r$ and $\overline{c} \in [c_\ell, c_h]$.*

The key result in Theorem 4 is that an (S, s) equilibrium cannot exist if the menu cost is sufficiently small. There is a simple intuition behind this result. As we lower the menu cost, the lower bound s of the price distribution becomes higher and higher because the seller's marginal cost of deferring a price adjustment, $r(V^* - c)$, grows relative to the marginal benefit, $R(s) = b(2 - \alpha)s$. Hence, as we lower the menu cost, the price distribution becomes more and more compressed towards the upper bound $S = Q$ of the distribution. When the menu cost is sufficiently small, the price distribution become so compressed at the top that an individual seller finds it optimal to deviate from the equilibrium by resetting the real value of its nominal price to some $p < S$ than to S. The seller finds this deviation profitable because, by resetting its price to p, it can sell not only to the captive buyers, but also to the many non-captive buyers who are in contact with a second seller charging a price between p and S. Thus, when the menu is sufficiently low, an (S, s) equilibrium with $S = Q$ does not exists.

Could there exist an (S, s) equilibrium with $S < Q$? The answer is negative. To see this suppose for a moment that there exists an (S, s) equilibrium in which every seller resets its price to some S smaller than Q. In this equilibrium, a seller's present discounted value of profits must be greater at S than at any lower price. It is easy to verify that this condition can be satisfied only if the seller's present value of profits at S is some $V^* \leq b\alpha S/r$. However, if $V^* \leq b\alpha S/r$, an individual seller finds it optimal to deviate from the equilibrium by resetting the real value of its nominal price to Q rather than to S. The seller find this deviation profitable because, by resetting its price to Q rather than to S, it still only sells to the captive buyers but it enjoys a higher profit margin. More precisely, by resetting its price to Q rather than to S, the seller enjoys a period of length $\log(Q/S)/\pi$ during which the flow profit is strictly greater than $b\alpha S$. Since $b\alpha S \geq rV^*$, the seller also enjoys a higher present discounted value of profits. Thus, an (S, s) equilibrium with $S < Q$ does not exist.

Overall, when the menu cost is sufficiently low, there cannot be an (S, s) equilibrium with $S = Q$ because an individual seller would want to deviate from this equilibrium by resetting its price below S. Moreover, there cannot be an (S, s) equilibrium with $S < Q$ because an individual seller would want to deviate from this equilibrium by resetting its price above S (to Q). The natural resolution of this tension is an equilibrium in which sellers randomize with respect to their reset price: a (Q, S, s) equilibrium. As it is clear from the discussion in the previous paragraphs, the economic forces that rule out an (S, s) equilibrium and ask for an equilibrium in which sellers randomize with respect to their reset price are exactly the same economic forces that, absent menu costs, rule out a unique price equilibrium and ask for an equilibrium in which sellers randomize with respect to their price.

Theorem 2 guarantees that there are no holes between the region of parameters for



(a) Example 1 (b) Example 2

Figure 6: Existence regions

which an (Q, S, s) equilibrium exists and the region of parameters for which an (S, s) equilibrium exists. However, Theorem 2 does not tell us whether the (Q, S, s) and the (S, s) existence regions overlap or not. Figure 6 shows that the answer depends on parameter values. Figure 6(a) shows the combinations of menu costs, c, and inflation rates, π, for which there exists a (Q, S, s) equilibrium and for which there exists an (S, s) equilibrium, given that the parameters (α, r, b, Q) take on the values $(.5, .03, 1, 1)$. In particular, a (Q, S, s) equilibrium exists for combinations of c and π in the dark gray area between the thin black line and the x-axis. Similarly, an (S, s) equilibrium exists for combinations of c and π in the dark gray area between the dashed red line and the blue line. In this example, the (Q, S, s) and the (S, s) existence regions are non-overlapping: if $c \leq c_\ell = \overline{c}$, only a (Q, S, s) equilibrium exists; if $c \in [c_\ell, c_h]$, only an (S, s) equilibrium exists; and if $c > c_h$, neither type of equilibrium exists.[6] Figure 6(b) shows the (Q, S, s) and (S, s) existence regions when the parameters (α, r, b, Q) take on the values $(.15, .03, 1, 1)$. In this example, the two existence regions are overlapping: if $c < c_\ell$, only a (Q, S, s) equilibrium exists; if $c \in [c_\ell, \overline{c})$, both a (Q, S, s) and an (S, s) equilibrium exist; if $c \in [\overline{c}, c_h]$, only an (S, s) equilibrium exists; and if $c > c_h$, neither type of equilibrium exists.[7]

[6] When $c > c_h$, a stationary equilibrium does not exist because the menu cost c is greater than the seller's maximum value V^* and, hence, sellers never have the incentive to adjust their nominal price. However, if we were to let sellers decide whether to enter the market at some cost $k > 0$, we would find an (S, s) equilibrium with a higher buyer-to-seller ratio b.

[7] To understand the magnitudes in Figure 6(a), notice that a menu cost of 1 is equivalent to 2 years of the seller's maximized profits. In Figure 6(b), a menu cost of 1 is equivalent to 6.6 years of the seller's maximized profits. Therefore, at relatively low inflation, all reasonable values of the menu cost are such that the unique equilibrium is of the (Q, S, s) type.

5 Calibration and Decomposition

In a (Q, S, s) equilibrium, both search frictions and menu costs contribute to create some price stickiness. Similarly, in a (Q, S, s) equilibrium, both search frictions and menu costs contribute to create some price dispersion. In this section, we calibrate our model and use it to understand how much of the observed stickiness of nominal prices and how much of the observed cross-sectional variation of prices are due to search frictions and how much are due to menu costs. Our main finding is that search frictions are the main source of both price stickiness and price dispersion.

5.1 Calibration

We calibrate a version of the model in which sellers face a cost k of producing the good. For this version of the model, the parameters to be calibrated are the real interest rate, r, the inflation rate, π, the arrival rate of buyers, b, the fraction of buyers who meet only one seller, α, the buyers' valuation of the good, Q, the sellers' cost of producing the good, k, and the sellers' menu cost, c.

We set the real interest rate r to 5% and the inflation rate π to 3%. We note that the equilibrium objects F, G, S and s increase by a factor $\lambda > 0$ whenever the parameters Q, k and c increase by the same factor λ. Hence, we can set the buyers' valuation of the good Q to 1 and interpret the equilibrium prices, the sellers' production cost and the sellers' menu costs as fractions of Q. Next, we note that the equilibrium objects F, G, S and s depend on c and b only through their ratio c/b. Hence, we can set the inflow b of buyers to 1 and interpret the sellers' menu cost as a fraction of b. Finally, we calibrate the parameters α, c and k to match empirical measures of price dispersion, price stickiness and markups.

Nakamura and Steinsson (2008) measure the extent of price stickiness for consumer goods using the Bureau of Labor Statistics microdata underlying the Consumer Price Index. They find that the mean duration of nominal prices depends on whether sales and product substitutions are included or excluded from the data. During the 1998-2005 period, they find that the average duration of nominal prices is 7.7 months if sales and product substitutions are included in the data, and it is 13 months when sales and product substitutions are excluded from the data. They also find that there is a great deal of heterogeneity in the duration of nominal prices across different types of goods.

For example, excluding sales and product substitutions, the median duration of nominal prices is 3.5 months for unprocessed food and 27.3 months for apparel.

Kaplan and Menzio (2014) measure the extent and source of price dispersion for consumer goods using the Kielts-Nielsen Consumer Panel Dataset. First, Kaplan and Menzio (2014) measure the standard deviation of log prices at which the same item is sold during the same quarter and in the same market. Over the period 2004-2009, they find that the average standard deviation of prices for identical items is 19%. Second, Kaplan and Menzio (2014) decompose the variance of log prices into three different sources. Yhey find that: (a) approximately 10% of the variance of prices for an identical item is caused by differences in the expensiveness of the stores at which that item was sold; (b) between 25 and 60% of the variance of prices is caused by differences in the average price at which the same item is sold across stores that are equally expensive; (c) the remaining variance of prices is caused by differences in the price at which the same item is sold at the same store across different transactions (because of, say, coupon use). Third, Kaplan and Menzio (2014) find that there is a great deal of heterogeneity in the extent of price dispersion across different types of goods.

Our model abstracts from temporary sales and product substitutions. Hence, it is natural to calibrate the model to measures of price stickiness that are constructed excluding sales and product substitutions. Our model also abstracts from any variation in fundamentals across sellers, as well as from coupon usage. Hence, it is natural to calibrate the model to the component of price dispersion that is due to differences in the average price of the good across equally expensive stores, and not to differences in the expensiveness of different stores, or to differences in the price at which the good is sold at the same store. Moreover, since there is a great deal of variation in the extent of price dispersion and price stickiness across different goods, it is natural to entertain different calibration targets.[8]

[8] In our model, we assume that all sellers have the same, constant real cost of production. This assumption is reasonable as long as all sellers purchase the good in a wholesale market at a common, relatively constant real price. If sellers face different and/or time-varying real prices, there is a mismatch between our model and the real world. A natural way to tackle the potential mismatch between model and data is to restrict attention to a subset of sellers facing the same, constant wholesale price. With respect to price dispersion, we accomplish this task by taking out the variance of prices that is due to sellers' fixed effects. With respect to price stickiness, we cannot accomplish this task because we do not have a way to assess which price changes are caused by changes in sellers' costs. One possibility would be to restrict attention to goods that have a fairly stable wholesale price. Another possibility would be to take out all negative price changes, which—through the lens of our model—must be caused by changes in sellers' costs. These corrections would lead to measures of price stickiness that are higher than those

Table 1 reports our calibration targets and the associated calibration outcomes. In the benchmark calibration (column 1), we target a relatively long duration of prices and a relatively small amount of price dispersion. Specifically, we target an average duration of nominal prices of 5 quarters, a standard deviation of log prices of 4% and an average markup of 5%. Given these targets, we find that the fraction α of buyers who contact only one seller is 21%, the sellers' cost k of producing the good is 0.81, and the sellers' cost c of changing the nominal price of the good is 0.091%, which is equivalent to 2.3% of the sellers' average annual profit. Given these parameter values, the model admits a unique equilibrium where sellers follow a (Q, S, s) pricing rule with $Q = 1$, $S = 0.847$ and $s = 0.832$.

In the second calibration (column 2 in Table 1), we target a lower duration of nominal prices than in the benchmark calibration. In particular, we target an average duration of nominal prices of 3 quarters, a standard deviation of log prices of 4%, and an average markup of 5%. Given these targets, we find that the fraction α of buyers who contact only one seller is 22%, the sellers' cost k of producing the good is 0.81, and the sellers' menu cost c is 0. The equilibrium of the calibrated model is the same as in Head, Liu, Menzio and Wright (2012). In particular, sellers enjoy the same profit for any price on the support of the distribution F. Hence, they are indifferent between changing and not changing their nominal price, as long as the real value of that price remains on the support of F. Similarly, they are indifferent between resetting their nominal price to take any real value on the support of F.

In the third calibration (column 3 in Table 1), we target a higher standard deviation of prices than in the benchmark calibration. In particular, we target a standard deviation of log prices of 8%, an average duration of nominal prices of 5 quarters, and an average markup of 5%. Given these targets, we find that the fraction α of buyers who contact only one seller is 7.8%, the sellers' cost k of producing the good is 0.61, and the sellers' menu cost c is 0. Again, the calibrated model involves no menu costs and the equilibrium is like in Head, Liu, Menzio and Wright (2012).

In general, we find that the calibrated model features positive menu costs only for goods that are characterized by an above-average degree of price stickiness and by a

reported by Klenow and Kryvstov (2008) or Nakamura and Steinsson (2008). However, as we shall see soon, our estimated menu costs are small even when we target a very high degree of price stickiness.

Table 1: Calibration Targets and Outcomes

Calibration Targets	(1)	(2)	(3)
Standard deviation of prices	4%	4%	8%
Average markup	5%	5%	5%
Average duration of prices	15 mo.	9 mo.	15 mo.

Calibration Outcomes	(1)	(2)	(3)
Search friction α	0.212	0.222	0.078
Production cost k	0.811	0.815	0.613
Menu cost c	0.093%	0%	0%
Interest rate r	5%	5%	5%
Inflation rate π	3%	3%	3%

below-average degree of price dispersion. For goods that are characterized either by a below-average degree of price stickiness or by an above-average degree of price dispersion, the calibrated model features no menu costs at all. The intuition behind this finding is straightforward. Search frictions create an interval $[S, Q]$ over which the sellers' profit is constant. When the targeted price dispersion is relatively high or when the targeted price stickiness is relatively low, the interval $[S, Q]$ is long enough that—even if sellers were to change their nominal price when it reaches the real value S—the model would be able to match the targeted duration of nominal prices. That is, when the targeted price dispersion is relatively high or when the targeted price stickiness is relatively low, menu costs are not needed to create more price stickiness than the one naturally created by search frictions.

5.2 Decomposing Stickiness and Dispersion

Using the calibrated version of the model, we can break down price stickiness into a component that is generated by menu costs—i.e., the stickiness that is due to the fact that the seller would like to change its nominal price but does not because of the menu cost—and a component that is generated by search frictions—i.e., the stickiness that is due to the fact that seller has nothing to gain from changing its nominal price.

The decomposition of price stickiness can be carried out in several ways. The first approach is to decompose the duration of the highest price Q. The duration of the price Q that is caused by search frictions is given by $T_1 = \log(Q/S)/\pi$, while the duration caused by menu costs is given by $T_2 = \log(S/s)/\pi$. The second approach is to decompose the average duration of all prices in the distribution F. The average price duration that is caused by search frictions is given by $\int_S^Q \log(p/S)dF(p)$, while the average price duration that is caused by menu costs is given by $\int_S^Q \log(S/s)dF(p) + \int_s^S \log(p/s)dF(p)$. A third approach is to measure the duration of prices caused by search frictions as the average duration of nominal prices if menu costs were equal to zero (i.e. $c = 0$) and, similarly, to measure the duration of prices caused by menu costs as the average duration of nominal prices if none of the buyers met multiple sellers (i.e. $\alpha = 1$) and, hence, search did not create any price stickiness.

Panel A in Table 2 reports the result of the decomposition of price stickiness. The table only reports the results for the benchmark calibration, as in the two alternative calibrations menu costs are equal to zero and, hence, all price stickiness is due to search frictions. The first column in Table 2 shows that the duration of the highest price is 6.09 years and that 90.7% of this duration is due to search frictions, while 9.3% is due to menu costs. The second column shows that the average duration of a price is 1.25 years and that 65% of this duration is due to search frictions, while 35% is due to menu costs. The last column shows that the average duration of prices would be 3% lower in a version of the model without menu costs, and 90% lower in a version of the model where search frictions do not create any price dispersion. Overall, the decomposition reveals that search frictions are the main source of price stickiness[9] even for goods that are characterized by the relatively high average price duration of 5 quarters and by the relatively low standard deviation of log prices of 4%. Since only the price stickiness caused by menu costs reflects an impediment to seller changing their nominal prices, our decomposition suggest that—as pointed out by Head, Liu, Menzio and Wright (2012)—sticky prices may play a limited role in transmitting monetary policy shocks to the real side of the economy.

We can also use the calibrated model to break down price dispersion into a component that is generated by menu costs—i.e., the dispersion due to the fact that sellers let the real value of their prices fall in order to avoid paying the menu cost—and a component

[9] These findings suggest that estimating menu costs using models that abstracts from the search frictions is likely to cause major biases. To illustrate this point, we consider a version of our model in which the parameter α is constrained to be equal to one, so that search frictions do not generate any price dispersion and any price stickiness. We calibrate the model so as to match an average duration of prices of 5 quarters. We find that the calibrated value of c is 8.5%, which is 90 times larger than the value estimated using the unconstrained version of our model.

Table 2: Decomposition of Price Stickiness and Dispersion

A: Decomposition of price stickiness	(1)	(2)	(3)
Price stickiness	73 mo.	15 mo.	15 mo.
Search friction	90.7%	65%	97%
Menu cost	9.3%	35%	10%

B: Decomposition of price dispersion	(1)	(2)	(3)
Price dispersion	0.167	0.16%	0.16%
Search friction	91%	75%	100%
Menu cost	9%	25%	0%

that is generated by search frictions—i.e., the dispersion that is due to the fact that sellers have an incentive to play mixed strategies with respect to their real prices.

The decomposition of price dispersion can be carried out in several ways. The first approach is to measure price dispersion as the length of the support of the distribution. According to this approach, the component of price dispersion that is due to search frictions is $Q - S$ and the component of price dispersion that is due to menu costs is $S - s$. The second approach is to measure the dispersion of prices caused by search frictions as the variance of log prices in the interval $[S, Q]$, and to measure the dispersion of prices caused by menu costs as the variance of log prices in the interval $[s, S]$. The third approach is to measure the dispersion of prices caused by search frictions as the variance of log prices if menu costs were equal to zero (i.e. $c = 0$) and, similarly, to measure the dispersion of prices caused by menu costs as the variance of log prices if none of the buyers met multiple sellers (i.e. $\alpha = 1$) and, hence, search did not give sellers any incentives to play mixed strategies.

Panel B in Table 2 reports the result of the decomposition of price dispersion. The first column shows that the length of the support of the distribution is 0.167 and that 91% of this length is caused by search frictions, while 9% is caused menu costs. The second column shows that the overall variance of log prices is 0.16% and that 75% of this variance is due to search frictions, while 25% is due to menu costs. The last column shows that the variance of log prices would be approximately the same in a version of the model without menu costs, and would be approximately zero in a version of the model without search frictions. Overall, the decomposition reveals that search frictions are the main source of price dispersion, even for goods that are characterized by an above-average degree of price stickiness.[10]

6 Conclusions

In this paper, we introduced menu costs in a standard search-theoretic model of price dispersion in the spirit of Butters (1977), Varian (1980) and Burdett and Judd (1983). We proved that, when menu costs are sufficiently small, the only equilibrium is such that sellers follow a (Q, S, s) pricing rule. According to this rule, sellers let inflation erode the real value of their price until it reaches some point s, then they pay the menu cost and change their nominal price so that the real value of the new price is randomly drawn from a distribution with support $[S, Q]$, where Q is the buyer's reservation price and S is some price between s and Q. A (Q, S, s) equilibrium looks like a hybrid between a standard menu cost model (as, e.g., B88) and a standard search-theoretic model of price dispersion (as, e.g., BJ83). In particular, as in BJ83, the seller's profits remain constant as the real value of the seller's nominal price falls from Q to S. As in B88, the seller's profits decline as the real value of seller's nominal price falls from S to s. Similarly, as in BJ83, the distribution of prices over the interval $[Q, S]$ is such that the seller's profits are constant. As in B88, the distribution of prices over the interval $[s, S]$ is log-uniform. These natural equilibrium outcomes obtain because sellers draw their new prices from a rather surprising mixing distribution, which involves mass points at both S and Q. We also argued that, when menu costs are sufficiently small, an equilibrium in which seller follow a standard (S, s) pricing rule does not exist for the same reasons why the equilibrium of our model features price dispersion absent menu costs. This last observation makes us conjecture that, for small menu costs, a (Q, S, s) equilibrium will emerge in other models that, absent menu costs, feature price dispersion (see, e.g., Prescott 1975, Eden 1994, Menzio and Trachter 2014).

We then calibrated the model to understand how much price stickiness is due to search frictions and how much is due to menu costs. We found that, for goods characterized by a below-average duration of prices or by an above-average cross-sectional dispersion of

[10] These findings suggest that estimating search frictions using models that abstracts from menu costs (see, e.g., Hong and Shum 2006 or Moraga-Gonzales and Wildenbeest 2008) is unlikely to cause major biases. To illustrate this point, we consider a version of our model in which the menu cost c is constrained to be equal to zero. We calibrate the model so as to match a 4% standard deviation of log prices. We find that the calibrated value of α is 0.198, which is rather close to the value of α estimated using the unconstrained version of our model (0.212).

prices, menu costs are zero. Only for goods characterized by an above-average duration of prices and by a below-average price dispersion, we found menu costs that are positive, albeit small. For example, for goods with a price duration of prices of 5 quarters and a standard deviation of prices of 4%, menu costs account for somewhere between 10 and 35% of price stickiness, depending on how the decomposition is carried out. Since only the price stickiness caused by menu costs reflects an actual impediment to seller changing their nominal prices, our calculations suggest that—as pointed out by Head, Liu, Menzio and Wright (2012)—sticky prices may play a limited role in transmitting monetary policy shocks to the real side of the economy.

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Appendix

A Proof of Theorem 1

(i) Part (i) of Theorem 1 states that a (Q,S,s) equilibrium exists if and only if $\varphi(Q)>0$. The function $\varphi(S)$ is defined as

$$\begin{aligned}\varphi(S)\equiv\ & \left[\frac{1-e^{-(r+\pi)T_2(S)}\left(1+(r+\pi)T_2(S)\right)}{(r+\pi)^2}\right]\frac{(2-\alpha)S-\alpha Q}{T_2(S)} \\ &+\left[\frac{1-e^{-(r+\pi)T_2(S)}}{r+\pi}\right]\alpha Q+e^{-rT_2(S)}\left(\frac{\alpha Q}{r}-\frac{c}{b}\right)-\frac{\alpha Q}{r}=0,\end{aligned}\tag{A1}$$

where

$$T_2(S)\equiv\frac{\log(S/s)}{\pi},\quad s=\frac{\alpha Q-rc/b}{2-\alpha}.\tag{A2}$$

It is straightforward to verify that $\varphi(S)$ has the following properties: (i) $\varphi(S)<0$ for all $S\in[s,\alpha Q/(2-\alpha)]$; (ii) $\varphi'(S)>0$ for all $S\in[\alpha Q/(2-\alpha),Q]$.

Suppose that $\varphi(Q)\leq 0$. Then a (Q,S,s) equilibrium does not exist. On the way to a contradiction, let (F,G,S,s,V) be a (Q,S,s) equilibrium. As proved in Section 3, if (F,G,S,s,V) is a (Q,S,s) equilibrium, then S must be a solution to the equation $\varphi(S)=0$ and it must belong to the interval (s,S). However, $\varphi(S)=0$ cannot admit any solution in the interval (s,Q) because $\varphi(S)<0$ for all $S\in(s,Q)$.

Conversely, suppose that $\varphi(Q)>0$. Then, the equation $\varphi(S)=0$ admits at most one solution in the interval (s,Q) because $\varphi(s)<0$ and $\varphi(Q)\leq 0$ and $\varphi(S)$ is strictly increasing in S. Let S^* denote this solution. Let s^* be defined as in equation (19) for $S=S^*$. Let F^* be defined as in equations (14) and (15) for $S=S^*$ and $s=s^*$. Let G^* be defined as in equations (16)-(18) for $S=S^*$ and $s=s^*$. Let V^* be defined as in equation (13). Moreover, let T_1^* and T_2^* be defined, respectively, as $\log(Q/S^*)/\pi$ and $\log(S^*/s^*)/\pi$ and let $R(p)$ be defined as in (2) for $F=F^*$. To prove that the tuple (F^*,G^*,S^*,s^*,V^*) constitutes a (Q,S,s) equilibrium, we need to verify that it jointly satisfies the optimality conditions (4)-(7) and the stationarity conditions (9), (11) and (12). In addition, we need to verify that $S^*\in(s^*,Q)$, $s^*\in[0,Q)$, F^* is a CDF with support $[s^*,Q]$ and G^* is a CDF with support $[S^*,Q]$.

The tuple (F^*,G^*,S^*,s^*,V^*) satisfies the stationarity condition (9) because, for all

$p \in (s^*, S^*)$, we have

$$F^{*\prime}(p)p \equiv \left[1 - \frac{\alpha}{2(1-\alpha)}\frac{Q - S^*}{S^*}\right]\frac{1}{\log(S^*/s^*)} \\ = F^{*\prime}(s^*)s^*. \tag{A3}$$

Similarly, the stationarity condition (11) is satisfied because, for all $p \in (S^*, Q)$, we have

$$\begin{aligned} F^{*\prime}(p)p &\equiv \frac{\alpha}{2(1-\alpha)}\frac{Q}{p} \\ &= \left[1 - \frac{\alpha}{2(1-\alpha)}\frac{Q - S^*}{S^*}\right]^{-1}\left[1 - \frac{\alpha}{2(1-\alpha)}\frac{Q - S^*}{S^*}\right]\frac{s^*}{\log(S^*/s^*)}\frac{\alpha\log(S^*/s^*)Q}{2(1-\alpha)s^*p} \\ &= F^{*\prime}(s)\left(1 - G^*(p)\right)s^*. \end{aligned} \tag{A4}$$

Moreover, notice that F^* is a CDF with support $[s^*, Q]$. In fact, $F^*(s^*) = 0$, $F^*(Q) = 1$ and $F^{*\prime}(p) > 0$ for all $p \in [s^*, Q]$.

The tuple $(F^*, G^*, S^*, s^*, V^*)$ satisfies the optimality condition (4) because

$$\begin{aligned} r\left(V^* - c\right) &= b\alpha Q - cr \\ &= b\left[\alpha + 2(1-\alpha)(1 - F^*(s^*))\right]s^* \\ &= R(s^*). \end{aligned} \tag{A5}$$

The optimality condition (5) is satisfied because, for all $t \in [0, T_1^*]$, we have

$$\begin{aligned} R(Qe^{-\pi t}) &= b\left[\alpha + 2(1-\alpha)(1 - F^*(Qe^{-\pi t}))\right]Qe^{-\pi t} \\ &= b\alpha Q = rV^*. \end{aligned} \tag{A6}$$

The optimality condition (6) is satisfied because $\varphi(S^*) = 0$ implies

$$\int_{T_1^*}^{T_1^*+T_2^*} e^{-r(x-T_1^*)}R(Qe^{-\pi x})dx + e^{-rT_2^*}\left(V^* - c\right) - V^* = 0. \tag{A7}$$

Moreover, $S^* \in (s^*, Q)$ and the assumption $c \in (0, b\alpha Q/r)$ implies

$$s^* \equiv \frac{\alpha Q - rc/b}{2 - \alpha} \in (0, Q)\,. \tag{A8}$$

The optimality condition (5) together with the optimality condition (6) guarantees that the seller's value $V(t)$ is equal to V^* for all $t \in [0, T_1^*]$. Now, we need to verify that the optimality condition (7) That is, we need to verify that the seller's value $V(t)$ is non-greater than V^* for all $t \in [T_1^*, T_1^* + T_2^*]$ To this aim, notice that $V(t)$ satisfies the differential equation

$$
\begin{aligned}
rV(t) &= \hat{R}(t) + V'(t), \\
\hat{R}(t) &\equiv R(Qe^{-\pi t}).
\end{aligned}
\tag{A9}
$$

Notice that, for $t \in [T_1^*, T_1^* + T_2^*]$, $\hat{R}(t)$ is given by

$$
\hat{R}(t) = e^{-\pi(t-T_1^*)} b \left\{ (2-\alpha)S^* - [(2-\alpha)S^* - \alpha Q] \left[1 - \frac{\pi(t-T_1^*)}{\log(S^*/s^*)} \right] \right\}. \tag{A10}
$$

The derivative of $\hat{R}(t)$ with respect to t is given by

$$
\hat{R}'(t) = \pi e^{-\pi(t-T_1^*)} b \left\{ [(2-\alpha)\, S^* - \alpha Q] \left[1 + \frac{1 - \pi(t-T_1^*)}{\log(S^*/s^*)} \right] - (2-\alpha)S^* \right\}. \tag{A11}
$$

Notice that $\hat{R}'(t)$ has the same sign as

$$
\sigma(t) = [(2-\alpha)\, S^* - \alpha Q] \left[1 + \frac{1 - \pi(t-T_1^*)}{\log(S^*/s^*)} \right] - (2-\alpha)S^*. \tag{A12}
$$

It is straightforward to verify that $\sigma(t)$ is strictly decreasing in t for all $t \in [T_1^*, T_1^* + T_2^*]$ and that $\sigma(T_1^*)$ is strictly positive (*negative*) if $(2-\alpha)S^*$ is strictly greater (*smaller*) than $\alpha Q\,(1 + \log(S^*/s^*))$. Therefore, if $(2-\alpha)S^* > \alpha Q\,(1 + \log(S^*/s^*))$, $\hat{R}(t)$ is first strictly increasing and then strictly decreasing over the interval $[T_1^*, T_1^* + T_2^*]$. If $(2-\alpha)S^* < \alpha Q\,(1 + \log(S^*/s^*))$, $\hat{R}(t)$ is strictly decreasing for all $t \in [T_1^*, T_1^* + T_2^*]$. Notice that $\hat{R}(t)$ cannot be increasing for all $t \in [T_1^*, T_1^* + T_2^*]$ because $\hat{R}(T_1^*) = b\alpha Q$ and $\hat{R}(T_1^* + T_2^*) = b\alpha Q - cr$.

Consider the phase diagram in Figure 7, which describes the differential equation (A9). The black line passing through the origin denotes the locus of points $(\hat{R}, V)$ such that $V = \hat{R}/r$ and, hence, $V' = 0$. Any point below the black line is such that $V < \hat{R}/r$ and, hence, $V' < 0$. Any point above the black line is such that $V > \hat{R}/r$ and, hence, $V' > 0$. From (5) and (6), it follows that $\hat{R}(T_1^*) = b\alpha Q$ and $V(T_1^*) = b\alpha Q/r$. Hence, the point $(\hat{R}(T_1^*), V(T_1^*))$ lies on the black line and $V'(T_1^*) = 0$. From (4) and (13), it follows that $\hat{R}(T_1^* + T_2^*) = b\alpha Q - cr$ and $V(T_1^* + T_2^*) = b\alpha Q/r - c$. Hence, the point $(\hat{R}(T_1^* + T_2^*), V(T_1^* + T_2^*))$ lies on the black line and $V'(T_1^* + T_2^*) = 0$.

Now, we want to find out the trajectory that the pair $(\hat{R}(t), V(t))$ follows as it goes from the point $(b\alpha Q, b\alpha Q/r)$ at $t = T_1^*$ to the point $(b\alpha Q - cr, b\alpha Q/r - c)$ at $t = T_1^* + T_2^*$. Notice that $(2-\alpha)S^*$ must be greater than $\alpha Q\,(1 + \log(S^*/s^*))$. In fact, if $(2-\alpha)S \leq$

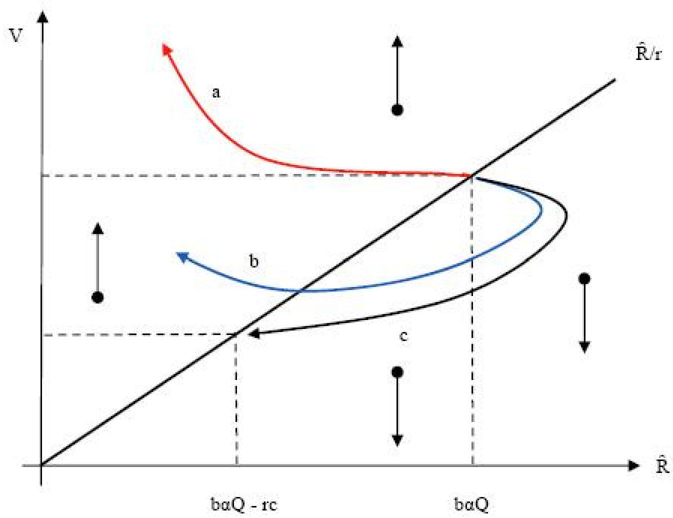

Figure 7: Joint Dynamics of $V(t)$ and $\hat{R}(t)$

$\alpha Q\left(1+\log(S^*/s^*)\right)$, $\hat{R}(t)$ is strictly decreasing for all $t \in [T_1^*, T_1^* + T_2^*]$. As illustrated by the trajectory (a) in the phase diagram, this implies that $(\hat{R}(t), V(t))$ exits the initial point $(b\alpha Q, b\alpha Q/r)$ from the left, enters the region where $V'(t) > 0$, and remains in that region for all $t \in (T_1^*, T_1^* + T_2^*)$. Thus, $V(T_1^* + T_2^*) > V(T_1^*) = b\alpha Q/r$, which contradicts the fact that $V(T_1^* + T_2^*) = b\alpha Q/r - c$.

Since $(2-\alpha)S > \alpha Q\left(1+\log(S^*/s^*)\right)$, $\hat{R}(t)$ is first increasing and then decreasing over the interval $[T_1^*, T_1^* + T_2^*]$. As illustrated by trajectories (b) and (c) in the phase diagram, this implies that $(\hat{R}(t), V(t))$ exits the initial point $(b\alpha Q, b\alpha Q/r)$ from the right, enters the region where $V'(t) < 0$, and remains in that region until it crosses the black line either at some $\hat{t} < T_1^* + T_2^*$, as in trajectory (b), or at $T_1^* + T_2^*$, as in trajectory (c). If $(\hat{R}(t), V(t))$ crosses the black line at some $\hat{t} < T_1^* + T_2^*$, then $\hat{R}'(\hat{t}) < 0$. This implies that $V'(t) > 0$ and $\hat{R}'(t) < 0$ for all $t \in (\hat{t}, T_1^* + T_2^*)$. Thus, $(\hat{R}(t), V(t))$ cannot reach the end point $(b\alpha Q - cr, b\alpha Q/r - c)$. Therefore, the pair $(\hat{R}(t), V(t))$ must follow the trajectory (c). Along this trajectory, $V(t)$ is strictly decreasing.

To complete the existence proof, we still need to verify that G^* is a CDF with support

$[S^*, Q]$. To this aim, recall that G^* is given by

$$G^*(p) \equiv \begin{cases} 0, & \text{if } p < S^*, \\ 1 - \left[1 - \dfrac{\alpha}{2(1-\alpha)} \dfrac{Q - S^*}{S^*}\right]^{-1} \dfrac{\alpha \log(S^*/s^*) Q}{2(1-\alpha)p}, & \text{if } p \in [S^*, Q), \\ 1, & \text{if } p \geq Q. \end{cases} \tag{A13}$$

The distribution function $G^*(p)$ has the following properties: (i) $G^*(p) = 0$ for all $p < S^*$; (ii) $G^*(S^*) \geq 0$ if and only if $(2-\alpha)S^* \geq \alpha Q\left[1 + \log(S^*/s^*)\right]$; (iii) $G^{*\prime}(p) > 0$ for all $p \in (S^*, Q)$ if and only if $S^* > \alpha Q/(2-\alpha)$; (iv) $G^*(Q-) \leq 1$ if and only if $S^* > \alpha Q/(2-\alpha)$; (iv) $G^*(p) = 1$ for all $p \geq Q$. Therefore, $G^*(p)$ is a proper CDF with support $[S^*, Q]$ if and only if $S^* > \alpha Q/(2-\alpha)$ and $(2-\alpha)S^* \geq \alpha Q\left[1 + \log(S^*/s^*)\right]$. We have already established that both conditions hold.

(ii) Part (ii) of Theorem 1 states that: (a) $V'(t) = 0$ for all $t \in (0, T_1^*)$ and $V'(t) < 0$ for all $t \in (T_1^*, T_1^* + T_2^*)$; (b) $\hat{R}'(t) = 0$ for all $t \in (0, T_1^*)$, $\hat{R}'(t) > 0$ for all $t \in (T_1^*, \hat{T})$ and $\hat{R}'(t) < 0$ for all $t \in (\hat{T}, T_1^* + T_2^*)$. Both properties have been established while proving part (i). ■

B Proof of Theorem 2

(i) In the proof of Theorem 1, we showed that a (Q, S, s) equilibrium exists if and only if the equation $\Phi(S, c) = 0$ admits a solution for $S \in (\alpha Q/(2-\alpha), Q)$. The function $\Phi(S, c)$ is given by

$$\begin{aligned} \varphi(S,c) = \quad & \left[\frac{1 - e^{-(r+\pi)T_2(S,c)}(1 + (r+\pi)T_2(S,c))}{(r+\pi)^2 T_2(S,c)}\right]\left[(2-\alpha)S - \alpha Q\right] \\ & + \left[\frac{1 - e^{-(r+\pi)T_2(S,c)}}{r+\pi}\right]\alpha Q + e^{-rT_2(S,c)}\left(\frac{\alpha Q}{r} - \frac{c}{b}\right) - \frac{aQ}{r}, \end{aligned} \tag{B1}$$

where $T_2(S, c)$ is given by

$$T_2(S,c) = \frac{\log\left(S/s(c)\right)}{\pi}, \quad s(c) = \frac{\alpha Q - rc/b}{2-\alpha}. \tag{B2}$$

Let $\varphi_S(S, c)$ and $\varphi_c(S, c)$ denote the derivatives of $\varphi(S, c)$ with respect to S and c.

The derivative $\varphi_S(S,c)$ is given by

$$\varphi_S(S,c) = \left[\frac{1-e^{-(r+\pi)T_2(S,c)}\left(1+(r+\pi)T_2(S,c)\right)}{(r+\pi)^2}\right]\frac{(2-\alpha)S\left[\log(S/s(c))-1\right]+\alpha Q}{\log(S/s(c))^2}. \tag{B3}$$

The derivative $\varphi_c(S,c)$ is given by

$$\begin{aligned}\varphi_c(S,c) = &-\frac{e^{-rT_2(S,c)}}{b} \\ &-\left[\frac{1-e^{-(r+\pi)T_2(S,c)}\left(1+(r+\pi)T_2(S,c)\right)}{(r+\pi)^2}\right]\left[\frac{(2-\alpha)S-\alpha Q}{b\log(S/s(c))^2}\right]\frac{\pi r}{(2-\alpha)s(c)}.\end{aligned} \tag{B4}$$

It is straightforward to verify that $\varphi_S(S,c)$ is strictly positive and $\varphi_c(S,c)$ is strictly negative for all $S \in [\alpha Q/(2-\alpha), Q]$.

Let $S^*(c)$ denote the solution to the equation $\varphi(S,c) = 0$ with respect to S. For $c = 0$, $S^*(c)$ is equal to $\alpha Q/(2-\alpha)$. Since $\varphi_S(S,c)$ is strictly positive and $\varphi_c(S,c)$ is strictly negative for all $S \in [\alpha Q/(2-\alpha), Q]$, it follows that $S^*(c)$ is strictly greater than $\alpha Q/(2-\alpha)$ and strictly increasing for all $c > 0$. Moreover, there exists a $\overline{c} > 0$ such that $S^*(\overline{c}) = Q$. It then follows that, for all $c \in (0,\overline{c})$, $S^*(c) \in (\alpha Q/(2-\alpha), Q)$ and a (Q,S,s) equilibrium exists. In contrast, for $c \geq \overline{c}$, $S^*(c) \geq Q$ and a (Q,S,s) equilibrium does not exist.

Let $(s^*(c), S^*(c), T_1^*(c), T_2^*(c))$ denote the cutoff prices and travelling times in the (Q,S,s) equilibrium associated with the menu cost $c \in (0,\overline{c})$. Above we proved that $S^*(c)$ is strictly increasing in c. Since $s^*(c) \equiv (\alpha Q - rc/b)/(2-\alpha)$, $s^*(c)$ is strictly decreasing and $Q - s^*(c)$ is strictly increasing in c. Since $T_1^*(c) \equiv \log(Q/S^*(c))/\pi$ and $T_2^*(c) \equiv \log(S^*(c)/s^*(c))/\pi$, $T_1^*(c)$ is strictly decreasing and $T_2^*(c)$ is strictly increasing in c. Moreover, since $T_1^*(c) + T_2^*(c)$ is strictly increasing, $T_1^*(c)/(T_1^*(c)+T_2^*(c))$ is strictly decreasing in c.

(ii) Using the fact that $S = s\exp(\pi T_2)$ and $s = (\alpha Q - rc/b)/(2-\alpha)$, we can write the equation $\varphi(S,\pi) = 0$ as

$$\begin{aligned}\hat{\varphi}(T_2,\pi) \quad &= \left[\frac{1-e^{-(r+\pi)T_2}(1+(r+\pi)T_2)}{(r+\pi)^2 T_2}\right]\left[(\alpha Q - rc/b)e^{\pi T_2} - \alpha Q\right] \\ &+ \left[\frac{1-e^{-(r+\pi)T_2}}{r+\pi}\right]\alpha Q + e^{-rT_2}\left(\frac{\alpha Q}{r} - \frac{c}{b}\right) - \frac{aQ}{r} = 0.\end{aligned} \tag{B5}$$

After some algebraic transformations, (B5) becomes

$$b\alpha Q M(T_2,\pi) = rcN(T_2,\pi), \tag{B6}$$

where the function $M(T_2,\pi)$ is defined as

$$\begin{aligned} M(T_2,\pi) = \quad & -r - \pi(r+\pi)T_2 + re^{-(r+\pi)T_2} \\ & + (\pi(r+\pi)T_2 - r)\, e^{-rT_2} + re^{\pi T_2}, \end{aligned} \tag{B7}$$

and the function $N(T_2,\pi)$ is defined as

$$N(T_2,\pi) = re^{\pi T_2} + (\pi(r+\pi)T_2 - r)\, e^{-T_2}. \tag{B8}$$

Let $M_{T_2}(T_2,\pi)$ and $N_{T_2}(T_2,\pi)$ denote the partial derivatives of $M(T_2,\pi)$ and $N(T_2,\pi)$ with respect to T_2. Similarly, let $M_\pi(T_2,\pi)$ and $N_\pi(T_2,\pi)$ denote the partial derivatives of $M(T_2,\pi)$ and $N(T_2,\pi)$ with respect to π.

Now, let $T_2^*(\pi)$ denote the solution to (B5) with respect to T_2. The derivative of $T_2^*(\pi)$ with respect to the inflation rate π is given by

$$\begin{aligned} T_2^{*\prime}(\pi) &= \frac{M(T_2^*,\pi)N_\pi(T_2^*,\pi) - M_\pi(T_2^*,\pi)N(T_2^*,\pi)}{N(T_2^*,\pi)M_{T_2}(T_2^*,\pi) - N_{T_2}(T_2^*,\pi)M(T_2^*,\pi)} \\ &= \frac{-T_2^*\left\{2 - (r+\pi)T_2^* - 4e^{-(r+\pi)T_2^*} + \left[2 + (r+\pi)T_2^*\right] e^{-2(r+\pi)T_2^*}\right\}}{\pi\left\{1 - (r+\pi)T_2^* + \left[(r+\pi)^2 T_2^{*2} - 2\right] e^{-(r+\pi)T_2^*} + \left[1 + (r+\pi)T_2^*\right] e^{-2(r+\pi)T_2^*}\right\}}. \end{aligned} \tag{B9}$$

It is easy to verify that the above expression is strictly negative.

Next, let $S^*(\pi)$ denote the solution to the equation $\varphi(S) = 0$ with respect to S, which is given by

$$S^*(\pi) = \frac{\alpha Q - rc/b}{2-\alpha} e^{\pi T_2^*(\pi)}. \tag{B10}$$

The derivative of $S^*(\pi)$ with respect to π has the same sign as the derivative of $\pi T_2^*(\pi)$ with respect to π, which is given by

$$\begin{aligned} & T_2^*(\pi) + \pi T_2^{*\prime}(\pi) \\ &= \frac{T_2^*\left\{e^{-(r+\pi)T_2^*}\left[2 + (r+\pi)^2 T_2^{*2}\right] - e^{-2(r+\pi)T_2^*} - 1\right\}}{1 - (r+\pi)T_2^* + \left[(r+\pi)^2 T_2^{*2} - 2\right] e^{-(r+\pi)T_2^*} + \left[1 + (r+\pi)T_2^*\right] e^{-2(r+\pi)T_2^*}} \end{aligned} \tag{B11}$$

It is easy to verify that the above expression is strictly positive and, hence, $S^*(\pi)$ is strictly increasing in π. Moreover, $S^*(\pi)$ has the following properties: (i) $S^*(\pi) = \alpha Q/(2-\alpha)$

for $\pi \to 0$; (ii) $S^*(\pi) > \alpha Q/(2-\alpha)$ for all $\pi > 0$; (iii) $S^*(\pi) > Q$ for $\gamma \to \infty$. Since $S^*(\pi)$ is a continuous and strictly increasing function of π, the above properties imply that there exists a $\overline{\pi} > 0$ such that, for all $\pi \in (0, \overline{\pi})$, $S^*(\pi) \in (\alpha Q/(2-\alpha), Q)$ and, hence, a (Q, S, s) equilibrium exists. In contrast, for $\pi \geq \overline{\pi}$, $S^*(\pi) \geq Q$ and a (Q, S, s) equilibrium does not exist.

Let $(s^*(\pi), S^*(\pi), T_1^*(\pi), T_2^*(\pi))$ denote the cutoff prices and travelling times in the (Q, S, s) equilibrium associated with the inflation rate $\pi \in (0, \overline{\pi})$. We have already established that $S^*(\pi)$ is strictly increasing and $T_2^*(\pi)$ is strictly decreasing in π. Since $s^*(\pi) \equiv (\alpha Q - rc/b)/(2-\alpha)$, $s^*(\pi)$ is independent of π and so is $Q - s^*(\pi)$. Since $T_1^*(\pi) \equiv \log(Q/S^*(c))/\pi$, it follows that $T_1^*(\pi)$ is strictly decreasing in π. Moreover, the ratio $T_1^*(\pi)/(T_1^*(\pi) + T_2^*(\pi))$ is strictly decreasing in π. ■

C Proof of Theorem 3

Suppose that $V^* \in (c, b\alpha Q/r]$ is a solution to the equation

$$\left[\frac{1 - e^{-(r+\pi)T(V^*)}\left(1 + (r+\pi)T(V^*)\right)}{(r+\pi)^2}\right]\frac{2b(1-\alpha)Q}{T(V^*)} + \left[\frac{1 - e^{-(r+\pi)T(V^*)}}{r+\pi}\right] b\alpha Q + e^{-rT(V^*)}\left(V^* - c\right) - V^* = 0, \tag{C1}$$

where

$$T(V^*) \equiv \frac{1}{\pi}\log\left(\frac{b(2-\alpha)Q}{r(V^*-c)}\right).$$

Let S^* be defined as Q. Let s^* be defined as in equation (27). Let F^* be defined as in equation (26) for $s = s^*$ and $S = S^*$. Moreover, let T^* be defined as $\log(S^*/s^*)/\pi$ and let $R(p)$ be defined as in (22) for $F = F^*$. In order to establish that the tuple (F^*, s^*, S^*, V^*) constitutes an (S, s) equilibrium, we need to verify that it jointly satisfies the optimality conditions (23)-(25) and the stationarity condition (26). In addition, we need to verify that $s^* \in (0, Q)$ and that F^* is a CDF with support $[s^*, S^*]$.

The tuple (F^*, s^*, S^*, V^*) satisfies the stationarity condition (26) by definition. Moreover, notice that F^* is a CDF with support $[s^*, S^*]$ because $F^*(s^*) = 0$, $F^*(S^*) = 1$ and $F^{*\prime}(p) > 0$ for all $p \in [s^*, S^*]$.

The optimality condition (23) is satisfied because $R(s^*) = b(2-\alpha)s^*$ and $s^* = r(V^* - c)/b(2-\alpha)$ imply $R(s^*) = r(V^* - c)$. Moreover, notice that $s^* \in (0, Q)$ because $V^* > c$

implies $s^* > 0$ and $V^* \leq b\alpha Q/r$ implies $s^* < Q$. The optimality condition (24) is satisfied because equation (C1) implies

$$\begin{aligned} & \int_0^{T^*} e^{-rt} R(Qe^{-\pi t}) dx + e^{-rT^*}(V^* - c) - V^* = 0, \\ \iff \quad & V(0) - V^* = 0. \end{aligned} \tag{C2}$$

Now, we need to verify that the tuple (F^*, s^*, S^*, V^*) satisfies the stationarity condition (25). That is, we need to verify that the seller's value $V(t)$ is non-greater than V^* for all $t \in [0, T^*]$. To this aim, notice that $V(t)$ satisfies the differential equation

$$\begin{aligned} & rV(t) = \hat{R}(t) + V'(t), \\ & \hat{R}(t) \equiv R(Qe^{-\pi t}). \end{aligned} \tag{C3}$$

The function $\hat{R}(t)$ is given by

$$\hat{R}(t) = e^{-\pi t} bQ \left\{ (2 - \alpha) - 2(1 - \alpha) \left[1 - \frac{\pi t}{\log(Q/s^*)} \right] \right\}. \tag{C4}$$

The derivative of $\hat{R}(t)$ with respect to t is given by

$$\hat{R}'(t) = \pi e^{-\pi t} bQ \left\{ 2\,(1 - \alpha) \left[1 + \frac{1 - \pi t}{\log(Q/s^*)} \right] - (2 - \alpha) \right\}. \tag{C5}$$

The derivative $\hat{R}'(t)$ has the same sign as the term in curly brackets in (C5). It is straightforward to verify that this term is strictly increasing in t. Hence, $\hat{R}(t)$ is either strictly decreasing in t over the entire interval $[0, T^*]$, or it is first strictly increasing and then strictly decreasing in t. Notice that $\hat{R}(t)$ cannot be increasing for all $t \in [0, T^*]$ because $\hat{R}(0) = b\alpha Q \geq rV^*$ and $\hat{R}(T^*) = r\,(V^* - c) < rV^*$.

Consider the phase diagram in Figure 8, which describes the differential equation (C3). The black line passing through the origin denotes the locus of points $(\hat{R}, V)$ such that $V = \hat{R}/r$ and, hence, $V' = 0$. Any point below the black line is such that $V < \hat{R}/r$ and, hence, $V' < 0$. Any point above the black line is such that $V > \hat{R}/r$ and, hence, $V' > 0$. From (24), it follows that $\hat{R}(0) = b\alpha Q$ and $V(0) = V^* \leq b\alpha Q/r$. Hence, the point $(\hat{R}(0), V(0))$ lies either on or below the black line and $V'(0) \leq 0$. From (23), it follows that $\hat{R}(T^*) = r(V^* - c)$ and $V(T^*) = V^* - c$. Therefore, the point $(\hat{R}(T^*), V(T^*))$ lies on the black line and $V'(T^*) = 0$.

Now, we want to find out the trajectory that the pair $(\hat{R}(t), V(t))$ follows as it travels

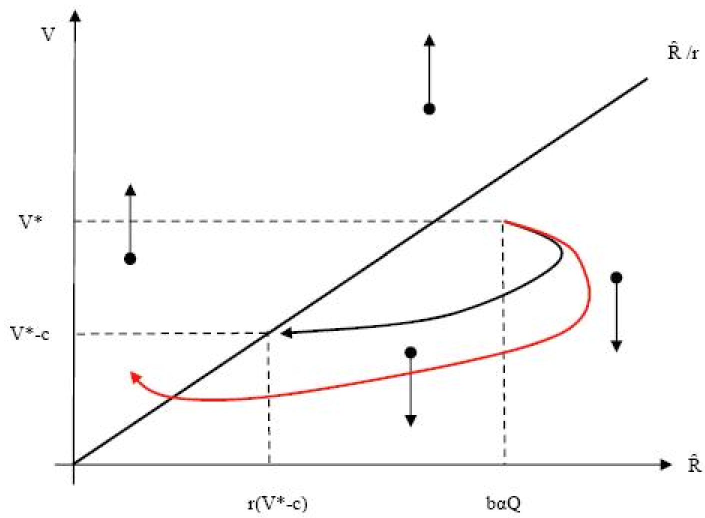

Figure 8: Joint Dynamics of $V(t)$ and $\hat{R}(t)$

from the initial point $(b\alpha Q, V^*)$ to the endpoint $(r(V^*-c), V^*-c)$. First, consider the case $V^* < b\alpha Q/r$. In this case, the initial point $(b\alpha Q, V^*)$ lies in the region where $V'(t) < 0$. For as long as $\hat{R}(t)$ increases, $(\hat{R}(t), V(t))$ moves to the south-east of the initial point $(b\alpha Q, V^*)$. When $\hat{R}(t)$ begins to decrease, $(\hat{R}(t), V(t))$ changes direction and moves towards the south-west and, eventually, it crosses the black line. Suppose that $(\hat{R}(t), V(t))$ crosses the black line at a time $\hat{T} < T^*$. Then, after time $\hat{T}$, $(\hat{R}(t), V(t))$ moves to the north-west and, since $\hat{R}(t)$ is decreasing and $V(t)$ is increasing, $(\hat{R}(t), V(t))$ does not reach the black line again. This contradicts the fact that $(\hat{R}(t), V(t))$ reaches the black line at the point $(r(V^*-c), V^*-c)$ at time T^*. Therefore, $(\hat{R}(t), V(t))$ must first cross the black line at time T^* and, hence, $V'(t) < 0$ for all $t \in [0, T^*]$.

Second, consider the case $V^* = b\alpha Q/r$. In this case, it is easy to verify that $(\hat{R}(t), V(t))$ must move first to the south-east, then to the south-west and reach the black line at time T^*. Also in this case, $V'(t) \leq 0$ for all $t \in [0, T^*]$. This completes the proof that the tuple (F^*, s^*, S^*, V^*) is an (S, s) equilibrium. The proof that there is no (S, s) equilibrium if equation (C1) does not admit a solution for $V^* \in (c, b\alpha Q/r]$ is straightforward and it is omitted for the sake of brevity. ■