Labor Market Friction, Firm Heterogeneity, and Aggregate Employment and Productivity*

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September 22, 2010
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Abstract

The paper is based on a synthesis of a "product variety" version of the firm life cycle model developed by Klette and Kortum (2004) and an equilibrium search model of the labor market with job to job flows introduced by Mortensen (2003). In the construction, a continuum of intermediate product and service varieties are produced with labor that serve as inputs in the production of a final good. Intermediate goods producers generally differ with respect to their productivity. New firms enter and continuing firms grow by developing new product varieties. The time required to match workers and jobs in the model depends on the total number of vacancies and possibly on the fraction of employed to unemployed worker. Workers receive job offers both while employed and unemployed. Wages are set through bargaining over marginal match surplus where the worker's bargaining position may improve with the arrival of outside job opportunities as in Dey and Flinn (2005) and Cahuc et al. (2006). A job separation occurs if either a worker quits or a job is destroyed. We show that a general equilibrium solution to the model exists and that the equilibrium is broadly consistent with observed dispersion in firm productivity, wages, and the relationship between them as well as patterns of worker flows. The model implies that frictions, both in the labor market and in the firm growth process, can be important determinants of aggregate productivity as well as aggregate employment.

*Financial supported of this research includes grants from the U.S. National Science Foundation and the Danish National Research Foundation. The research assistance of Jesper Bagger is also gratefully acknowledged.

1 Introduction

Firm productivity differentials are large and persistent. Average wages paid by firms are positively correlated with firm productivity and more productive firms are larger and are more likely to export. Empirical evidence supports the view that workers move from lower to higher paying jobs. These differences imply that the reallocation process may be an important determinate of aggregate productivity as well as employment. The purpose of this paper is to develop a tractable equilibrium model that explains these and other stylized facts relating worker flows, wages, and productivity across firms. The equilibrium solution to the model also provides a framework for studying the determinants of the distribution of productivity across firms as well as the level of aggregate employment and productivity when frictions in the labor market are present.[1]

The paper is based on a synthesis of a "product variety" version of the firm life cycle model developed by Klette and Kortum (2004), the Melitz (2003) model of heterogeneous firms, and an equilibrium search model of the labor market in the spirit of that introduced in Mortensen (2003). Households value future streams of consumption and leisure. A continuum of intermediate product and service varieties are produced with labor input. These serve as inputs in the production of a final good which can be used either for consumption or investment.

Both potential and continuing firms invest in costly R&D. Potential firms that innovate enter the economy as operating firms. Continuing firms grow by creating and developing new product lines. The creator of a new variety is the sole supplier of the variety which through monopoly rents allows the needed surplus to motivate the investment required to innovate. The firm's derived demand for labor is limited by the demand for the firm's portfolio of products. Existing products are destroyed at an exogenous rate and a firm dies when all of its products lines are destroyed.

Time is required to match workers and jobs in the model. Workers search for job opportunities both while employed and unemployed. Wages are set as the outcome of a strategic bargaining problem over marginal match surplus where the worker's bargaining position may improve with the arrival of outside job opportunities similar to Dey and Flinn (2005) and Cahuc et al. (2006).

[1] In this respect, the paper is closely related to Lagos (2006).

A job separation occurs if either a worker quits or a job is destroyed. Firms with vacancies also invest in recruiting workers. The rates at which workers and vacancies meet are determined by a matching function that depends on aggregate worker search and recruiting effort.

In general steady state equilibrium, consumption, the total number of product lines supplied by each firm type, the number of worker employed by each firm type, and the number of employed workers are all stationary. The steady state equilibrium distributions of both labor productivity and wages across firms and the employment rate reflect these steady state conditions as well as optimality requirements. A steady state equilibrium is defined as a consumption flow, a firm entry rate, a product creation rate and a measure of products supplied for each firm type, a firm recruiting strategy, and a worker search strategy that satisfy optimality and the steady state conditions. The existence of at least one steady state equilibrium solution is established. The equilibrium solution illustrates the importance of both labor market frictions and the cost of firm growth as determinants of aggregate productivity and employment.

2 Danish Wage and Productivity Dispersion

Both productivity and wage dispersion are large and persistent in the micro data. (Early work documenting productivity dispersion is reviewed in Bartelsman and Doms (2000). See Davis and Haltiwanger (1991) on wage dispersion.) Foster et al. (2001) present reduced form evidence that workers are reallocated from less to more productive firms as well. Finally, Lentz and Mortensen (2008) estimate a structural model of firm dynamics, closely related to that studied here, using Danish longitudinal firm data. It implies that about half of productivity growth can be attributed to reallocation within firm birth cohort as reflected in the higher relative growth rates of more productive firms.

Nagypál (2004) finds that over half of all U.S. prime age full time workers who separate in a month are reemployed with another employer in the next month and that 70% of those who don't leave the labor force experience such a job-to-job movement. That workers who quit to move to another employer typically receive an increase in earnings has long been know. (Bartel and Borjas (1981) and Mincer (1986) represent early work on the finding.) Recently structural models of job to job movements have added more details. For example, Jolivet et al. (2006) show that an estimated off-the-shelf search on-the-job model does a good job of explaining the observed extent to which the distribution of wages offered to new employees is stochastically dominated by the distribution of wages earned in 9 out of 11 OECD countries. Christensen et al. (2005) estimate a structural model that allows for an endogenous choice of search intensity using Danish matched worker-employer data. All of these studies supports the basic view that wage dispersion exits in the sense that different firms pay similar workers differently and that workers respond to these differences by moving from lower to higher paying employers.

This paper is based on evidence that wage dispersion is induced by productive firm heterogeneity through "rent sharing." Given search friction, match rents are larger at more productive firms. Hence, if the wage is determined by some form of rent sharing, then we should see a positive cross section relationship between the average wage paid and firm productivity.

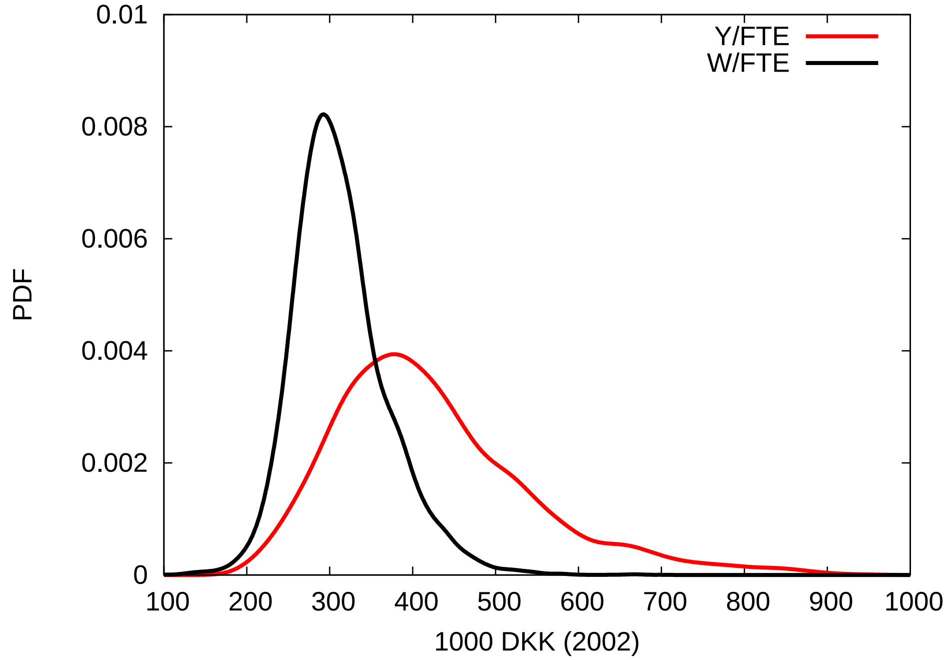
In this section, we present Danish evidence for wage and productivity dispersion as well as a positive association between them drawn from an longitudinal files on the value added (Y), full time equivalent (FTE) employment (N) and the wage bill (W) paid by privately owned Danish firms. These firm accounting data are collected annually in a survey conducted by Statistics Denmark and are supplemented from tax records. The survey is a rolling panel and the sampling of firms is based on firm size and revenue.[2]

In Figure 1, cross firm probability density functions for wage cost per worker, as reflected in the ratio of the wage bill to FTE employment (W/N), and for average productivity per worker, as measured by the ratio of valued added to employment (Y/N), are plotted for the year 2002.[3] Note that dispersion in both the average wage paid and labor productivity are large. Specifically, four fold productivity differentials lie within the 5-95 percentile support of the data as well as over 100% differences in the average wages paid. The relative positions of the two distributions suggest that average labor productivity exceeds the average wage paid, as one should expect. In Figure 2, a non-parametric regression of firm labor productivity on average firm wage cost

[2]Only firms with 5 or more employees and revenue above 500M DKK are included. All firms with labor force at or above 50 are included in the survey. The sampling proportions for 20-49 workers is 50%, for 10-19 is 20%, and for 5-9 is 10%.

[3]At the moment, we have data for 1999-2002 inclusive. All the properties characteristics of the data pointed out here are virtually identical for each of the other years as well.

Figure 1: Danish Firm Wage and Productivity Distributions



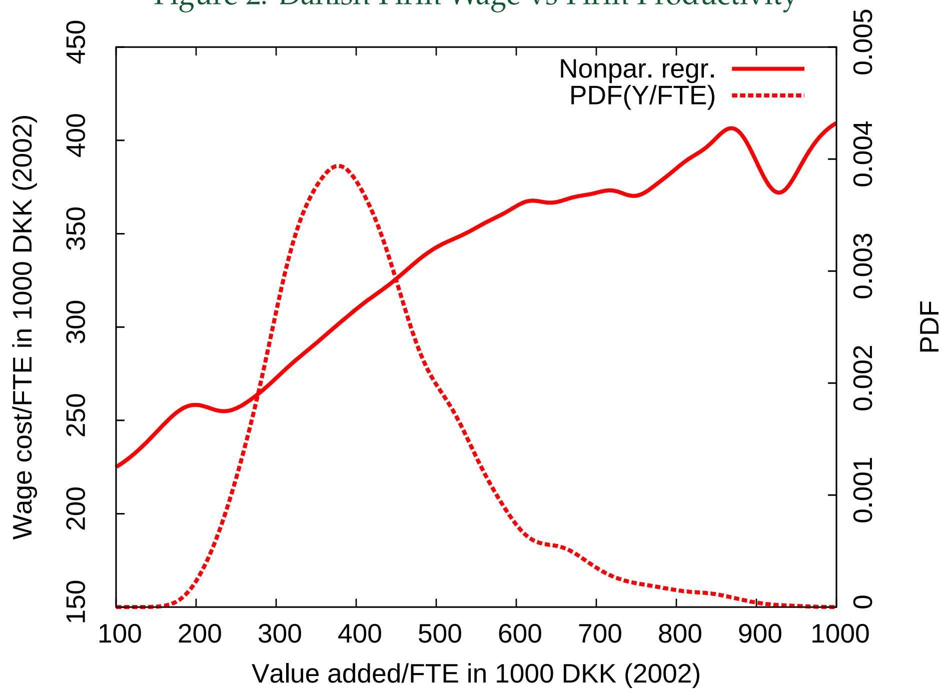
is plotted. This evidence suggests that the relationship is close to linear. The strong positive relationship between firm wage and productivity supports "rent-sharing" theories of the wage determination such as the one embodied in this paper.

3 The Model

3.1 Household Preferences

There is a continuum L of identical households each composed of a unit measure of individuals who are either employed or not.. There is a single final good used for either consumption or investment and produced by a competive market sector using a continuum of intermediate goods. Each intermediate good is supplied by a single firm. Employed workers produce intermediate goods and those not employed produce a substitute for the final good at rate b. Household utility is the expected present value of consumption where the fixed discount rate r is the equilibrium rate of interest.

Figure 2: Danish Firm Wage vs Firm Productivity



3.2 Intermediate Product Demand

A single final good is supplied by a competitive sector and the aggregate quantity produced is a function of a continuum of intermediate inputs each supplied by a monopolist. Specifically, final output is determined by the following constant returns to scale Dixit-Stigliz production function,

$$Y = \frac{A}{K}\left[\int_0^K \alpha(j)^{\frac{1}{\sigma}} x(j)^{\frac{\sigma-1}{\sigma}} dj\right]^{\frac{\sigma}{\sigma-1}}, \tag{1}$$

where $x(j)$ is the quantity of intermediate product j available, K is the measure of intermediate goods, and σ is the elasticity of substitution between inputs. The demand for each input given final output is

$$x(j) = \alpha(j) Y \left(\frac{A}{K}\right)^{\sigma-1} p(j)^{-\sigma},$$

where $p(j)$ is the price of input j expressed in units of the final output. Each intermediate good is produced with labor and supplied by a single firm. Specifically, $x(j) = q(j)n(j)$ where $q(j)$ is the productivity of each worker employed to produce product j and $n(j)$ is employment. The demand and productivity pair parameters (α, q) define the supplier of product j in the model.

Given this pair of parameters, the supplying firm's revenue is

$$px = R_n(\alpha, q) = (\alpha Y)^{\frac{1}{\sigma}} \left(\frac{Aqn}{K}\right)^{\frac{\sigma-1}{\sigma}}. \tag{2}$$

Equation (2) makes clear that in this model with monopolistic pricing, demand and productivity shocks work in the same way by shifting the revenue function. It is useful to rewrite the revenue as a function of a single random variable, $\mu = \alpha^{\frac{1}{\sigma}} (Aq)^{\frac{\sigma-1}{\sigma}}$,

$$R_n(\mu) = \mu Y^{\frac{1}{\sigma}} \left(\frac{n}{K}\right)^{\frac{\sigma-1}{\sigma}}. \tag{3}$$

Absent direct observation of quantity produced, the model cannot distinguish value productivity from quantity productivity when σ is finite. That is, we generally cannot distinguish if a firm looks highly productive because it has a high demand realization α or a high productivity realization q. In the perfect substitutes case ($\sigma = \infty$), however, there are no demand constraints and we have $\mu = Aq$, that is the revenue function shifts across product lines only as a result of q dispersion. We will in the main part of the exposition focus on the perfect substitutes case, where productive heterogeneity is attributed to differences in q realizations. In the appendix, we treat the general case. Here, we refer to heterogeneity across product lines by μ emphasizing the fundamental lack of identification between value productivity and physical productivity in the model.

3.3 Bargaining and Search

When an individual worker meets an employer, the two bargain over their match surplus. Given that the value of the worker's current employment status is the threat point in the bilateral bargaining game, the generalized Nash solution allocates the share $\beta \in (0, 1)$ to the worker and the residual to the employer. A match is formed if its value exceeds the worker's outside option.

For example, if the worker is unemployed, then match surplus to be divided is $S = X - U$ where X is the match value and U is the value of unemployment. Given the Nash solution to the bargaining problem, the worker's value of unemployment is

$$rU = b + \lambda_0 \beta \int_U^\infty (X - U) dF(X - U) = b + \lambda_0 \beta E[S], \tag{4}$$

where b is the flow value of home production, λ_0 is the rate a which an unemployed worker meets vacancies, $F(S)$ is the offer c.d.f., the distribution of match surplus over vacancies, and $E[S] = \int_0^\infty X dF(X)$.

In the model, employed workers have the opportunity to transit voluntarily to alternative employment at frequency λ_1 and an exogenous transition occurs with frequency λ_2. Consistent with this interpretation, a worker actually moves from one job to the other in the first case only if the transition is value improving but not necessarily in the second. As in Jolivet et al. (2006), the principal purpose of allowing for "involuntary" job-to-job transitions is to account for the fact that the significant fraction of workers in the data experience wage loss when making job-to-job transitions. More generally, involuntary transitions also account for job-to-job movements made for non-economic reasons.

We adopt a bargaining model that resembles that of Dey and Flinn (2005) and Cahuc et al. (2006) where firms match outside offers, and the employment contract specifies that the worker receives the share $\beta \in (0,1)$ of the surplus value of the match. It can be viewed as the outcome of a bilateral bargaining problem between worker and employer when they meet and the worker's outside option is full surplus extraction from the outside employer. In this case, a worker voluntarily moves from job-to-job if and only if the total value of the new match exceeds that of the old. Hence, the worker's value of employment at the beginning of a new job is $W = W_0 + \beta(X - W_0)$ where X is the value of the marginal match with the new employer when employed and W_0 is equal to the value of the previous match if the worker was hired voluntary and the value of unemployment if either the worker was previously unemployed or the move was involuntary.

3.4 Vacancy Creation and Marginal Match Surplus

Firms are composed of product lines, each of which faces a destruction risk equal to δ_1. Each product line has its own labor force; direct reallocation across lines within the firm has no cost advantage relative to worker reallocation through the labor market. Hence, the employing unit in the model is the product line, a coalition composed of an employer and its labor force.

The size of the labor force employed in a product line, denoted as n, is a discrete variable defined on the non-negative integers. As the model is formulated in continuous time and hires and separations are individual decisions, only a one unit change in n can occur in any instant if the line continues into the future. Only in the case of product destruction, an event which occurs infrequently at rate δ_1, is this rule violated. In this case, all workers are laid off. Hence, employment in a product line is a birth-death stochastic process with transition rate from n to $n+1$ equal its hire frequency, from $n-1$ to n equal to the frequency with which workers quit, and from any n to $n=0$ with frequency δ_1.

Define V_n as the expected present value of the future income of the members of a product line composed of the employer and the n employees. Furthermore, define $S_n = V_n - nU$ as the value of the coalition net of the worker's combined value of unemployment. The marginal coalition surplus of a worker is $\Delta S_n = S_n - S_{n-1} = \Delta V_n - U$. If a worker receives a job offer from another firm where the marginal value of a worker X exceeds ΔV_n, the worker will quit and receive value $W = \Delta V_n + \beta(X - \Delta V_n)$ with the new firm. In this case, the total value of the remainder of the coalition moves from V_n to V_{n-1}. The asset equation for the total surplus of the coalition can be written as,

$$\begin{aligned}
rS_n(\mu) &= R_n(\mu) + \pi(\Delta V_{n+1} - U) + \delta_1(nU - V_n(\mu)) + \delta_0 n(U - \Delta V_n) \\
&\quad + \lambda_1 n \int_{\Delta V_n}^{\infty} [\Delta V_n + \beta(X - \Delta V_n) + V_{n-1} - V_n]\, dF(X - U) \\
&\quad + \lambda_2 n \int_{U}^{\infty} [U + \beta(X - U) + V_{n-1} - V_n]\, dF(X - U) - nU \\
&= R_n(\mu) - nb - (\lambda_0 - \lambda_2)\, n\beta E[S] + \pi(\Delta S_{n+1}) - \delta_1 S_n - (\delta_0 + \lambda_2)\, n\Delta S_n \\
&\quad + \lambda_1 n\beta \int_{\Delta S_n}^{\infty} (X - \Delta S_n) dF(X), \qquad (5)
\end{aligned}$$

where λ_1 is the rate at which employed workers generate outside offers, λ_2 represents the quit rate of those moving to another job for other reasons with a bargaining position equal to that of unemployment, δ_0 is the transition rate to unemployment, δ_1 denotes the product destruction rate, $F(S)$ is again the offer distribution, and finally $\pi(\Delta V_{n+1} - U)$ is the value of the product line's recruiting operation.

The firm's recruiting strategy determines the number of vacancies posted contingent on the current level of employment in a firm. An optimal strategy maximizes the value that accrues to the current coalition from recruitment. Upon meeting and hiring a worker with outside option of $W_0 \leq \Delta V_{n+1}$, the value of the coalition changes from V_n to V_{n+1}. The outside worker receives value $W_0 + \beta\left(\Delta V_{n+1} - W_0\right)$. Hence, the gain to the coalition in this case is $\Delta V_{n+1} - W_0 - \beta\left(\Delta V_{n+1} - W_0\right) = (1-\beta)\left(\Delta V_{n+1} - W_0\right)$. The value of the recruitment operation maximizes the product of the firm's share of the match surplus and the rates at which workers of different types are met summed over the types of worker as characterized by employment status. As all unemployed workers but only those employed at smaller match values accept, the number of vacancies posted by a product line with n employees is the solution to the optimization problem,

$$\pi(\Delta S_{n+1}) = \max_{v \geq 0} \left\{ (1-\beta) v \left[\left(\eta_0 + \eta_2\right) \Delta S_{n+1} + \eta_1 \int_0^{\Delta S_{n+1}} (\Delta S_{n+1} - X) dG(X) \right] - c_0(v) \right\}, \tag{6}$$

where $G(X)$ denotes the fraction of workers employed at firms with marginal match surplus X or less, and $c_0(v)$ represents the cost of posting v vacancies, an increasing convex function. The parameters η_0, η_1, and η_2 are rates per vacancy posted at which employers meet unemployed workers, employed workers who have the option of turning down an outside offer, and employed workers who move for reasons exogenous to the model.

As all offers are acceptable to an unemployed worker and one who otherwise transitions to unemployment but only those employed worker in a match with a lower value that the match will voluntarily accept an offer, the employment size contingent hire frequency is

$$h_n = h(\Delta S_{n+1}) = \left[\eta_0 + \eta_1 G(\Delta S_{n+1}) + \eta_2\right] v(\Delta S_{n+1}), \tag{7}$$

where $v(X)$ is the solution to the optimization problem defined in (6) when $X = \Delta S_{n+1}$. Note that the optimal number of vacancies does not depend directly on either productivity or market size. In short, the value of the match is a sufficient statistic.

By equation (5) one obtains the expression for the marginal coalition surplus,

$$\begin{aligned} \left(r + \delta_1 + n\left(\delta_0 + \lambda_2\right)\right) \Delta S_n\left(\mu\right) = \Delta R_n - b - \left(\lambda_0 - \lambda_2\right) \beta E\left[S\right] + \pi(\Delta S_{n+1}) - \pi(\Delta S_n) \\ + \left(\delta_0 + \lambda_2\right)\left(n-1\right) \Delta S_{n-1} + \lambda_1 n \beta \int_{\Delta S_n}^{\infty} (X - \Delta S_n) dF(X) \\ - \lambda_1 \left(n-1\right) \beta \int_{\Delta S_{n-1}}^{\infty} (X - \Delta S_{n-1}) dF(X), \end{aligned} \tag{8}$$

were $\Delta R_n = R_n - R_{n-1}$ is the revenue product of the marginal worker.

3.5 Wages

The future wage stream for any new employee must equal the value of the worker's previous employment status, represented as W_0, plus the share β of the difference between it and the new match surplus. Consequently, the value of employment given that the worker is hired when there are n employees in the firm is $W_n^0 - U = W_0 - U + \beta(\Delta S_n - (W_0 - U))$, where $W_0 = U$ if the worker is hired while unemployed or in the process of an involuntary job-to-job move and the value of employment in the worker's previous match otherwise. Although there are a continuum of payment streams that are consistent with W_n^0, it is unique under the additional restriction that the flow wage paid must be the same for all current employees with the same market experience as embodied in the value of their employment status when hired. Both legal requirements and fairness considerations provide reasons for this equal pay condition.

By the fact that separation is efficient, $W_0 - U$ is necessarily less than the surplus of the match at its inception, however as time passes in the match, the worker's share of the surplus of the marginal worker may drop and possibly below $W_0 - U$. However, efficient separation requires the lifetime wage is revised downward if the marginal surplus is positive but is less than $W_0 - U$. Hence, if $W_0 - U > \Delta S_n \geq 0$, we assume that the employment contract awards the worker with the full surplus of the match, ΔS_n. Hence, conditional on the worker's labor market experience and the number of employees in the product line, the worker's continuation value is given by,

$$\hat{W}_n = W_n - U = \min\left[\Delta S_n, W_0 - U + \beta\left(\Delta S_n - (W_0 - U)\right)\right]. \tag{9}$$

Given this specification, the worker only quits voluntarily if offered a job that offers surplus in excess of ΔS_n.

4 Perfect Substitutes ($\sigma = \infty$)

Solving the model for the value of the marginal match and the associated optimal recruiting strategy is complicated in the general case in which intermediate good are imperfect substitutes, but doing so in the limiting case of perfect substitutes is quite straight forward. Since the conceptual issues are the same but the structure is much simpler, we study the special case in the main exposition. Later, we reintoduce the general case for the purpose of empirical estimation.

4.1 The Surplus Value of a Job-Worker Match

As noted earlier, the critical property of the special case is the proportionality of product line revenue to employment. That is $R_n = Aqn/K$ in the limit from equation (2). Therefore, $\Delta R = Aq/K$ is a constant which together with equation (8) implies that the surplus value of the marginal worker is also constant. Indeed, its value, denoted as $\Delta S(q)$ where the dependence on productivity q is emphasized, is the unique solution to

$$(r+\delta_0+\delta_1)\Delta S(q) = Aq/K - b - \lambda_0\beta\int_0^{\overline{X}} XdF(X) + \\ \lambda_1\beta\int_{\Delta S(q)}^{\overline{X}}(X-\Delta S(q))dF(X) + \lambda_2\beta\int_0^{\overline{X}}(X-\Delta S(q))dF(X)$$

An integration by part yields

$$(r+\delta_0+\delta_1+\lambda_2)\Delta S(q) = Aq/K - b \\ +\lambda_1\beta\int_{\Delta S(q)}^{\overline{X}}[1-F(X)]dX - (\lambda_0-\lambda_2)\beta\int_0^{\overline{X}}[1-F(X)]dX. \quad (10)$$

It is equal to the present value of marginal product, Aq/K, minus the worker's flow value if not matched, plus the expected gain in the coalitions joint value attributable to on-the-job search. It is straight forward to verify that $\Delta S(q)$ is differentiable and increasing in q. Indeed,

$$\Delta S'(q) = \frac{A/K}{r+d(q)} > 0 \quad (11)$$

if $F(X)$ is continuous where

$$d(q) = \delta_0+\delta_1+\lambda_1[1-F(\Delta S(q))]+\lambda_2. \quad (12)$$

is the separation rate from a product line of productivity q. Note that $\Delta S(q)$ is unique solution to the ODE (11) consistent with the boundary condition $\Delta S(\underline{q}) = 0$ where $\underline{q}$, the productivity of the marginal product line, is the unique solution to

$$A\underline{q}/K = b + (\lambda_0-\lambda_1-\lambda_2)\beta\int_0^{\overline{X}}[1-F(X)]dX \quad (13)$$

Equation (6) implies that firms optimal choice of vacancies, denoted $v(q)$, is the solution to the problem on the right side of

$$\pi(q) = \max_{v\geq 0}\left\{(1-\beta)\left[(\eta_0+\eta_2)\Delta S(q) + \eta_1\int_0^{\Delta S(q)} G(X)dX\right]v - c(v)\right\} \quad (14)$$

where $\pi(q)$ represents the net return to recruiting activity per worker. As the value of a match $\Delta S(q)$ is increasing in productivity and the FONC for an optimal choice of vacancies, more productive firms post more vacancies,

$$c_0'(v(q)) = (1-\beta)\left[(\eta_0+\eta_2)\,\Delta S(q) + \eta_1 \int_0^{\Delta S(q)} G(X)dX\right]$$

more productive product lines engage in more recruiting effort. Indeed, the hire frequency

$$h(q) = [\eta_0 u + (\eta_1 G(\Delta S(q)) + \eta_2)(1-u)]\, v(q) \tag{15}$$

is increasing in worker productivity while the separation rate $d(q)$, defined in (12) is decreasing in q. As the employment process is mean reverting and tends toward the value of n that equates the flow of hires $h(q)$ to the quit flow $d(q)n$, more productive product lines will grow to become larger on average.

The expected present value of future income for any employee of a product line of productivity q is

$$W(q) = W_0 + \beta(\Delta S(q) + U - W_0) \tag{16}$$

where W_0 is reflects the worker's employment history. Namely, W_0 is value of an outside offer that resulted in a raise in lifetime wage relative to that earned at the time of employment. In the case of no such raise, then W_0 is value of employment in the worker's previous job if the worker transferred volutarily to the firm or the value of unemployment if either hired from unemployed or an involutary transfer from another firm. As W is also the solution to the following forward looking Bellman equation

$$\begin{aligned}
rW &= w + \lambda_1 \int_{\Delta S}^{\overline{X}} [U + \Delta S + \beta(X - \Delta S) - W]\, dF(X) \\
&\quad + \lambda_2 \int_0^{\overline{X}} [U + \beta(X-U) - W]\, dF(X) - (\delta_0+\delta_1)(U-W) \\
&\Updownarrow \\
(r+d(q))(W-U) &= w - b - (\lambda_1 - \lambda_2)\beta \int_0^{\overline{X}} [1-F(X)]dX \\
&\quad + \lambda_1 \int_{\Delta S}^{\overline{X}} [\Delta S + \beta(X - \Delta S)]\, dF(X),
\end{aligned}$$

the wage flow satisfies,

$$w(q,A) = \Delta R - (1-\beta)\left\{[r+d(q)](\Delta S(q) - A) + \lambda_1 \int_A^{\Delta S(q)} (S-A)\, dF(S)\right\}, \tag{17}$$

where $W_0 - U = A$.

4.2 Product Innovation and Entry

Firm size as reflected in the number of products supplied is generated by a research and development process as introduced in Klette and Kortum (2004). At any point in time, a firm has k product lines can grow only by creating new product varieties. Investment in R&D is required to create new products. Specifically, the firm's R&D investment flow generates new product arrivals at frequency γk where γ represents the firm's innovation rate per product line. The total R&D investment cost expressed in terms of output is $c_1(\gamma)k$ where $c_1(\gamma)$ is assumed to be strictly increasing and convex function. The assumption that the total cost of R&D investment is linearly homogeneous in the new product arrival flow, γk, and the number of existing product, k, "captures the idea that a firm's knowledge capital facilitates innovation," in the words of Klette and Kortum (2004). The specification assumption also implies that Gibrat's law holds in the sense that innovation rates are size independent contingent on type, a property needed to match the data on firm growth. Finally, every product is subject to destruction risk with exogenous frequency δ_1. Given this specification, the number of products supplied by any firm is a stochastic birth-death process characterized by "birth rate" γ and "death rate" δ_1.

The productivity of a product line, q, is realized when a new product is created. For a particular productivity realization, the product line faces a revenue function $R(q) = Aqn/K$. Firm type heterogeneity enters the model through type dependence of the distribution of product line productivity, q. Let $\Gamma_\tau(q), \tau \in \{1, 2...., T\}$, denote the cumulative distribution of q for any firm of type τ. Finally, we assume that the magnitude of the type index reflects the productivity rank of the type in the sense of first order stochastic dominance. That is, $\tau > \tau'$ implies that product lines of firms of τ are likely to have greater employment conditional revenue than firms of type τ', i.e., $\Gamma_\tau(q) \leq \Gamma_{\tau'}(q)$ for all $q \in \left[0, \underline{q}\right]$.

Let Ψ represent the value of research per existing product line. The innovation frequency per product line is the choice variable γ. If an innovation arrives in next instant, the employer will have a product line with no workers, which generates a revenue flow equal to the net profit associated with attempting to fill the first job, which is $\pi(\Delta S(q))$. As every product line faces destruction risk δ_1, its present value at the moment of discovery is $\pi(\Delta S(q))/(r+\delta_1)$. In addition, the option to create still another, which again has value Ψ, arrives with each product created. Because the option to create a new product is lost if the existing product line is destroyed, the productivity of the line is realized only after the product is created, and there are no employees at the date of creation, the value of a new product for firm of type τ per worker solves

$$(r+\delta_1)\Psi_\tau = \max_\gamma \left\{ \gamma \left(\frac{\int_{\underline{q}}^{\overline{q}} \pi(x) d\Gamma_\tau(x)}{r+\delta_1} + \Psi_\tau \right) - c_1(\gamma) \right\}. \tag{18}$$

As $\Gamma_\tau(\underline{q})$ may be positive, this formulation accounts for the possibility that an innovation will be marketed only if it yields a positive surplus.

From equation (18), the innovation frequency satisfies the FONC

$$c_1'(\gamma_\tau) = \int_{\underline{q}}^{\overline{q}} \frac{\pi(x)}{r+\delta_1} d\Gamma_\tau(x) + \Psi_\tau \tag{19}$$

Note that the choice is independent of both the number of products currently supplied and of the number of workers employed to supply each line.

Proposition 1. *If the cost of R&D, $c_1(\gamma)$, is increasing, strictly concave and $c_1(0) = c_1'(0) = 0$, and the measure of products supplied, K, is sufficiently large, then the optimal product creation rate is positive and less than the product destruction rate, δ_1. Furthermore, more productive firms innovate more frequently (γ_τ is increasing in τ).*

Proof. If a solution to (19) exists, then the value of an additional product line is defined by

$$\Psi_\tau = \max_{\gamma \geq 0} \left\{ \frac{\gamma E_\tau S_0(\tilde{q}) - c_1(\gamma)}{r+\delta_1-\gamma} \right\}. \tag{20}$$

Hence, the first order condition for the optimal innovation rate can be written as

$$f(\gamma) = (r+\delta_1-\gamma)\left(E_\tau S_0(\tilde{q}) - c_1'(\gamma)\right) + \gamma E_\tau S_0(\tilde{q}) - c_1(\gamma) = 0 \tag{21}$$

and the second order condition requires $f'(\gamma) = -(r+\delta_1-\gamma)\, c''(\gamma) \leq 0$ at a maximal solution. As $f'(0) = (r+\delta_1)E_\tau S_0(\tilde{q}) > 0$, the first order condition has a unique solution satisfying $0 < \gamma <$

δ_0 and the sufficient second order condition is satisfied if $f'(\delta_1) = (r+\delta_1)E_\tau S_0(\tilde{q}) - rc_\gamma(\delta_1) - c_\gamma(\delta_1) < 0$. As $S_0(\tilde{\mu})$ is bounded above by the largest value of a new product line and that bound converges to zero as $K \to \infty$ by equations (2) and (5) where $\sigma > 1$, the claim follows. □

Assume that entry requires a successful innovation, that the cost of innovation activity by a potential entrant is $c_1(\gamma)$, and that firm type is unknown to any entrepreneur prior to entry. The entry rate is the product $v = \kappa\gamma_0$ where κ is a given measure of entrepreneurs and γ_0 is the frequency with which any one of them creates a new product. As the optimal innovation rate by a potential entrant maximizes the unconditional expected value of a job-worker match, the optimal choice is

$$v = \kappa\gamma_0 = \kappa \arg\max_{\gamma \geq 0} \left\{ \sum_\tau \left[\frac{\int_{\underline{q}}^{\overline{q}} \pi(x) d\Gamma_\tau(x)}{r+\delta_1} + \Psi_\tau \right] \phi_\tau \gamma - c_1(\gamma) \right\}. \quad (22)$$

where ϕ_τ is the exogenous probability of becoming a type τ firm.

5 Steady State Market Equilibrium

5.1 The Meeting Process

The aggregate rate at which workers and vacancies meet is determined by increasing concave and homogeneous of degree one matching function of aggregate vacancies and search effort. Aggregate search effort is equal to $\lambda_0 u + (\lambda_1 + \lambda_2)(1-u)$ where the parameters λ_0 and $\lambda_1 + \lambda_2$ reflect the search intensities of unemployed and employed workers respectively. Hence, the aggregate meeting rate is $m(\theta)\left(u + (a_1 + a_2)(1-u)\right)L$ where by an appropriate normalization

$$\lambda_0 = m(\theta) \text{ and } \lambda_i = a_i\lambda_0, : i \in \{1, 2\} \quad (23)$$

represent the meeting rates, u is the unemployment rate, a_1 and a_2 are the relative search intensity of voluntary and involuntary search while employed, assumed to be constants. Because unemployed workers find jobs at rate $\lambda_0 = m(\theta)$ and lose them at rate $\delta = \delta_0 + \delta_1$, the steady state unemployment rate is

$$\frac{u}{1-u} = \frac{\delta_0 + \delta_1}{\delta_0 + \delta_1 + m(\theta)} = \frac{\delta}{\delta + m(\theta)}. \quad (24)$$

By assumption the function $m(\theta)$ is increasing and concave. The rate a vacancy meets some worker is $\eta = m(\theta)/\theta$. The assumption that workers of each type are met at rates proportional to their relative search intensities implies that the vacancy meeting rates by worker type are functions of tightness:

$$\begin{aligned}
\eta_0 &= \left(\frac{u}{u+(1-u)(a_1+a_2)}\right)\frac{m(\theta)}{\theta} = \left(\frac{\delta}{\delta+(\delta+m(\theta))a}\right)\frac{m(\theta)}{\theta} \\
\eta_1 &= \left(\frac{(1-u)a_1}{u+(1-u)(a_1+a_2)}\right)\frac{m(\theta)}{\theta} = \left(\frac{(\delta+m(\theta))a_1}{\delta+(\delta+m(\theta))a}\right)\frac{m(\theta)}{\theta} = \eta \\
\eta_2 &= \left(\frac{(1-u)a_2}{u+(1-u)(a_1+a_2)}\right)\frac{m(\theta)}{\theta} = \left(\frac{(\delta+m(\theta))a_2}{\delta+(\delta+m(\theta))a}\right)\frac{m(\theta)}{\theta}
\end{aligned} \tag{25}$$

where $a = a_1 + a_2$.

5.2 Product Distribution and Average Firm Size

The measure of products supplied by the set of type τ firms evolves according to the law of motion,

$$\dot{K}_\tau = (\upsilon\phi_\tau + \gamma_\tau K_\tau)\left[1 - \Gamma_\tau(\underline{q})\right] - \delta_1 K_\tau$$

where υ is the innovation rate of new entrants, ϕ_τ is the fraction of entrant who are of type τ, γ_τ is the innovation rate of type τ firms per product line, δ_1 is the product destruction rate, and $\underline{q}$ is the reservation productivity of a new product. In other words, the net rate of change in the measure is equal to the sum of the flows of products supplied by new entrants and continuing firms respectively less the flow of product lines currently supplied by the type that are destroyed. Hence, in steady state, the measure of products supplied by type τ firms and the aggregate measure of products are

$$K_\tau = \frac{\upsilon\phi_\tau[1-\Gamma_\tau(\underline{q})]}{\delta_1 - \gamma_\tau[1-\Gamma_\tau(\underline{q})]} \tag{26}$$

and

$$K = \sum_\tau K_\tau = \sum_\tau \frac{\upsilon\phi_\tau[1-\Gamma_\tau(\underline{q})]}{\delta_1 - \gamma_\tau[1-\Gamma_\tau(\underline{q})]} \tag{27}$$

Hence, more productive firms supply more products in steady state relative to their share at entry by Proposition 1.

Each firm's number of product lines k evolves according to a birth-death process where new product lines are added at rate $k\gamma_\tau$ and product lines are destroyed at rate $k\delta_1$. When a firm is born, it enters with one product. The product line dynamic is identical to the one in Klette and Kortum (2004). Denote by $M_{\tau,k}$ the mass of type τ incumbent firms that have k product lines. It evolves according to,

$$\begin{aligned}\dot{M}_{\tau,k} &= (k-1)\,\gamma_\tau M_{\tau,k-1} + (k+1)\,\delta_1 M_{\tau,k+1} - k\left(\gamma_\tau + \delta_1\right) M_{\tau,k},\ k = 2,\dots \\ \dot{M}_{\tau,1} &= \upsilon\phi_\tau + 2\delta_1 M_{\tau,2} - \left(\gamma_\tau + \delta_1\right) M_{\tau,1}.\end{aligned}$$

Define $\sum_{k=1}^{\infty} M_{\tau,k} = M_\tau$ and denote by $m_{\tau,k} \equiv M_{\tau,k}/M_\tau$ the probability that a type τ incumbent has k product lines. As shown in Klette and Kortum (2004), in steady state $m_{\tau,k}$ is distributed according the logarithmic distribution with parameter γ_τ/δ_1,

$$m_{\tau,k} = \frac{\left(\frac{\gamma_\tau}{\delta_1}\right)^k}{\ln\left(\frac{\delta_1}{\delta_1 - \gamma_\tau}\right) k}. \tag{28}$$

By implication, the average type conditional firm size is,

$$E_\tau\left[k\right] = \frac{\gamma_\tau}{\delta_1} \frac{\frac{\delta_1}{\delta_1 - \gamma_\tau}}{\ln\left(\frac{\delta_1}{\delta_1 - \gamma_\tau}\right)}. \tag{29}$$

The steady state mass of firms M_τ is given by,

$$M_\tau = \frac{\upsilon\phi_\tau}{\gamma_\tau} \ln\left(\frac{\delta_1}{\delta_1 - \gamma_\tau}\right). \tag{30}$$

In sum, more productive firms create more products, represent a larger fraction by equations (29), and (30) and Proposition 1.

5.3 Wage Distributions and Market Tightness

Given the constructs in the previous section and the fact that the optimal number of vacancies posted depends only on the marginal value, the market steady state distribution of vacancies over match values is

$$\begin{aligned}F(q) &\equiv F\left(\Delta S(q)\right) = \frac{\sum_\tau K_\tau \int_{\hat{q}}^{q} v(x) d\Gamma_\tau(x)}{\sum_\tau K_\tau \int_{\underline{q}}^{\overline{q}} v(x) d\Gamma_\tau(x)} \\ &= \frac{\sum_{\tau=1}^{T} \frac{\upsilon\phi_\tau[1-\Gamma_\tau(\underline{q})]}{\delta_1 - \gamma_\tau[1-\Gamma_\tau(\underline{q})]} \int_{\hat{q}}^{q} v(x) d\Gamma_\tau(x)}{\sum_{\tau=1}^{T} \frac{\upsilon\phi_\tau[1-\Gamma_\tau(\underline{q})]}{\delta_1 - \gamma_\tau[1-\Gamma_\tau(\underline{q})]} \int_{\underline{q}}^{\overline{q}} v(x) d\Gamma_\tau(x)}\end{aligned} \tag{31}$$

where $\digamma(q)$ is the fraction of vacancies posted by product lines that are no more productive than q. Because the flows in and out of the set of workers employed in product line of productivity q or less are equal in steady state,

$$\lambda_0 F(\Delta S(q))u = (\delta + (\lambda_1 + \lambda_2)\left[1 - F(\Delta S(q))\right])\, G(\Delta S(q))(1-u). \tag{32}$$

where the odds of being unemployed are

$$\frac{u}{1-u} = \frac{\delta}{m(\theta)} \tag{33}$$

for (24) by $F(U) = G(U) = 0$. Hence, the steady state distribution of employment is

$$G\left(\Delta S(q)\right) = \frac{\delta \digamma(q)}{\delta + am(\theta)[1-\digamma(q)]} \tag{34}$$

by equations (24) and (23) where $a = a_1 + a_2$. The ratio of the aggregate vacancies to aggregate search effort, market tightness, is

$$\theta = \frac{\sum_\tau K_\tau \int_{\underline{q}}^{\overline{q}} v(x) d\Gamma_\tau(x)}{\left(u + (a_1+a_2)(1-u)\right) L} = \frac{(\delta + m(\theta)) \sum_{\tau=1}^{T} \frac{v\phi_\tau[1-\Gamma_\tau(\underline{q})]}{\delta_1 - \gamma_\tau[1-\Gamma_\tau(\underline{q})]} \int_{\underline{q}}^{\overline{q}} v(x) d\Gamma_\tau(x)}{(\delta + am(\theta))\, L}. \tag{35}$$

5.4 Definition and Existence

Define

$$R(q) = (1-\beta)\frac{m(\theta)}{\theta}\left[\left(\frac{\delta + m(\theta)a_2}{\delta + m(\theta)a}\right)\Delta S(q) + \left(\frac{m(\theta)a_1}{\delta + m(\theta)a}\right)\int_{\underline{q}}^{q} \frac{\delta \digamma(x)\Delta S'(x)dx}{\delta + am(\theta)[1-\digamma(x)]}\right]. \tag{36}$$

Definition 1. A steady state labor market equilibrium is an optimal vacancy posting strategy

$$v(q) = \arg\max_{v} \left\{R(q)v - c_0(v)\right\}, \tag{37}$$

an optimal creation rate for each firm type,

$$\gamma_\tau = \arg\max_{\gamma} \left\{ \frac{\left(\int_{\underline{q}}^{\overline{q}} \frac{\max_v\{R(q)v - c_0(v)\} d\Gamma_\tau(q)}{r+\delta}\right)\gamma - c_1(\gamma)}{r + \delta_1 - \gamma} \right\}, \tau = 1, ..., T, \tag{38}$$

and entry rate

$$v = \kappa \arg\max_{\gamma \geq 0} \left\{ \frac{\left(\sum_\tau \phi_\tau \left(\int_{\underline{q}}^{\overline{q}} \frac{\pi(x)d\Gamma_\tau(x)}{r+\delta}\right)\right)\gamma - c_1(\gamma)}{r + \delta_1 - \gamma} \right\}, \tag{39}$$

where $R(q)$ and $\digamma(q)$ are the solutions to the ODE system

$$R'(q) = \frac{\left(\frac{\delta+m(\theta)a_2}{\delta+m(\theta)a} + \frac{m(\theta)a_1}{\delta+m(\theta)a} \cdot \frac{\delta\digamma'(q)}{\delta+am(\theta)[1-\digamma(q)]} \right)(1-\beta)\frac{m(\theta)}{\theta}\frac{A}{K}}{r+\delta+a_1 m(\theta)\left[1-\digamma(q)\right]+a_2 m(\theta)} \tag{40}$$

$$\digamma'(q) = \frac{\sum_\tau K_\tau v(q)\Gamma'_\tau(q)}{\sum_\tau K_\tau \int_{\underline{q}}^{\overline{q}} v(x)d\Gamma_\tau(x)} \tag{41}$$

associated with the boundary conditions $R(\overline{q}) = F(\overline{q}) = 0$ where

$$\underline{q} = \frac{1}{A}\left(b + (1-a_1-a_2)\beta m(\theta)\int_0^{\overline{X}}[1-F(X)]dX\right)\sum_{\tau=1}^{T} K_\tau \tag{42}$$

is the reservation productivity together a distribution of products over firm type K_τ, $\tau = 1, ..., T$, and a market tightness parameter θ that satisfy equations (26) and (35).

The following two assumptions are standard. The third places reasonable restrictions on parameter values.

Assumption 1: The job finding rate $m(\theta)$ is strictly increasing and concave, and $\lim_{\theta\to 0}\{\theta/m(\theta)\} = 0$.

Assumption 2: The cost functions, $c_i(\cdot)$, $i \in \{0,1\}$, are both continuous, strictly convex, and satisfy $c_i(0) = c_i'(0) = 0$.

Assumption 3: The value of home production is strictly positive ($b > 0$) and the sum of the offer arrival rates when employed are less than the rate when unemployed ($a = a_1 + a_2 \leq 1$).

Proposition 2. *If the conditional productivity distributions $\Gamma_\tau, \tau = 1, ..., T$, are Lipschitz continuous for all $\tau = 1, 2, ..., T$ and the union of their supports is bounded above by $\overline{q} < \infty$, a steady state labor market equilibrium with strictly positive tightness exists.*

Proof: See Appendix I.

6 Imperfect Substitutes ($\sigma < \infty$)

When intermediate goods are imperfect substitutes, the model is complicated by the fact that marginal revenue is decreasing in employment size n because demand for each good is downward sloping and differences in both productivity q and product market size α contribute to dispersion in value TFP, denoted as $\mu = \alpha^{\frac{1}{\sigma}} (Aq)^{\frac{\sigma-1}{\sigma}}$. Formally, the total revenue of a product line of size n is

$$R_n(\mu) = \mu Y^{\frac{1}{\sigma}} \left(\frac{n}{K}\right)^{\frac{\sigma-1}{\sigma}} \tag{43}$$

where again Y represents the total output of the final good. In this section, we consider the complications implied by this fact.

6.1 Recruiting and the Maximum Labor Force Size

As in the perfect substitutes case, the optimal number of vacancies posted when there are n employees is the solution to the problem on the right side of equation (10). The difference is that the optimal choice

$$\begin{aligned} v_n(\mu) \quad = \quad & v(\Delta S_{n+1}(\mu)) = \arg\max_{v \geq 0} \Bigg\{ (1-\beta) \Bigg[(\eta_0 + \eta_2)\, \Delta S_{n+1}(\mu) \\ & + \eta_1 \int_0^{\Delta S_{n+1}} (\Delta S_{n+1}(\mu) - X) dG(X) \Bigg] v - c_0(v) \Bigg\}, \end{aligned} \tag{44}$$

depends on the marginal surplus value associated with the the next hire in the general case where $\Delta S_n(\mu)$ solves equation (8) for any given value of μ.

Because marginal revenue product declines with n, marginal match surplus, ΔS_n, diminishes as well. As a consequence, a maximum labor force size exists that depends on value TFP μ. Define $m(\mu)$ as the upper product line size for a given μ realization. It is defined by $\Delta S_{m(\mu)}(\mu) = 0$. Since $\Delta S_n(\mu)$ is decreasing in n for any finite σ, there is no option value from hiring at the upper firm size bound. Hence, by equation (8) and integration by parts

$$\begin{aligned} \Delta R_m(\mu) = b + (\lambda_0 - \lambda_2 - \lambda_1)\, \beta \int_0^{\overline{X}} [1 - F(X)] dX \\ - \lambda_1 (m(\mu) - 1)\, \beta \int_0^{\Delta S_{m-1}(\mu)} [1 - F(X)] dX - (\delta_0 + \lambda_2)(m-1)\, \Delta S_{m-1}. \end{aligned} \tag{45}$$

Since ΔR_m is monotonically decreasing in ΔS_{m-1} one can bound the expression for ΔR_m by,

$$\Delta R_m(\mu) \leq \Delta R_{\hat{m}} = b + (\lambda_0 - \lambda_1 - \lambda_2)\, \beta \int_0^{\overline{X}} [1 - F(X)] dX. \tag{46}$$

Equation (46) states that ignoring labor hoarding incentives, labor is hired to the point where the marginal revenue product of an additional worker equals an unemployed workers value flow

$\Delta R_{\hat{m}} = rU$, which is independent of value TFP μ.Denote this value of n as $\hat{m}(\mu)$. Generally, $m(\mu) > \hat{m}(\mu)$ reflecting that firms may choose to hoard labor in anticipation of the costly replacement of future quits. Of course in the perfect substitutes case ($\sigma = \infty$), there is no upper size bound and by definition no labor hoarding. In this case, firm size is finite purely as a result of labor market frictions.

The Wage Function

The possibility of a lifetime wage cut on a job arises in the general case as a consequence of the fact that the marginal marginal declines with labor force size. A cut occurs when another worker is added to the coalition if the value of $W_0 - U$, the surplus currently earned by an employee, exceeds the marginal match surplus ΔS_n and the later is still positive. In this case, the employer would fire the worker were the wage not to change but such a separation would be inefficient because the worker's only alternative option is unemployment. In this case, we assume that the wage is revised down to the marginal match surplus. Of course, a diminishing marginal revenue product also implies that the worker get a raise in lifetime income if some other worker leaves the coalition. Formally, the lifetime wage is always

$$\widehat{W}_n = \max \langle \Delta S_n, W_0 + \beta(U + \Delta S_n - W_0 \rangle \tag{47}$$

where W_0 reflects the worker's past labor market experience as previously defined.

Because a worker's wage depends on the size of the firm in the general case, the asset equation for employment in a product line of size n, W_n is given by,

$$\begin{aligned} r\widehat{W}_n = {} & w_n + h_n \Delta \widehat{W}_{n+1} - d_n(n-1)\beta \Delta \widehat{W}_n + (\delta_0 + \delta_1)(U - \widehat{W}_n) \\ & + \lambda_1 \int_{\Delta S_n}^{\infty} \left[U + \Delta S_n + \beta(X - \Delta S_n) - \widehat{W}_n \right] dF(X) \\ & + \lambda_2 \int_0^{\overline{X}} \left[U + \beta(X - U) - \widehat{W}_n \right] dF(X) \end{aligned} \tag{48}$$

where $\Delta \widehat{W}_n = \widehat{W}_n - \widehat{W}_{n-1}$,

$$h_n(\mu) = \left[\eta_0 u + (\eta_1 G(\Delta S_{n+1}(\mu)) + \eta_2)(1-u) \right] v(\Delta S_{n+1}(\mu)) \tag{49}$$

is the hire frequency, the transition rate from n to $n+1$ workers, $d(\Delta S_n)(n-1)$ is the frequency with which some other worker in the coalition will quit and size will transit from n to $n-1$ given that workers quit for all reasons at rate

$$d_n(\mu) = \delta_0 + \lambda_1[1 - F(\Delta S_n(\mu))] + \lambda_2. \tag{50}$$

Computing $\widehat{W}_n(q)$ and the associated value of the flow wage w_n is a straight forward given a numerical solution for ΔS_n obtained using (8) and (4).

As emphasized in both Dey and Flinn (2005) and Postel-Vinay and Robin (2002) one may observe both wage increases and decreases between jobs in this model. However, unlike these two previous examples, wages can both decrease and increase within the job as well. Wages will increase as a result of an arrival of an outside offer that increases W_0 but is below the marginal coalition surplus of the worker's current job. However, wages will also rise and fall as changes in the coalitions labor force size impact the coalition's marginal match surplus. In the early life of a product line, the expansion of its labor force size will tend to imply decreasing wage profiles which will be impacted in the opposite direction as workers improve their bargaining position as a result of outside offer arrivals.

6.2 Recruiting, Innovating, and Entering

The productivity of a product line, q, and the demand, α, are realized independent from each other when a new product is created. The distribution of market size is assumed independent of type. Firm type heterogeneity enters the model through type dependence of the distribution of product line productivity, q. Consequently, the distribution of the value of TFP random variable $\mu = \alpha^{\frac{1}{\sigma}} (Aq)^{\frac{\sigma-1}{\sigma}}$ is also type dependent. Let $\Gamma_\tau(\mu), \tau \in \{1, 2...., T\}$, denote the cumulative distribution of μ for any firm of type τ. Of course, the magnitude of the type index reflects the value productivity rank of the type in the sense of first order stochastic dominance. That is, $\tau > \tau'$ implies that product lines of firms of τ are likely to have greater employment conditional revenue than firms of type τ', i.e., $\Gamma_\tau(\mu) \leq \Gamma_{\tau'}(\mu)$ for all $\mu \in [0, \bar{\mu}]$.

Because there are no workers when a new product line is created, the optimal choice of the creation rate depends on the marginal surplus value of the first worker. Formally,

$$\gamma_\tau = \arg\max_{\gamma} \left\{ \frac{\gamma \left(\int \frac{\pi(\Delta S_1(\tilde{\mu}))d\Gamma_\tau(\tilde{\mu})}{r+\delta_1} \right) - c_1(\gamma)}{r+\delta_1-\gamma} \right\} \tag{51}$$

where

$$\begin{aligned} \pi(\Delta S_1(\mu)) \quad = \quad & \max_{v\geq 0} \Bigg\{ (1-\beta) \Bigg[(\eta_0+\eta_2)\, \Delta S_1(\mu) \\ & + \eta_1 \int_0^{\Delta S_{n+1}} (\Delta S_1(\mu) - X) dG(X) \Bigg] v - c_0(v) \Bigg\}, \end{aligned} \tag{52}$$

Because there are no workers yet in the line, only the future return to recruiting the first worker is relevant and that accrues to the employer. Hence, the innovation frequency optimal for the firm is optimal for the coalition as well.

Finally, the entry rate is the product $v = \kappa\gamma_0$ where κ is a given measure of entrepreneurs and γ_0 is the frequency with which any one of them creates new product. As the optimal innovation rate by a potential entrant is chosen before value TFP is realized,

$$v = m\gamma_0 = m \arg\max_{\gamma\geq 0} \left\{ \frac{\gamma \sum_{\tau=1}^{T} \left(\int \frac{\pi(\Delta S_1(\tilde{\mu}))d\Gamma_\tau(\tilde{\mu})}{r+\delta_1} \right) \phi_\tau - c_1(\gamma)}{r+\delta_1-\gamma} \right\}. \tag{53}$$

Match Surplus Distributions, Market Tightness and Final Output

The distribution of offered match surplus $F(X)$ is the fraction of vacancies posted by product lines with marginal surplus match values less than or equal to X. As the number of vacancies posted by each product line depends on the number of worker employed, the size of the market, and the productivity of the line, one needs to calculate the distribution of employment over product lines of each type. Let $P_n(\mu)$ represent the fraction of product lines with employment equal to n. As the flow out of the state $n = 0$ equal hires plus product destruction and the flow into the state is equal to the flow of all newly created products,

$$\sum_{\tau} (v\phi_\tau + \gamma_\tau) \Gamma'_\tau(\mu) + \big(\delta_0 + d_n(\mu)\big) P_1(\mu) = [\delta_1 + h_n(\mu)]\, P_0(\mu). \tag{54}$$

Since only transition from $n-1$ to n, $n+1$ to n, and n to zero can occur in an instant, all the other measures satisfy the difference equation

$$\big(\delta_0 + d_{n+1}(\mu)\big)(n+1)P_{n+1}(\mu) + h_{n-1}(\mu)P_{n-1}(\mu) = \big[\delta_1 + h_n(\mu) + \big(\delta_0 + d_n(\mu)\big)n\big]\, P_n(\mu), \tag{55}$$

where the functions $d_n(\mu)$ and $h_n(\mu)$ are those defined by equations (50) and (49). This difference equation together with the requirementment that $\sum_{n=0}^{\infty} P_n(\mu) = 1$ can be used to solve for the distribution of labor force size for every μ as characterized by the sequence $\{P_n(\mu)\}_0^{\infty}$.

Given these constructs in the previous section and the fact that the optimal number of vacancies posted depends only on surplus match value, the market steady state distribution of vacancies over match value offers is

$$F\left(\Delta S(\mu)\right) = \frac{\sum_\tau K_\tau \left[\sum_{n=0}^{\infty} \int_0^{\mu} v_n(x) P_n\left(x\right) d\Gamma_\tau(x)\right]}{\sum_\tau K_\tau \left[\sum_{n=0}^{\infty} \int_0^{\overline{\mu}} v_n x P_n\left(x\right) d\Gamma_\tau(x)\right]} \tag{56}$$

where the measure of products supplied by type τ firms is

$$K_\tau = \frac{v\phi_\tau}{\delta_1 - \gamma_\tau}, \tau = 1, .., T,$$

as in the perfect substitutes case. Similarly, the distribution of employment over surplus match values is given by

$$G\left(\Delta S(\mu)\right) = \frac{\sum_\tau K_\tau E_\tau \left[\sum_{n=0}^{\infty} \int_0^{\mu} n P_n\left(x\right) d\Gamma_\tau(x)\right]}{\left(1-u\right) L} \tag{57}$$

where the unemployment rate is again

$$u = \frac{\delta}{\delta + m(\theta)}.$$

Of course, the ratio of the aggregate vacancies to aggregate search effort, market tightness, is

$$\theta = \frac{\sum_\tau K_\tau \sum_{n=0}^{\infty} \int_0^{\mu} v_n(x) P_n\left(x\right) d\Gamma_\tau(x)}{\left(u + (a_1 + a_2)(1-u)\right) L}. \tag{58}$$

Given the production function specified in equation (1) and the fact that input j is supplied in quantity $x(j) = q(j)n(j)$ where $q(j)$ is the productivity and $n(j)$ is employment of product j, final market output in steady state is produced at rate

$$Y = \frac{1}{K}\left[\sum_\tau K_\tau \sum_{n=1}^{\infty} \int_0^{\overline{\mu}} x n^{\frac{\sigma-1}{\sigma}} P_n(x) d\Gamma_\tau(x)\right]^{\frac{\sigma}{\sigma-1}} \tag{59}$$

by the law of large numbers where $K = \sum_\tau K_\tau$. Note that equation (1) yields the following upper bound on final good output

$$\overline{Y} = AL\overline{q} \tag{60}$$

which is equal to what output would be if the entire labor force were employed in a product line with productivity equal to the sup of the upper supports, $\overline{q}$, of the distributions of productivity conditional on firm type.

7 Simulations

We solve the model numerically. In the following example, there are two separate firm types. Both firm types face the same support of possible μ realizations, $\tilde{\mu}_\tau \in \left[\underline{\mu}, \overline{\mu}\right]$, where $\tilde{\mu}_\tau \sim \Gamma_\tau(\cdot)$. The type conditional μ distribution function is parameterized as a beta distribution with parameters $\left(\mu_{\alpha,\tau}, \mu_{\beta,\tau}\right)$. Firm types only differ in terms of their μ realization distributions. The vacancy cost function is specified by $c_0(v) = c_{v,0}v^{c_{v,1}}$. The product innovation cost function is specified as $c_1(\gamma) = c_{\gamma,0}v^{c_{\gamma,1}}$. It is assumed that matched and unmatched search are equally efficient, $\lambda_0 = \lambda_1 + \lambda_2$, and $\lambda_1 = \lambda_2$. The matching function is given by $m(\theta) = A\sqrt{\theta}$. The model parameterization is given in Table 1. The firm type conditional value productivity realizations, $\Gamma_\tau(\mu)$ are chosen parameterized so that the type 2 distribution stochastically dominates the type 1 distribution. The income flow during unemployment is normalized at unity. The labor force is also normalized at unity. Both the innovation and hiring cost functions are specified to be quadratic with constant terms to put the unemployment rate in the vicinity of 6% and the average firm size around 10-15 workers when the product destruction rate is about, $\delta_1 = .09$. The bargaining power parameter β is set at a half. The interest rate is set at $r = 0.04$, reflecting that all rates are annual rates. Finally, the elasticity of substitution is set at $\sigma = 1.5$.

Solving the model then produces the steady state equilibrium outcome shown in Table 2 and in the following figures. First of all, it is immediately seen that the type heterogeneity results in a positive selection effect in that the steady state representation of type 2 product lines is $K_2/K = 0.37$ which is greater than the type 2 representation at birth, $\phi_2 = 0.2$. This is a direct result of type 2 firms choosing a greater product innovation rate than type 1 firms, driven by the greater ex ante product line profitability for type 2 firms. Another reflection of the positive selection is that type 2 firms on average are larger than type 1 firms in terms of number of product lines. Since, average labor force size is increasing in the value productivity realization, μ, as seen in Figure 5, Type 2 firms will on average also be larger both in terms of input and output.

The following figures show a few more of the characteristics of the steady state equilibrium. Figure 3 shows the offer and match distributions of match surplus. Dispersion in match surplus

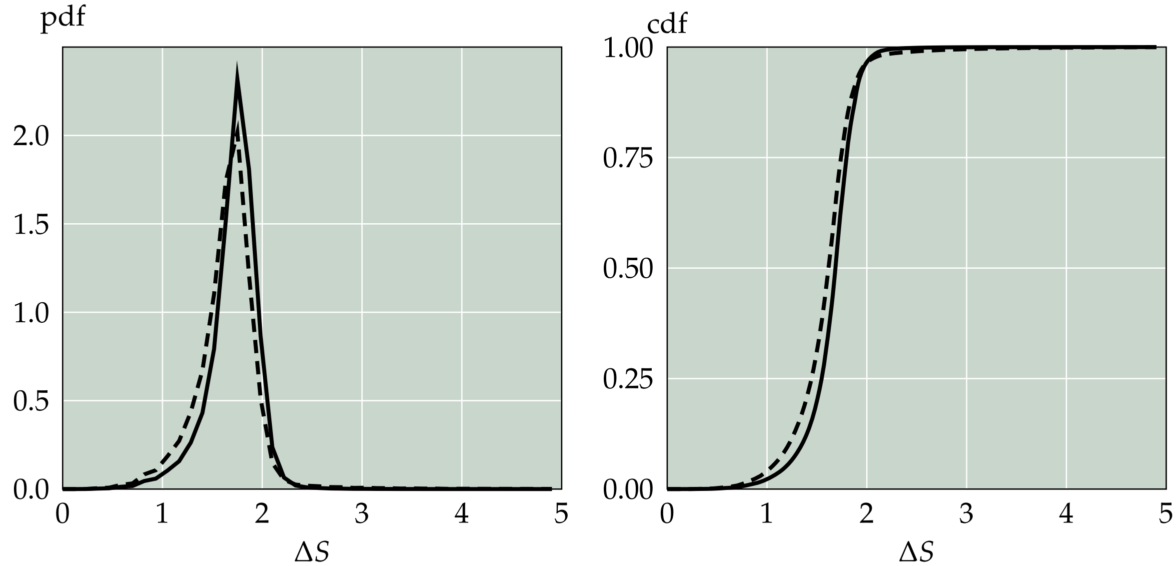
Table 1: Model parameters

		Type 1	Type 2
$\mu_{\alpha,\tau}$		2.000	4.000
$\mu_{\beta,\tau}$		3.000	3.000
$\underline{\mu}$	0.100		
$\overline{\mu}$	10.000		
b	1.000		
$c_{v,0}$	0.001		
$c_{v,1}$	2.000		
$c_{\gamma,0}$	7,000.000		
$c_{\gamma,1}$	2.000		
r	0.040		
β	0.500		
σ	1.500		
ϕ_τ		0.800	0.200
δ_0	0.100		
δ_1	0.089		
L	1.000		
A	1.000		

Table 2: Steady state equilibrium

		Type 1	Type 2
K	0.100		
λ_0	2.759		
λ_1	1.379		
λ_2	1.379		
ν	0.004		
θ	7.611		
u	0.064		
Y	11.600		
γ_τ		0.039	0.067
K_τ		0.063	0.037
M_τ		0.047	0.017
Ψ_τ		75.222	227.594
$E_\tau[k]$		1.347	2.189
U	63.777		

Figure 3: Offer and Match Value Distributions

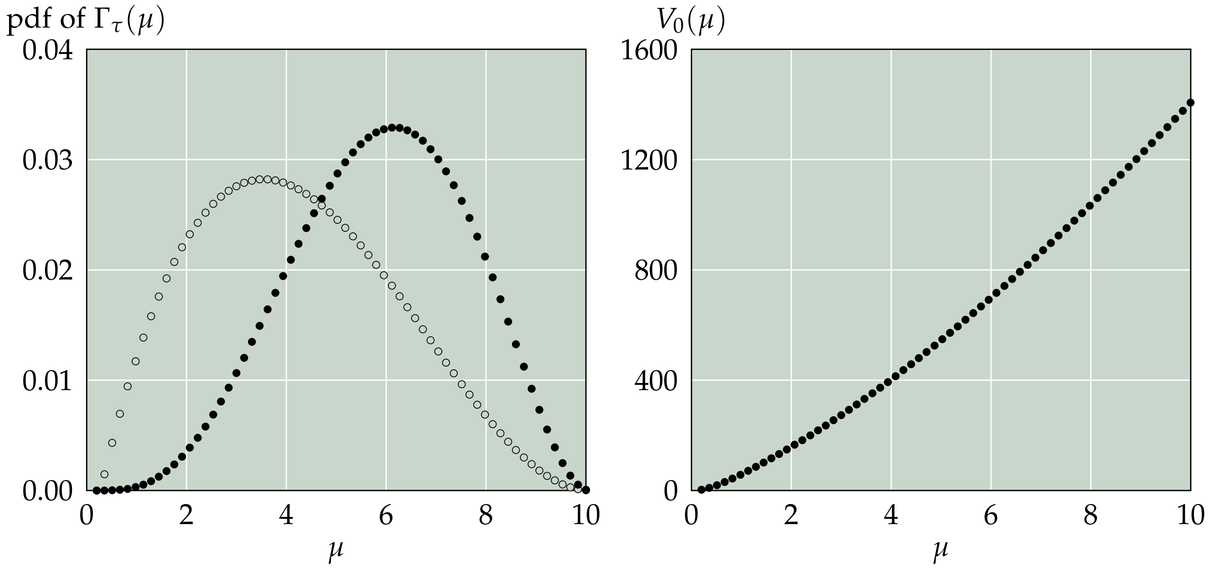


Note: Match distribution $G(\cdot)$ drawn in solid line. Offer distribution $F(\cdot)$ drawn in dashed line. Upper bound on distribution support is 5.72.

in both offers and matches is in the current specification crucially supported by labor market frictions. Absent labor market frictions, firms adjust their labor force sizes such that marginal surplus equalizes across product lines. The labor force size of a product line is determined as the combination of labor market frictions and the product demand implied by the value productivity realization, μ. Which of these two forces dominate reduces to a comparison of hiring costs and the elasticity of substitution, σ. In the perfect substitutes case, $\sigma = \infty$, labor force size is purely determined by labor market frictions, since the revenue function in this case is constant returns. The right hand panel in Figure 5 shows the μ conditional average labor force size realization relative to the frictionless labor demand, $\hat{m}(\mu)$. In the current specification, the elasticity of substitution is chosen fairly high relative to the level of labor market frictions and so for the average μ realization, the associated labor force realization is typically 20-30% of the frictionless level. A possibly more interesting measure is the average marginal revenue realization relative to that of the frictionless level. Future versions of the paper will discuss this measure as well.

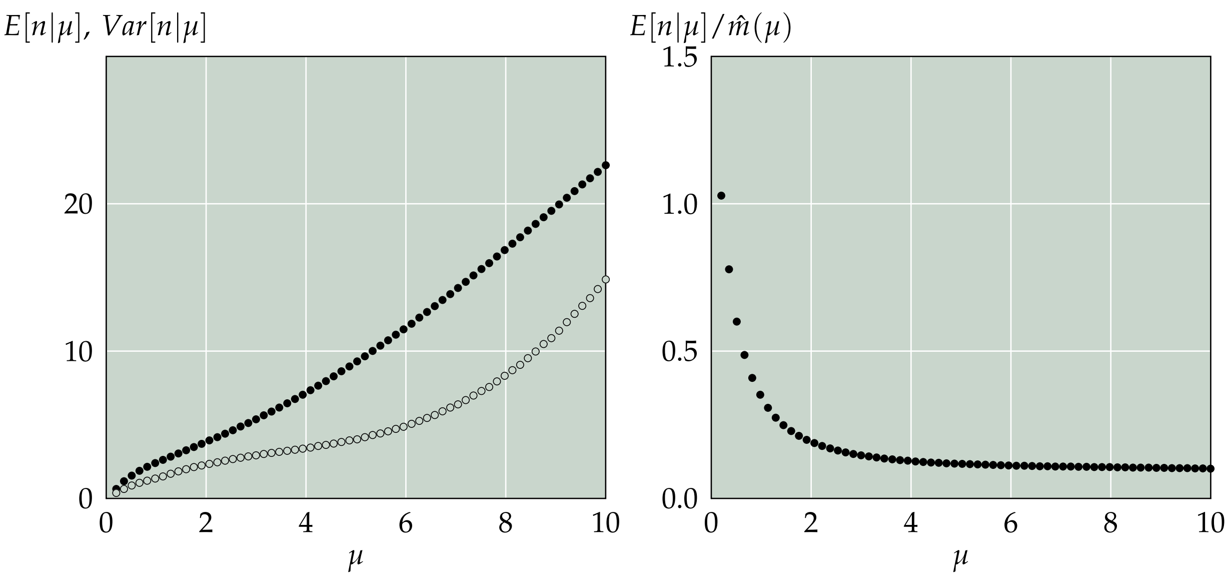
Given convex hiring costs, higher μ realization product lines will on average offer greater match surplus matches, as is reflected in Figure 6. Hence dispersion in μ across product lines

Figure 4: Firm type conditional product realization distribution and value



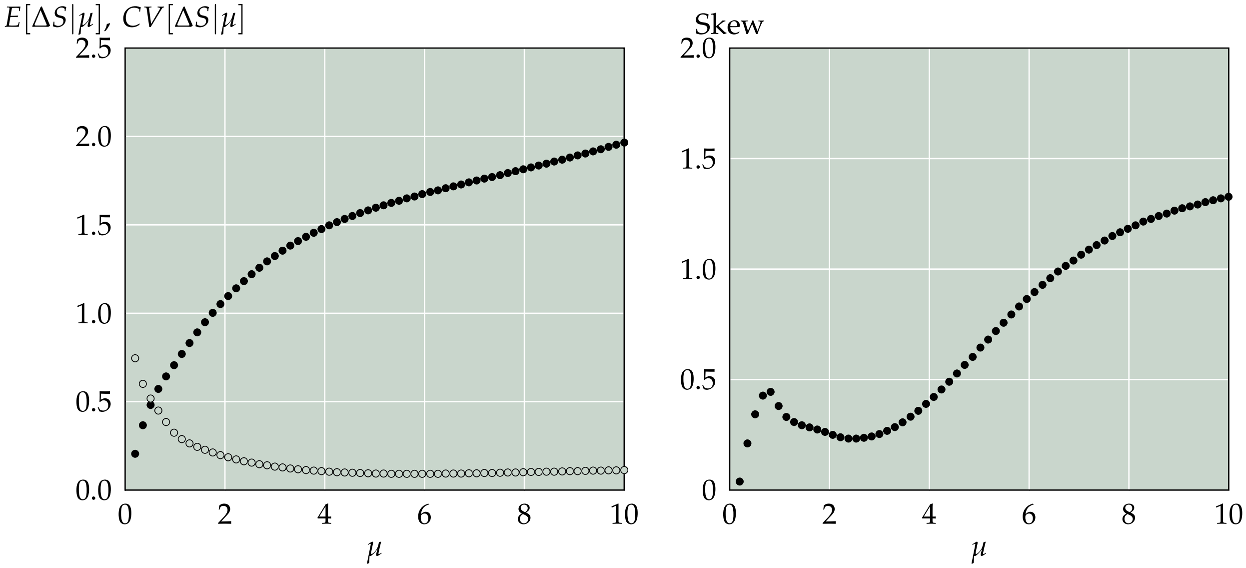
Note: The firm type 1 conditional probability distribution over μ is plotted with hollow dots. The firm type 2 conditional distribution is plotted with solid dots.

Figure 5: Product type conditional labor force size statistics



Note: Product type conditional average labor force size plotted with solid dots. Product type conditional labor force size variance plotted with hollow dots.

Figure 6: Product type conditional marginal match value statistics

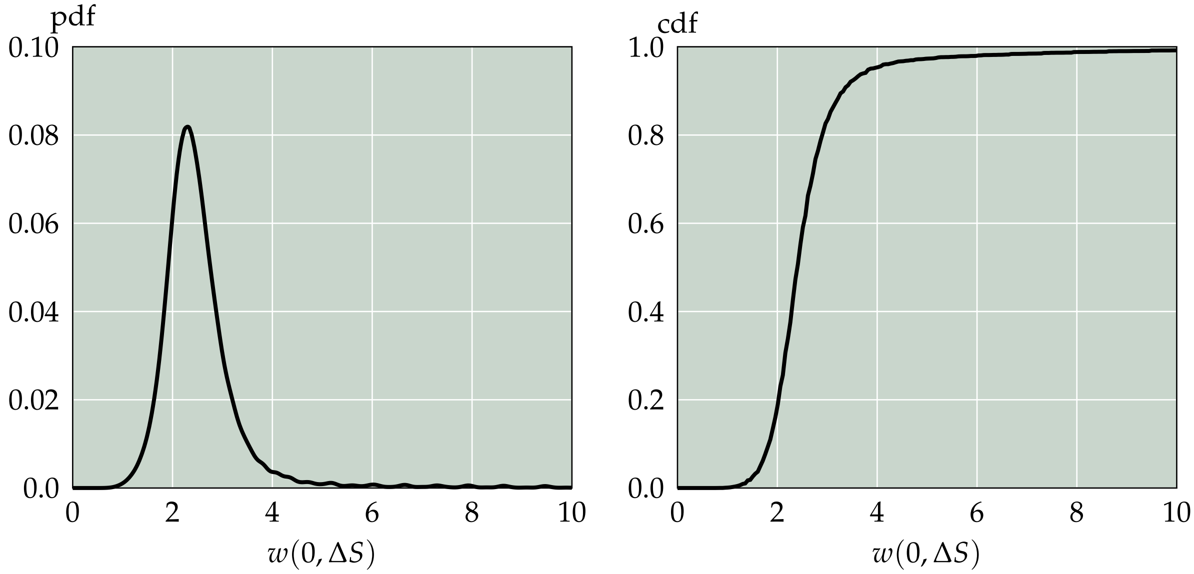


Note: Average marginal match surplus plotted with solid dots. Coefficient of variation plotted with hollow dots.

is one source of ΔS dispersion in both the offer and match distributions. However, even in the absence of μ dispersion, the equilibrium will produce ΔS dispersion as a result of the stochastic nature of the labor force hiring and separation process. A newly created product line will initially offer high ΔS matches. As the product line is populated, marginal revenue falls and along with it both the ΔS offer but also the marginal value of all its existing matches. This is why the model has a feature unlike most on-the-job search models that the match distribution does not necessarily stochastically dominate the offer distribution. This is only true for the finite elasticity of substitution case. In the $\sigma = \infty$ case, match surplus dispersion is purely a result of μ dispersion (or more appropriately in that case, q dispersion).

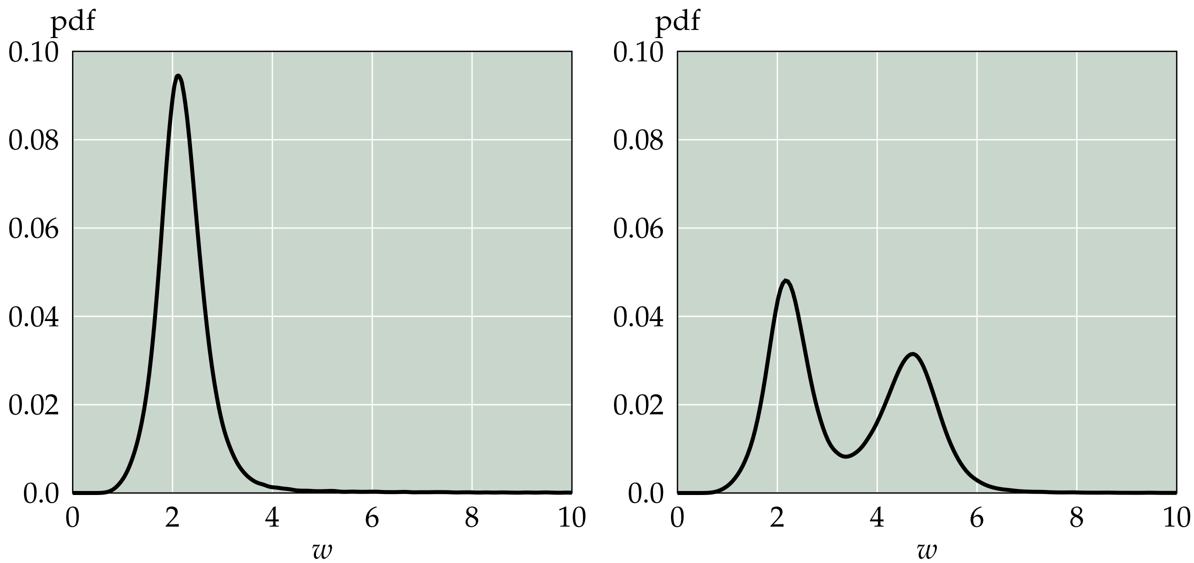
Figures 7 and 8 show the steady state match wage distribution and the distribution of wages directly out of unemployment. The model has a capacity to generate wage distributions with long right tails. In the current specification, the particular bargaining process is emphasized in the difference between match wages and initial wages out of unemployment. The match wage distribution reflects that as the employed worker receives outside offers, her bargaining position improves.

Figure 7: Distribution of initial wage out of unemployment



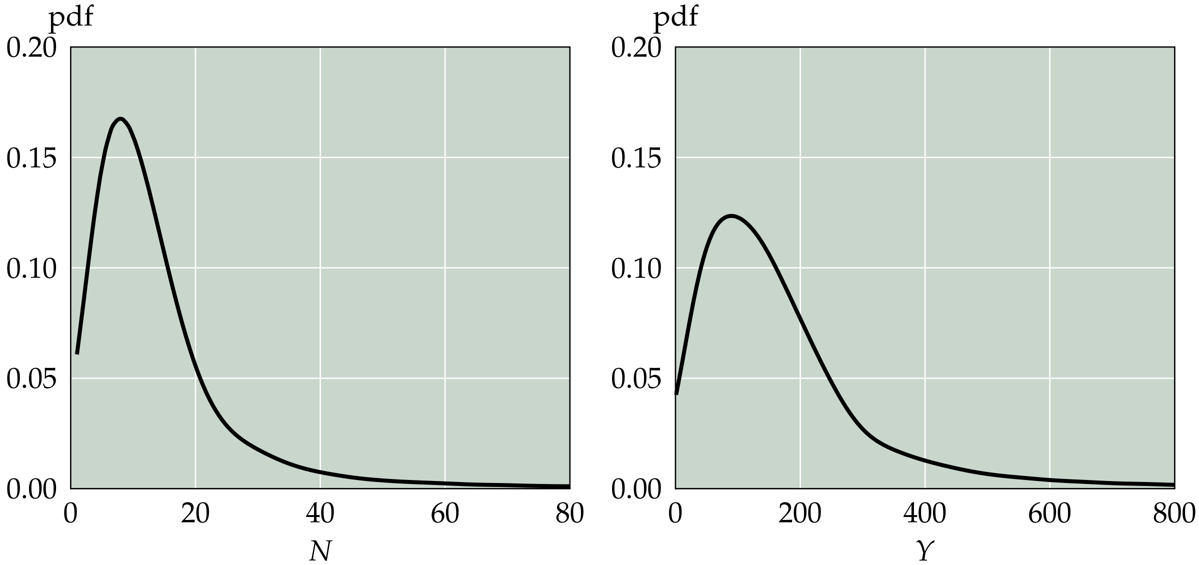
Note: Wage density estimate based on a Gaussian kernel estimation with bandwidth of 0.1. Wage support truncated at 10. Income flow during unemployment is $b = 1$.

Figure 8: Distribution of initial wage out of unemployment and match wage distribution.



Note: Wage density estimate based on a Gaussian kernel estimation with bandwidth of 0.1. Wage support truncated at 10. Income flow during unemployment is $b = 1$.

Figure 9: Firm size distribution



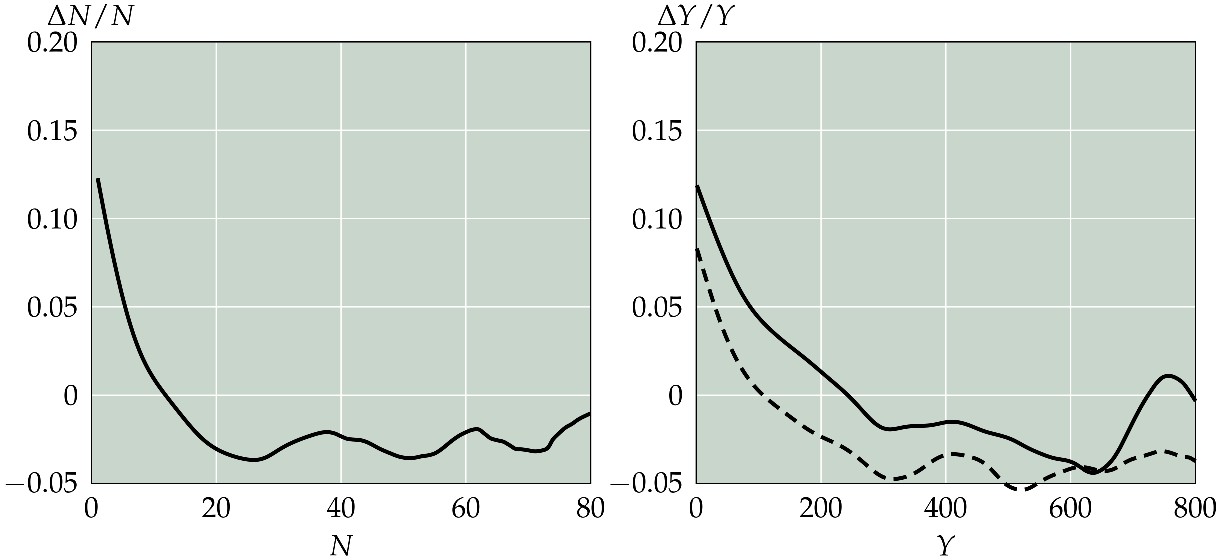
Note: Labor force size density estimate based on a Gaussian kernel estimation with bandwidth of 3. Output size density estimate based on a Gaussian kernel estimation with a bandwidth of 30.

Figures 9-12 show statistics for the firm side of the simulation. Not surprisingly, the firm size distribution is skewed with a long right tail, driven by the logarithmic distribution of the number of product lines in a firm. The model can produce plenty of labor productivity dispersion as measured by output per worker. While the underlying creation and destruction of products implies a somewhat increasing relationship between firm size and growth, the labor market frictions produce a strongly negative relationship at the lower size end, where understaffed product lines are over represented.

The right hand panel in Figure 11 shows the relationship between firm labor productivity and the growth rate. It is strongly positive in the case of the current growth rate. However, the growth rate one year into the future is largely uncorrelated with labor productivity. We know that type 2 firms have greater growth rates than type 1 firms. This result suggests that labor market frictions can introduce substantial noise into the labor productivity measure for the purpose of letting it predict firm type.

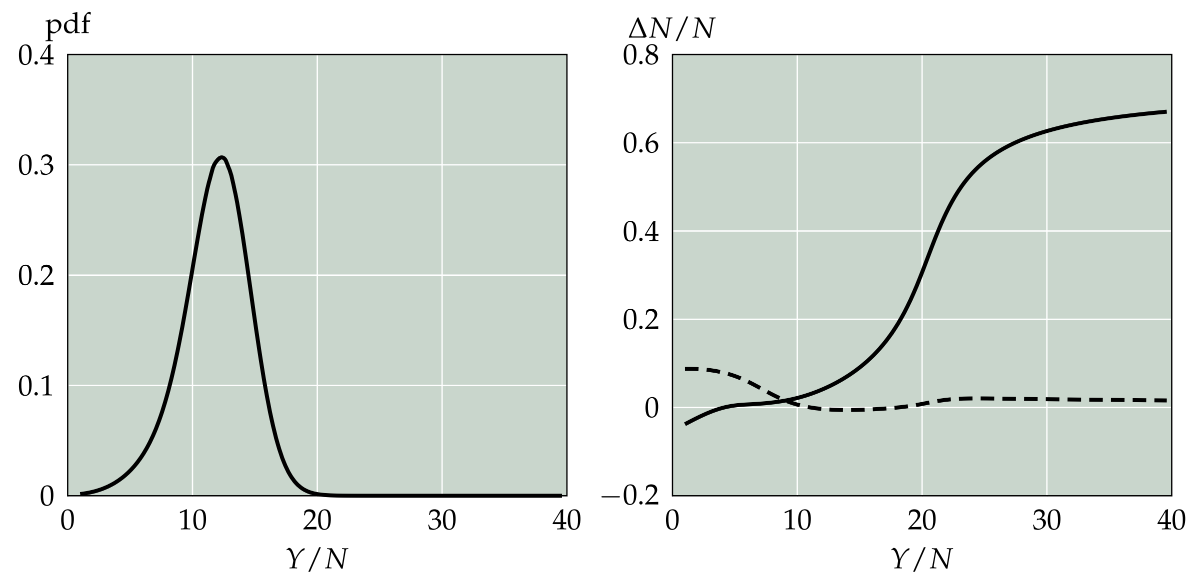
Figure 12 shows the relationship between labor productivity and size. By and large the relationship is positive as one would expect, however, we also see the impact of labor market

Figure 10: Firm growth rate and size



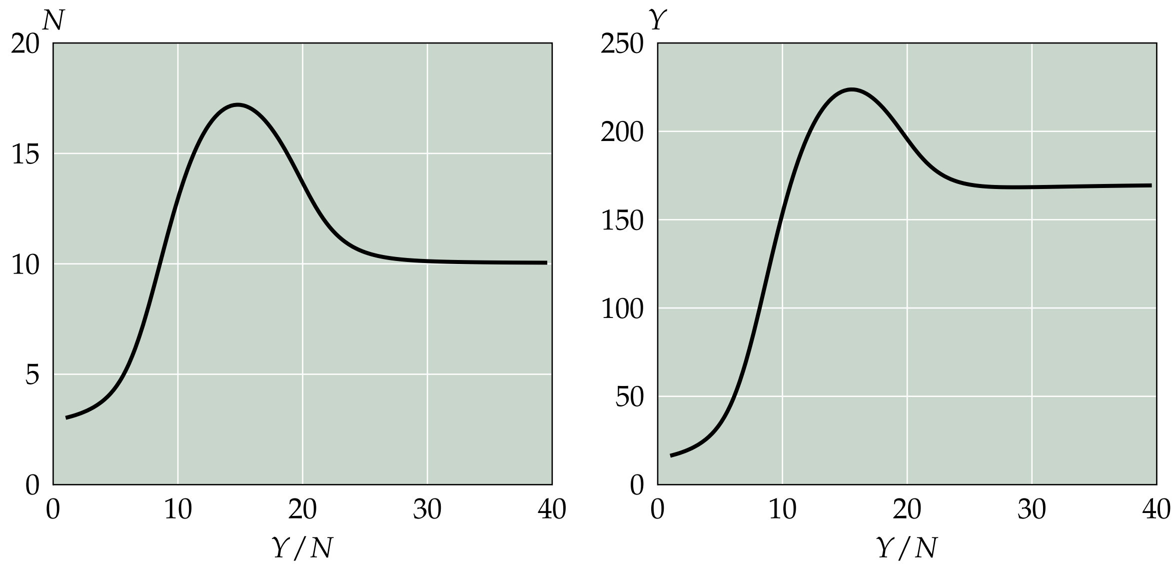
Note: The dashed line in the right hand panel is the output growth rate leaded by one year. Labor force size statistic based on Gaussian nonparametetric regression with bandwidth of 3. Output size statistic based on Gaussian regression with a bandwidth of 30.

Figure 11: Firm growth rate and labor productivity



Note: The dashed line in the right hand panel is the growth rate leaded by one year. Nonparametric Gaussian regressions with bandwidth of 2.

Figure 12: Labor productivity and size



Note: Nonparametric Gaussian regressions with bandwidth of 2.

frictions here in that the understaffed product lines will have high labor productivity measures and tend to produce small firms, which is why we see a non-monotone relationship between size and productivity. It is the presence of the understaffed product lines in the upper end of the labor productivity distribution that is confounding identification of firm heterogeneity.

8 Identification

To be completed

9 Estimation

To be completed

10 Counter factuals

To be completed

11 Conclusion

To be completed

Appendix

Proof of Existence

Let the triple $(\overline{\pi}, \overline{R}, \overline{\gamma})$ represent the unique solutions to

$$\frac{\kappa\overline{\gamma}}{\delta_1 - \overline{\gamma}} = \frac{A\overline{q}}{b}, \tag{61}$$

$$\overline{\gamma} = \max_{\gamma} \left\{ \frac{\overline{\pi}\gamma - c_1(\gamma)}{r + \delta_1 - \gamma} \right\} \tag{62}$$

$$\overline{\pi} = \max_{v} \left\{ \overline{R}v - c_0(v) \right\}. \tag{63}$$

Given $\overrightarrow{\gamma} = (\gamma_0, \gamma_1, ..., \gamma_T) \in \mathbb{R}_+^{T+1}$, $\overrightarrow{K} = (K_1, K_2, ..., K_T) \in \mathbb{R}_+^T$, and $(\underline{q}, \theta, K,) \in \mathbb{R}_+^3$, let $z = (\overrightarrow{\gamma}, \overrightarrow{K}, \theta, K, \underline{q}) \in \mathbb{Z}$ where

$$\mathbb{Z} = \left\{ \begin{array}{c} z \in \mathbb{R}_+^{2T+4} | \gamma_\tau \leq \overline{\gamma} \text{ for } \tau = 0, 1, ...T, K_\tau \leq \frac{A\overline{q}}{Tb} \text{ for } \tau = 1, ...T, \\ \theta \leq \frac{A\overline{q}}{bL} v(\overline{R}) \text{ and } (1-\beta) 2A(r+\delta) \leq \overline{R} \frac{\theta}{m(\theta)} \sum_{\tau=1}^{T} K_\tau \end{array} \right\}. \tag{64}$$

Note that $\mathbb{Z}$ is a bounded real vector space which is closed and convex given that $m(\theta)$ is increasing and concave.

The unique solution to the ODE system composed of (40) and (41), represented by $R(q, z)$ and $F(q, z)$, consistent with the boundary conditions $R(\underline{q}, 0) = F(\underline{q}, 0) = 0$ is continous and satisfy $0 \leq R(q) \leq \overline{R}$ and $0 \leq F(q) \leq 1$ on $[\underline{q}, \overline{q}] \times \mathbb{Z}$. Continuity follows from the continuity of the RHS of both (40) and (41) in z. Furthermore,

$$\begin{aligned} R(q,z) \;&\leq\; R(\overline{q}, z) = \int_{\underline{q}}^{q} \left(\frac{\left(\begin{array}{c} \left(\frac{\delta + m(\theta)a_2}{\delta + m(\theta)a}\right) \frac{m(\theta)}{\theta} \\ + \left(\frac{m(\theta)a_1}{\delta + m(\theta)a}\right) \frac{m(\theta)}{\theta} \frac{\delta F'(q)}{\delta + am(\theta)[1 - F(x)]} \end{array} \right) (1-\beta) \frac{A}{\sum_{\tau=1}^{T} K_\tau}}{r + \delta + a_1 m(\theta) [1 - F(q)] + a_2 m(\theta)} \right) dq \\ &\leq\; \frac{2A(1-\beta) \frac{m(\theta)}{\theta}}{(r+\delta) \sum_{\tau=1}^{T} K_\tau} = \overline{R}. \end{aligned}$$

and

$$F(q) \leq F(\overline{q}) = \frac{\int_{\underline{q}}^{\overline{q}} \sum_\tau K_\tau v(q) d\Gamma_\tau(q)}{\sum_\tau K_\tau \int_{\underline{q}}^{\overline{q}} v(x) d\Gamma_\tau(x)} = 1$$

Equations (37), (38), (39), (26), (35), and (42) define the map $M : \mathbb{Z} \to \mathbb{Z}$ where

$$
\begin{aligned}
(M\gamma_0)(z) &= \kappa \arg\max_{\gamma} \left\{ \frac{\left(\sum_{\tau=1}^{T} \phi_\tau \left(\int_{\underline{q}}^{\overline{q}} \frac{\pi(R(q,z))d\Gamma_\tau(q)}{r+\delta}\right) dz\right) \gamma - c_1(\gamma)}{r+\delta_1-\gamma} \right\} \leq \kappa\overline{\gamma} \\
(M\gamma_\tau)z &= \arg\max_{\gamma} \left\{ \frac{\left(\int_{\underline{q}}^{\overline{q}} \frac{\pi(R(q,z))d\Gamma_\tau(q)}{r+\delta} dz\right) \gamma - c_1(\gamma)}{r+\delta_1-\gamma} \right\} \leq \overline{\gamma}, \tau = 1, ..., T
\end{aligned}
$$

$$
\begin{aligned}
(MK_\tau)(z) &= \frac{\kappa\gamma_0\phi_\tau[1-\Gamma_\tau(\underline{q})]}{\delta_1 - \gamma_\tau[1-\Gamma_\tau(\underline{q})]} \leq \frac{\kappa\gamma_T}{\delta_1-\gamma_T} \leq \frac{\kappa\overline{\gamma}}{\delta_1-\overline{\gamma}} = \frac{A\overline{q}}{Tb}, \tau = 1, ..., T \\
(M\theta)(z) &= \frac{(\delta+m(\theta))}{(\delta+am(\theta))\,L} \sum_{\tau=1}^{T} K_\tau \int_{\underline{q}}^{\overline{q}} v(R(q,z))d\Gamma_\tau(q) \leq \frac{A\overline{q}}{bL} v(\overline{R}) \\
(M\underline{q})(z) &= \left(\begin{array}{c} b + (1-a_1-a_2)\beta m(\theta) \\ \times \int_0^{\overline{q}} \frac{[1-F(x)]dx}{r+\delta+a_1 m(\theta)[1-F(x)]+a_2 m(\theta)} \end{array} \right) \frac{\sum_{\tau=1}^{T} K_\tau}{A} \leq b\frac{A\overline{q}}{Ab} = \overline{q}.
\end{aligned}
$$

As $\mathbb{Z}$ is compact and convex real vector space, M is continuous, and any fixed point is an equilibrium existence follows by Brouwer's fixed point theorem.

Labor Hoarding

Define $m(\mu)$ as the upper product line size for a given μ realization. It is defined by $\Delta S_{m(\mu)}(\mu) = 0$. Since $\Delta S_n(\mu)$ is decreasing in n for any finite σ, there is no option value from hiring at the upper firm size bound. Hence, by equation (8),

$$
\begin{aligned}
0 &= \Delta R_m - b - (\lambda_0 - \lambda_2)\beta E[S] + (\delta_0 + \lambda_2)(m-1)\Delta S_{m-1} + \lambda_1 m \beta E[S] \\
&\quad - \lambda_1 (m-1)\beta \int_{\Delta S_{m-1}}^{\infty} (S - \Delta S_{m-1}) dF(S) \\
&\Updownarrow \\
\Delta R_m &= b + (\lambda_0 - \lambda_2 - \lambda_1)\beta E[S] - \lambda_1(m-1)\beta \left[E[S] - \int_{\Delta S_{m-1}}^{\infty} (S - \Delta S_{m-1}) dF(S) \right] \\
&\quad - (\delta_0 + \lambda_2)(m-1)\Delta S_{m-1}. \qquad (65)
\end{aligned}
$$

Since ΔR_m is monotonically decreasing in ΔS_{m-1} one can bound the expression for ΔR_m by,

$$
\Delta R_m \leq \Delta R_{\hat{m}} = b + (\lambda_0 - \lambda_1 - \lambda_2)\beta E[S]. \qquad (66)
$$

Equation (66) states that ignoring labor hoarding incentives, labor is hired to the point where the marginal revenue product of an additional worker equals a searching worker's payoff flow,

$b+(\lambda_0-\lambda_1-\lambda_2)\,\beta E\,[S]$. The last term reflects the efficiency of off the job search relative to on the job search. We denote this level by $\hat{m}\,(\mu)$. Generally, $m\,(\mu) > \hat{m}\,(\mu)$ reflecting that firms may choose to hoard labor in anticipation of the costly replacement of future losses of workers to quits or exogenous separation. Of course in the perfect substitutes case ($\sigma = \infty$), there is no upper bound firm size bound and by definition no labor hoarding. In this case, firm size is finite purely as a result of labor market frictions.

The Wage Function for Finite σ

The asset equation for $\hat{W}_n$ is given by,

$$\begin{aligned}(r+\delta_0+\delta_1+\lambda_2)\,\hat{W}_n = w_n - b + h_n\Delta\hat{W}_{n+1} - (n-1)\,(\delta_0+d_n)\,\Delta\hat{W}_n \\ + \lambda_1\beta\int_{\Delta S_n}^{\infty}(S-\Delta S_n)dF(S) \\ + \lambda_1\,[1-F\,(\Delta S_n)]\left(\Delta S_n - \hat{W}_n\right) \\ + \lambda_1\,(1-\beta)\int_{A}^{\Delta S_n}(S-A)\,dF(S) \\ + (\lambda_2-\lambda_0)\,\beta E\,[S]\,,\end{aligned}$$

where

$$d_n = d_n(\Delta S_n) = \lambda_1[1-F(\Delta S_n)] + \lambda_2. \tag{67}$$

It follows that

$$\begin{aligned}w_n &= b + (\lambda_0-\lambda_1-\lambda_2)\,\beta E\,[S] \\ &\quad + [r+\delta_1+n\,(\delta_0+s_n)+h_n]\,\hat{W}_n - h_n\hat{W}_{n+1} - (n-1)\,(\delta_0+s_n)\,\hat{W}_{n-1} \\ &\quad + \lambda_1\left[\beta\int_0^{\Delta S_n}SdF(S) - (1-\beta)\left[(1-F\,(\Delta S_n))\,\Delta S_n + \int_A^{\Delta S_n}(S-A)\,dF(S)\right]\right].\end{aligned} \tag{68}$$

Equation (68) is a straightforward evaluation given a numerical solution for ΔS_n. Equation (8) provides one strategy for solving for the marginal coalition surplus. Equation (68) has a simpler form for the case of a worker's wages directly out of unemployment, that is for $A = 0$,

$$
\begin{aligned}
w_n = b &+ (\lambda_0 - \lambda_1 - \lambda_2)\, \beta E[S] \\
&+ [r + \delta_1 + n(\delta_0 + s_n) + h_n]\, \beta \Delta S_n - h_n \beta \Delta S_{n+1} - (n-1)(\delta_0 + s_n)\, \beta \Delta S_{n-1} \\
&+ \lambda_1 \left[\beta \int_0^{\Delta S_n} S dF(S) - (1-\beta) \left[(1 - F(\Delta S_n))\, \Delta S_n + \int_0^{\Delta S_n} S dF(S) \right] \right] +
\end{aligned}
$$

The model has interesting implications for wage dynamics. As emphasized in both Dey and Flinn (2005) and Postel-Vinay and Robin (2002) one may observe both wage increases and decreases between jobs in this type of model. However, unlike these two previous examples, wages can both decrease and increase within the job in the model. Wages will increase as a result of an arrival of an outside offer that increases A but is below the marginal coalition surplus of the worker's current job. However, wages will also rise and fall as changes in the coalitions labor force size impact the coalition's marginal match surplus. In the early life of a product line, the expansion of its labor force size will tend to imply decreasing wage profiles which will be impacted in the opposite direction as workers improve their bargaining position as a result of outside offer arrivals.

Firm Heterogeneity and Innovation for Finite σ

The productivity of a product line, q, and the demand, α, are realized independent from each other when a new product is created. The distribution of market size is assumed independent of type. Firm type heterogeneity enters the model through type dependence of the distribution of product line productivity, q. Consequently, the distribution of the random variable $\mu = \alpha^{\frac{1}{\sigma}} (Aq)^{\frac{\sigma-1}{\sigma}}$ is also type dependent. Let $\Gamma_\tau(\mu), \tau \in \{1, 2...., T\}$, denote the cumulative distribution of μ for any firm of type τ. We assure that the magnitude of the type index reflects the productivity rank of the type in the sense of first order stochastic dominance. That is, $\tau > \tau'$ implies that product lines of firms of τ are likely to have greater employment conditional revenue than firms of type τ', i.e., $\Gamma_\tau(\mu) \leq \Gamma_{\tau'}(\mu)$ for all $\mu \in [0, \bar{\mu}]$.

Value of R&D

Let Ψ represent the value of research per existing product line. The innovation frequency per product line is the choice variable γ. If an innovation arrives in next instant, the employer will have a product line with no workers, which generates a revenue flow equal to the net profit associated with attempting to fill the first job, which is $\pi(\Delta V_1)$. As every product line faces destruction risk δ_1, its present value at the moment of discovery is $\pi(\Delta V_1)/(r+\delta_1)$. In addition, the option to create still another, which again has value Ψ, arrives with each product created. Because the option to create a new product is lost if the existing product line is destroyed and the market size and cost of production are realized only after the product is created, the value of a new product for firm of type τ solves

$$(r+\delta_1)\Psi_\tau = \max_\gamma \left\{ \gamma \left(\int \frac{\pi(\Delta V_1(\tilde{\mu}))d\Gamma_\tau(\tilde{\mu})}{r+\delta_1} + \Psi_\tau \right) - c_1(\gamma) \right\}. \tag{69}$$

Note that the contribution of a product line to the expected present value of future income of the coalition composed of the firm and all currently employed worker is $V_0 = \pi(\Delta V_1)/(r+\delta_1)$ from equation (5). Because there are no workers yet in the line, only the future return to recruiting the first worker is relevant and that accrues to the employer. Hence, one can show that the innovation frequency optimal for the firm, the solution to the problem specified on the right side of (69), is optimal for the coalition as well.

At a point in time, the coalition is supplying k different products. Let $V_{n_k}(\mu_k)$ represent the value of the k^{th} product line were n_k represents the number of employees and μ_k characterizes the revenue function of the product line. Given these specifics, it solves equation (5) when $R_n = R_{n_k}(\mu_k)$ as defined in equation (2). Let $(\vec{n}, \vec{\mu}, k)$ represent the state of the coalition where $\overrightarrow{x} = (x_1, ..., x_k)$ represents a vector of length k. Under the assumption that the innovation rate is chosen to maximize the value of the coalition given that it is of type τ, denoted $\Lambda_\tau(\vec{n}, \vec{\mu}, k)$, is the solution to the following continuous time Bellman equation,

$$\begin{aligned} r\Lambda_\tau(\vec{n}, \vec{\mu}, k) = k \max_{\gamma \geq 0} & \left\{ \gamma \int \left[\Lambda_\tau((\vec{n}, 0), (\vec{\mu}, \tilde{\mu}), k+1) - \Lambda_\tau(\vec{n}, \vec{\mu}, k) \right] d\Gamma_\tau(\tilde{\mu}) - c_1(\gamma) \right\} \\ & + (r+\delta_1) \sum_{j=1}^{k} V_{n_j}(\mu_j) + \delta_1 \sum_{j=1}^{k} \left(\Lambda_\tau(\vec{n}_{\langle j \rangle}, \vec{\mu}_{\langle j \rangle}, k-1) - \Lambda_\tau(\vec{n}, \vec{\mu}, k) \right), \end{aligned} \tag{70}$$

where $\vec{x}_{\langle j\rangle}$ is an $k-1$ vector constructed from $\vec{x}$ by deleting its j^{th} element. The first term on the right is the expected gain associated with the creation of a new product, the second term is simply the employer's gross revenue flow from its current product portfolio, and the last is the expected capital loss in value attributable to the destruction of an existing product.

Proposition 3. *The value of a firm-worker coalition takes the form*

$$\Lambda_\tau\left(\vec{n},\vec{\mu},k\right)=\sum_{j=1}^{k}V_{n_j}(\mu_j)+k\Psi_\tau. \tag{71}$$

Proof. Insert the conjecture into equation (70) to obtain

$$\begin{aligned}
r\left[\sum_{j=1}^{k}V_{n_j}(\mu_j)+k\Psi_\tau\right] &= k\max_{\gamma\geq 0}\left\{\gamma\int\left(V_0\left(\tilde{\mu}\right)+\Psi_\tau\right)d\Gamma_\tau(\tilde{\mu})-c_1\left(\gamma\right)\right\}\\
&\quad+\left(r+\delta_1\right)\sum_{j=1}^{k}V_{n_j}(\mu_j)-\delta_1\sum_{j=1}^{k}\left(V_{n_j}(\mu_j)-\Psi_\tau\right)\\
&\Updownarrow\\
\left(r+\delta_1\right)k\Psi_\tau &= k\max_{\gamma\geq 0}\left\{\gamma\int\left(\frac{\pi(\Delta V_1(\tilde{\mu}))}{r+\delta_1}+\Psi_\tau\right)d\Gamma_\tau(\tilde{\mu})-c_1\left(\gamma\right)\right\}.
\end{aligned}$$

Hence, equation (71) holds by (5) where Ψ_τ is the solution to (69). Since, equation (70) can be reformulated as a contraction, the form in equation (71) is also the unique solution. ☐

Innovation Frequency

Equation (5) defines the value of innovation activity per product line for a type τ firm. It is the present value of the maximal difference between two parts, the product of the return to innovation and the chosen innovation rate less the R&D investment required to sustain that rate. The return is the sum of two parts, the expected present value of the profit of a new product line plus that value of the opportunity to create another product facilitated by the existence of a new product line, the value of the knowledge embodied in the creation of a new product. From equation (70), the innovation frequency satisfies

$$c_1'(\gamma_\tau)=E_\tau V_0\left(\tilde{\mu}\right)+\Psi_\tau \tag{72}$$

when positive where E_τ represents the expectation operator conditional on firm type. Note that the choice is independent of both the number of products currently supplied and of the number of workers employed to supply each line.

Proposition 4. *If the cost of R&D, $c_1(\gamma)$, is increasing, strictly concave and $c_1(0) = c_1'(0) = 0$, intermediate products are imperfect substitutes, and the measure of products supplied, K, is sufficiently large, then the optimal product creation rate is positive and less than the product destruction rate, δ.*

Proof. If a solution to (72) exists, then the value of an additional product line is defined by

$$\Psi_\tau = \max_{\gamma \geq 0} \left\{ \frac{\gamma E_\tau V_0(\tilde{\mu}) - c_\gamma(\gamma)}{r + \delta_1 - \gamma} \right\}. \tag{73}$$

Hence, the first order condition for the optimal innovation rate can be written as

$$f(\gamma) = (r + \delta_1 - \gamma)\left(E_\tau V_0(\tilde{\mu}) - c_\gamma'(\gamma)\right) + \gamma E_\tau V_0(\tilde{\mu}) - c_\gamma(\gamma) = 0 \tag{74}$$

and the second order condition requires $f'(\gamma) = -(r + \delta_1 - \gamma)\, c''(\gamma) \leq 0$ at a maximal solution. As $f'(0) = (r + \delta_1) E_\tau V_0(\tilde{\mu}) > 0$, the first order condition has a unique solution satisfying $0 < \gamma < \delta_0$ and the sufficient second order condition is satisfied if $f'(\delta_1) = (r + \delta_1) E_\tau V_0(\tilde{\mu}) - r c_\gamma(\delta_1) - c_\gamma(\delta_1) < 0$. As $V_0(\tilde{\mu})$ is bounded above by the largest value of a new product line and that bound converges to zero as $K \to \infty$ by equations (70) and (5) where $\sigma > 1$, the claim follows. $\square$

Entry

Assume that entry requires a successful innovation, that the cost of innovation activity by a potential entrant is $c_1(\gamma)$, and that firm type is unknown to an entrepreneur prior to entry. The entry rate is the product $v = m\gamma_0$ where m is a given measure of entrepreneurs and γ_0 is the frequency with which any one of them creates new product. As the optimal innovation rate by a potential entrant maximizes expected value, equal to $\gamma_0 \sum_\tau \left[E_\tau V_0(\tilde{\mu}) + \Psi_\tau\right] \phi_\tau - c(\gamma_0)$ where ϕ_τ is the exogenous probability of being a type τ firm, the optimal choice is defined

$$v = m\gamma_0 = m \arg\max_{\gamma \geq 0} \left\{ \left(\sum_\tau \left[E_\tau V_0(\tilde{\mu}) + \Psi_\tau\right] \phi_\tau \right) \gamma - c_1(\gamma) \right\}. \tag{75}$$

Product Line Size and Product Distributions

The offer c.d.f. $F(X)$ is the fraction of vacancies posted by product lines with marginal match values less than or equal to X. As the number of vacancies posted by each product line depends on the number of worker employed, the size of the market, and the productivity of the line, one needs to calculate the distribution of employment over product lines of each type. Let $P_n(q, \alpha)$ represent the fraction of product lines with employment equal to n. As the flow out of the state $n = 0$ equal hires plus product destruction and the flow into the state is equal to the flow of newly created products,

$$\sum_{\tau} (v\phi_\tau + \gamma_\tau)\, \Gamma'_\tau(\mu) + \left(\delta_0 + d_n\,(\mu)\right) P_1\,(\mu) = \left[\delta_1 + h_n(\mu)\right] P_0(\mu),$$

where $\Gamma'_\tau(\mu)$ is the μ p.d.f.. Since only transition from $n-1$ to n, $n+1$ to n, and n to zero can occur in an instant, all the other measures satisfy the difference equation

$$\left(\delta_0 + d_{n+1}(\mu)\right)(n+1) P_{n+1}(\mu) + h_{n-1}(\mu) P_{n-1}(\mu) = \left[\delta_1 + h_n(\mu) + \left(\delta_0 + d_n(\mu)\right) n\right] P_n(\mu) \tag{76}$$

where the functions $d_n(\mu)$ and $h_n(\mu)$ are those defined by equations (12) and (15).

The measure of products supplied by the set of type τ firms evolves according to the law of motion, $\dot{K}_\tau = v\phi_\tau + \gamma_\tau K_\tau - \delta_1 K_\tau$ where v is the innovation rate of new entrants, ϕ_τ is the fraction of entrant who are of type τ, and γ_τ is the innovation rate of type τ firms per product line, and δ_1 is the product destruction rate. In other words, the net rate of change in the measure is equal to the sum of the flows of products supplied by new entrants and continuing firms respectively less the flow of product lines currently supplied by the type that are destroyed. Hence, in steady state, the measure of products supplied by type τ firms and the aggregate measure of products are

$$K_\tau = \frac{v\phi_\tau}{\delta_1 - \gamma_\tau} \text{ and } K = \sum_\tau K_\tau = \frac{\nu + \sum_\tau K_\tau \gamma_\tau}{\delta_1}. \tag{77}$$

Each firm's number of product lines k evolves according to a birth-death process where new product lines are added at rate $k\gamma_\tau$ and product lines are destroyed at rate $k\delta_1$. When a firm is born, it enters with one product. The product line dynamic is identical to the one in Klette and Kortum (2004). Denote by $M_{\tau,k}$ the mass of type τ incumbent firms that have k product lines. It evolves according to,

$$\begin{aligned} \dot{M}_{\tau,k} &= (k-1)\,\gamma_\tau M_{\tau,k-1} + (k+1)\,\delta_1 M_{\tau,k+1} - k\,(\gamma_\tau + \delta_1)\, M_{\tau,k},\ k = 2, \ldots \\ \dot{M}_{\tau,1} &= v\phi_\tau + 2\delta_1 M_{\tau,2} - (\gamma_\tau + \delta_1)\, M_{\tau,1}. \end{aligned}$$

Define $\sum_{k=1}^{\infty} M_{\tau,k} = M_{\tau}$ and denote by $m_{\tau,k} \equiv M_{\tau,k}/M_{\tau}$ the probability that a type τ incumbent has k product lines. As shown in Klette and Kortum (2004), in steady state $m_{\tau,k}$ is distributed according the logarithmic distribution with parameter γ_{τ}/δ_1,

$$m_{\tau,k} = \frac{\left(\frac{\gamma_{\tau}}{\delta_1}\right)^k}{\ln\left(\frac{\delta_1}{\delta_1 - \gamma_{\tau}}\right) k}. \tag{78}$$

By implication, the average type conditional firm size is,

$$E_{\tau}\left[k\right] = \frac{\frac{\gamma_{\tau}}{\delta_1 - \gamma_{\tau}}}{\ln\left(\frac{\delta_1}{\delta_1 - \gamma_{\tau}}\right)}. \tag{79}$$

The steady state mass of firms M_{τ} is given by,

$$M_{\tau} = \frac{\upsilon \phi_{\tau}}{\gamma_{\tau}} \ln\left(\frac{\delta_1}{\delta_1 - \gamma_{\tau}}\right) = \frac{K_{\tau}}{E_{\tau}\left[k\right]}. \tag{80}$$

Offer and Employment Distributions

Given the constructs in the previous section and the fact that the optimal number of vacancies posted depends only on the marginal value, the market steady state distribution of vacancies over match value offers is

$$F\left(X\right) = \frac{\sum_{\tau} K_{\tau} E_{\tau}\left[\sum_{n=0}^{\infty} \mathbb{I}\left[\Delta V_n\left(\tilde{\mu}\right) \le X\right] v_n\left(\tilde{\mu}\right) P_n\left(\tilde{\mu}\right)\right]}{\sum_{\tau} K_{\tau} E_{\tau}\left[\sum_{n=0}^{\infty} v_n\left(\tilde{\mu}\right) P_n\left(\tilde{\mu}\right)\right]}. \tag{81}$$

where $v_n(\mu)$ is the number of vacancies posted and $\Delta V_n(\mu)$ is the value of the marginal match in a product line characterized by μ when employment is n, $\mathbb{I}(\cdot)$ is the indicator function equal to unity if the argument is true and zero otherwise, K_{τ} is the number of products supplied by firms of type τ, and $E_{\tau}\{\cdot\}$ is the expectation operator taken with respect to μ distribution. Similarly, the distribution of employment over match values is given by

$$G\left(X\right) = \frac{\sum_{\tau} K_{\tau} E_{\tau}\left[\sum_{n=0}^{\infty} \mathbb{I}\left[\Delta V_n\left(\tilde{\mu}\right) \le X\right] n P_n\left(\tilde{\mu}\right)\right]}{\left(1-u\right) L}, \tag{82}$$

Market Tightness and Unemployment

Of course, the ratio of the aggregate vacancies to aggregate search effort, market tightness, is

$$\theta = \frac{\sum_{\tau} E_{\tau}\left\{\sum_{n=0}^{\infty} v_n\left(\tilde{\mu}\right) P_n\left(\tilde{\mu}\right)\right\} K_{\tau}}{\left(u + (a_1 + a_2)(1-u)\right) L}. \tag{83}$$

Finally, because unemployed workers find jobs at the rate $m(\theta)$ and lose them at rate equal to $\delta_0 + \delta_1$, the steady state unemployment rate, that which balances inflow and outflow, is the solution to

$$\frac{u}{1-u} = \frac{\delta_0 + \delta_1}{\delta_0 + \delta_1 + m(\theta)}. \tag{84}$$

Aggregate Output

Given the production function specified in equation (1) and the fact that input j is supplied in quantity $x(j) = q(j)n(j)$ where $q(j)$ is the productivity and $n(j)$ is employment of product j, final market output in steady state is produced at rate

$$Y = \frac{1}{K}\left[\sum_{\tau}\sum_{n=1}^{\infty} E_{\tau}\left\{\tilde{\mu} n^{\frac{\sigma-1}{\sigma}} P_n\left(\tilde{\mu}\right)\right\} K_{\tau}\right]^{\frac{\sigma}{\sigma-1}} \tag{85}$$

by the law of large numbers. Note that equation (1) yields the following upper bound on final good output

$$\overline{Y} = AL\overline{q} \tag{86}$$

which is equal to what output would be if the entire labor force were employed in a product line with productivity equal to the sup of the upper supports, $\overline{q}$, of the distributions of productivity conditional on type.

Simulation Algorithms

Simulating the match wage distribution

The sufficient statistic in order to understand the evolution of the worker's wage is (μ, n, A), that is the type and state of the worker's current product line along with his outside option. $\Delta S\left(\mu, n\right)$ is sufficient in order to determine the worker's acceptance rejection decisions of outside offers. However, the worker current match surplus also evolves according to the hiring and separation rates of the product line as defined in equations (12) and (15) which require (μ, n) for evaluation. In order to simulate the evolution of the worker's state, we combine the following independent Poisson processes:

- The product line may hire an additional worker, $h_n\left(\mu\right)$. In this case, the state transitions to $(\mu, n+1, A)$.

- The product line may lose a worker that is not the worker himself, $(n-1)(\delta_0 + s_n(\mu))$. In this case, the state transitions to $(\mu, n-1, A)$.

- The worker may be laid off, $\delta_0 + \delta_1$. In this case, the state transitions to unemployment.

- The worker may receive an outside offer, λ_1. Denote the outside offer by (μ', n'). If $\Delta S(\mu', n') > \Delta S(\mu, n)$ then the worker transitions to $(\mu', n', \Delta S(\mu, n))$. If $A < \Delta S(\mu', n') < \Delta S(\mu, n)$, then the worker transitions to $(\mu, n, \Delta S(\mu', n'))$. Otherwise the state does not change.

- The worker may be reallocated on the ladder at rate λ_2. In this case, the state transitions to $(\mu', n', 0)$.

An unemployed worker receives a job offer at rate λ_0. When that happens, the state transitions to $(\mu', n', 0)$.

For a given state (μ, n, A) determine the sum of all the hazards, $\lambda(\mu, n)$. The minimum of the independent poisson arrival times is exponentially distributed with parameter $\lambda(\mu, n)$. For a given monte carlo draw, $x \sim U[0,1]$, the duration is then found simply by using the inverse of the exponential CDF, $dur = -ln(1-x)/\lambda(\mu, n)$. The probability that poisson process i has the minimum duration is simply $\Pr(X_i = \min(X_1, \ldots, X_n)) = \lambda_i / (\lambda_1 + \cdots + \lambda_n)$. Hence, one monte carlo draw is used to determine the duration until one of the five events happen and another is used to determine which one of the events took place. If either an outside offer is drawn or a job-to-job reallocation happens, a third monte carlo draw is required to determine the new state.

In order to draw an offer from the offer distribution of (μ, n) simply run through all the product line types and states and determine the vacancy intensity associated with that particular (μ, n) pair. A vector is put together and the the vacancy intensities added up. This is a CDF and a monte carlo draw picks a (μ, n) pair by inverting this CDF.

Simulating firm dynamics

A firm's state and type is given by the tuple $(\vec{\mu}, \vec{n}, k, \tau)$ where k is the number of product lines and also the dimension of the $\vec{\mu}$ and $\vec{n}$ vectors which describe the type and state of each of the firm's product lines. Finally, τ describes the type of the firm which governs the creation rate of new products as well as the distribution of μ realizations. The formulation of the firm's state disregards information necessary to compute wages. In order to compute wages, one must keep track of each worker's outside option. In first cut, we avoid this complication and consequently, the firm dynamics statistics disregard wages.

In steady state, the number of product lines of a type τ incumbent is given by $m_{\tau,k}$ as defined in equation (78). It is consequently a straightforward matter to simulate the type conditional number of product lines given an initialization by steady state. The type distribution of each of the product lines is drawn from $\Gamma_\tau(\mu)$. In possible violation of the steady state initialization, we initialize the number of workers in each product line by the steady state distribution of $P_n(\mu)$ as defined in equation (76). The age distribution of the firm's product lines may differ from that of the overall product line population since it depends on the particular realization of product lines of that firm. To correct for this, we let the simulation run for a number periods prior to recording firm dynamics in order to let it run into steady state should the initialization be off in any significant way.

Thus, initialization is done by,

1. Firm type is determined by weighted sampling from the distribution of steady state firm types, M_τ defined in equation (80).

2. Conditional on the firm type draw, the number of products is sampled from the steady state product line distribution in equation (78). Specifically, this is done by inverting the CDF of the logarithmic distribution. Hence, for a particular monte carlo draw from the standardized uniform distribution $\tilde{x}$, the type conditional product realization draw is given by, $\tilde{k} = \min_k k \in N_+ | B(\gamma_\tau/\delta_1 | k+1, 0) \leq \ln(1-\gamma_\tau/\delta_1)(\tilde{x}-1)$, where $B(\cdot)$ is the incomplete beta distribution.

3. For each product line, the type of the product line is determined by weighted sampling from the $\Gamma_\tau(\mu)$ distribution.

4. Finally, for each product line, the labor force size is initialized by weighted sampling from

the steady state distribution $P_n(\mu)$ defined in equation (76).

Given the initialization, a firm is simulated forward in a manner similar to how simulation is performed for workers. Specifically, monte carlo draws determine durations until the next event occurs as a result of the many independent Poisson processes that the firm faces:

- At rate $k\delta_1$ a product line is destroyed. If this event occurs, a subsequent monte carlo draw determines which product line is destroyed.
- At rate $k\gamma_\tau$ a product line is created. If this event occurs, a subsequent monte carlo draw determines the μ realization of the new product line which starts with zero workers.
- Each of the product lines gain and lose workers according to the hire and separation poisson proceses,
  - at rate $n_j\left(\delta_0 + s_{n_j}(\mu_j)\right)$ product line j loses a worker. In this case, product line j moves from n_j workers to $n_j - 1$.
  - at rate $h_{n_j}(\mu_j)$ product line j gains a worker. In this case, product line j moves from n_j workers to $n_j + 1$.

Hence, define the combined arrival rate of events by,

$$\lambda(\vec{\mu}, \vec{n}, k, \tau) = k(\delta_1 + \gamma_\tau) + \sum_{j=1}^{k}\left[h_{n_j}(\mu_j) + n_j\left(\delta_0 + s_{n_j}(\mu_j)\right)\right].$$

The time until next event is exponentially distributed with parameter $\lambda(\vec{\mu}, \vec{n}, k, \tau)$. Conditional on arrival of an event, the probability of any one event is proportional to the the event's contibution to $\lambda(\vec{\mu}, \vec{n}, k, \tau)$.

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