Cash-in-the-Market Pricing in a Model with Money and Over-the-Counter Financial Markets

Fabrizio Mattesini
University of Rome, "Tor Vergata"

Ed Nosal
Federal Reserve Bank of Chicago

August 2011
This version: June 2014

Abstract

Entrepreneurs need cash to finance their investments. Since cash is costly to hold, entrepreneurs underinvest. If entrepreneurs are able to access secondary financial markets, then they can sell some of their less liquid assets for cash and invest at a higher level. When the secondary financial markets are over-the-counter (OTC), the amount of liquidity (cash) that is in the market affects asset prices. In particular, higher levels of liquidity lead to higher asset prices. Since the amount of cash in the market affects assets prices, asset prices can fluctuate over time even though asset fundamentals are unchanged. Bid and ask prices naturally arise in an OTC market. We find that the bid-ask spread is negatively correlated with asset returns—which is well documented in the literature—and positively correlated with inflation. Importantly, we also find that an increase in inflation *increases* returns and, hence, *decreases* asset prices.

1 Introduction

June 3, 2014

In the world of Modigliani and Miller, it is irrelevant how an entrepreneur finances investment, whether it be by cash, debt or equity. When there are frictions, however, a pecking order among the financing alternatives may emerge. For example, if an entrepreneur is better informed about his opportunities than the market, cash is the cheapest way to finance, followed by debt and then equity, (Myers and Majluf (1984)). In this paper, we appeal to other frictions—lack of commitment (Kiyotaki and Moore (2001)) and lack of record keeping (Kocherlakota (1998))—that imply the only feasible finance instrument is money.[1] Investment will be inefficiently low because it is constrained by an entrepreneur's money holdings.

[1] In Kiyotaki and Moore (2001), if the collateral constraint is sufficiently tight, then money emerges as a means of payment.

This follows since entrepreneurs accumulate money prior to investing and money is costly to hold. The existence of secondary financial markets may improve matters. In particular, if entrepreneurs can access financial markets prior to investing, then they can sell some of their (illiquid) non-monetary assets for money. Entrepreneurs can then undertake larger and more efficient investments. Once again, owing to frictions, secondary financial markets are not "perfectly competitive." If these markets are over-the-counter (along the lines of Duffie, Garleanu and Pederson (2005)), then asset prices may depend on the amount of liquidity in the market. A more liquid financial market may give rise to higher asset prices and more efficient investments by the entrepreneur. In this paper, we present and analyze a dynamic general equilibrium model that seeks to understand the basic interactions and relationships between real investments, assets prices and money.

The basic model is a standard one in monetary economics (Lagos and Wright (2005)). Money is essential and the framework is designed to address policy issues, such as the impact that a change in inflation has on economic activity. To this model, we add an over-the-counter (OTC) financial market. Following Duffie, Garleanu and Pederson (2005) and Lagos and Rocheteau (2009), the hallmark of the OTC financial market consists of: (1) a middleman—the dealer—who intermediates asset trade between 'buyers' and 'sellers' and (2) asset prices determined by bargaining. Since the dealer has bargaining power, bid and ask prices naturally arise. The sellers of assets are the fraction of entrepreneurs, σ, that have investment opportunities and want more money. The buyers are the fraction $1-\sigma$ that don't have the opportunities and don't need the money. In addition to bargaining powers of the various agents, asset prices also depend on the amount of liquidity (cash) in the OTC market, which equals the money holdings of the $1-\sigma$ entrepreneurs that do not have investment opportunities. As in Allen and Gale (2005, 2007), there is "cash-in-the-market" pricing in the sense that higher levels of cash/liquidity are associated with higher asset prices. In contrast to Allen and Gale, there actually is something that resembles "cash"—fiat money—in the market as opposed to a short term real asset that is simply labelled 'cash.'

Each period the fraction of entrepreneurs who get an investment opportunity, σ, is drawn from a probability distribution. Hence, the amount cash in the OTC market varies over time. This means that asset prices can fluctuate over time even though asset fundamentals are unchanging. The amount of cash-in-the-market has implications for the bid-ask spread: Higher levels of liquidity in the OTC market are associated with lower bid-ask spreads. Hence, changes in liquidity in the OTC market brought about by either changes in σ or changes in inflation will affect asset prices, asset returns and bid-ask spreads. Our model predicts a positive relationship between asset returns and bid-ask spreads, and this relationship is well documented in the literature, (e.g., Amihud and Mendelson (1986) and Amihud, Mendelson, Pedersen (2005)). Our model also predicts a *positive* relationship between inflation and asset returns or, equivalently, a negative relationship between inflation and asset prices. This prediction is consistent with the observation that periods of low inflation are usually associated with periods of high asset prices (Christiano, Ilut, Motto and Rostagno (2010)).

This paper mainly contributes to the literature on asset pricing in OTC markets that starts with Duffie, Garleanu and Pederson 2005. Lagos and Rocheteau (2009) relax their indivisible asset assumption. Both papers assume that assets can be purchased by "transferable utility," or that buyers purchase the asset by producing a numeraire good that enters the seller's utility function in an additive and linear manner. Hence, liquidity—the object that buys assets—can be interpreted as being potentially in infinite supply. This is unappealing. To limit the amount of liquidity, Geromichalos and Herrenbrueck (2012) and Lagos and Zhang (2013), like this paper, embed an OTC financial market in a monetary model. Geromichalos and Herrenbrueck (2012) have their buyers and sellers directly meeting (with some probability) in the OTC financial market—there is no dealer—which implies there is no bid-ask spread. In Lagos and Zhang (2013), buyers and sellers can either meet one another or a dealer in the OTC financial market. Dealers have access to a competitive interdealer market where they can off load their positions. In both of these papers, like this one, asset prices deviate from their fundamental values. However, neither of these papers features a phenomenon that resembles cash-in-the-market pricing as an equilibrium outcome. As well, a novel prediction of our model is that asset prices and inflation are negatively correlated. In Geromichalos and Herrenbrueck (2012) and Lagos and Zhang (2013), the effect of inflation on OTC asset prices is ambiguous. We attribute this to the differences between the OTC bargaining environments in the three papers.

The paper is organized as follows. The next section presents the model environment. In Section 3, the bargaining model is presented and analyzed. Section 4 derives agents' value functions and characterizes equilibrium. The relationship between liquidity and assets prices, returns and bid-ask spreads is explored in Section 5. Section 6 analyzes the effect that inflation has on assets prices, returns and bid-ask spreads. Section 7 concludes.

2 Environment

Time is discrete and continues forever. Each time period has 3 subperiods: a financial market, FM, followed by a decentralized real investment market, DM, and finally by a competitive rebalancing market, CM. There is a unit measure of infinitely-lived agents called *entrepreneurs* that participate in all subperiods/markets; a unit measure of infinitely-lived agents called investment good producers, or simply *producers*, that participate in the DM and CM; and an infinitely-lived agent called a *dealer* that participates only in the FM. Financial services are produced in the FM; real investment goods are produced in the DM; and a numeraire consumption good is produced in the CM.

There are 2 assets, money and a real asset. The quantity of money, M_t, grows at constant gross rate μ so that $M_{t+1} = \mu M_t$. New money is injected—$\mu > 1$—or withdrawn—$\mu < 1$—by lump-sum transfers $T_t = M_t(\mu - 1)$ in the CM. Let p_t^m represent the amount of the numeraire good that one unit of money buys in the CM of period t and $z_t = p_t^m m_t$ is real cash balances (measured in terms of the date t numeraire good). Since M_t grows at a constant rate, in the steady state $p_{t+1}^m M_{t+1} = p_t^m M_t$, which implies that $p_t^m / p_{t+1}^m = \mu$. Each real asset provides a single dividend payout of δ numeraire goods and then "dies."

Entrepreneurs need money to purchase real investment goods in the DM. To this end, they accumulate money balances in the CM. At the beginning of the FM, entrepreneurs learn if they get an investment opportunity in the subsequent DM. Those who get an opportunity would like to hold more money balances and less real assets, and those who do not, would like to hold less money balances and more real assets. A dealer operates an over-the-counter market that allows entrepreneurs to reallocate their money and real asset holdings. Entrepreneurs who have investment opportunities enter the DM and search for them. Those who are successful undertake the investment and consume the investment payoff. All entrepreneurs and producers re-enter the CM to consume and rebalance their money holdings, and so on.

More formally, in period $t - 1$, entrepreneurs exit the CM with m_t units of money and ownership of a_{t-1} units of the real asset. At the beginning of period t, entrepreneurs enter the FM, and the following sequence of events occur:

- each unit of real assets pays a dividend of δ, which the asset owner consumes;
- all entrepreneurs receive ownership of $\bar{a}$ new assets, where each asset pays a single

dividend payment δ at the beginning of period $t+1$;

- entrepreneurs learn if they get investment opportunities in the subsequent DM. A fraction $\sigma \in [0,1]$ of entrepreneurs get an investment opportunity and a fraction $1-\sigma$ do not.

- all entrepreneurs contact the dealer;

- the dealer intermediates asset trade between entrepreneurs who have investment opportunities and those who don't, where asset ownership is exchanged for money.

As in Lagos and Rocheteau (2009), we assume that entrepreneurs can only buy and sell assets in the FM through a dealer.[2] In the FM and DM we must distinguish between the measure σ of entrepreneurs who get investment opportunities (and want to sell assets) and the measure $1-\sigma$ of entrepreneurs who do not (and want to buy assets). Call the former *investors* and the latter *liquidity providers.* The dealer bargains with investors and liquidity providers to determine the prices at which assets are bought and sold, as well as the quantities of assets that are bought and sold. (The bargaining environment is described in Section 3.) The dealer buys τ_t^b assets from an investor at a unit price of p_t^b, where p_t^b is the amount of assets that the dealer purchases per unit of money.[3] One can interpret $p_t^m/p_t^b \equiv P_t^b$ as the real *bid price*—the amount of real balances the dealer needs to buy one unit of the asset. The dealer sells τ_t^a assets to a liquidity provider at a unit price of p_t^a, where p_t^a is the amount of assets that a liquidity provider receives per unit of money. One can interpret $p_t^m/p_t^a \equiv P_t^a$ as the real *ask price*—the amount of real balances obtained by the dealer for one unit of the asset.

Investors enter the DM of period t and are randomly matched with producers.[4] The probability that an investor is matched with a producer is σ_D. Matched investors and producers bargain over the amount of the investment good, q_t, the producer exchanges for the investor's real balances, τ_t^m. The cost of producing q_t units of the investment good is $c(q_t)$. For simplicity we assume that $c(q_t) = q_t$. The investor invests the good and consumes the output of the investment, $q_t R$, $R \geq 1$, which he values as $u(q_t R)$, where $u' > 0$, $u'' < 0$, $u'(0) = \infty$ and $u'(\infty) = 0$. The investment output is perishable and is realized in the DM

[2] Duffie, Garleanu and Pedersen (2005) assume that entrepreneurs can also directly meet one another and trade.

[3] Both Lagos and Rocheteau (2009) and Duffie, Garleanu and Pedersen (2005) are non-monetary models, where non-investors purchase assets with with transferable utility.

[4] Liquidity providers do not enter the DM since they do not have any investment opportunities.

that the investment was undertaken. We assume that the investor makes a take-it-or-leave-it offer to the producer, which means that the investor can extract all of the match surplus, where the match surplus is given by $u(qR) - q$. Match surplus is maximized at q^*, where q^* solves $Ru'(q^*R) = 1$. For convenience, and without loss of generality, we assume that $R = 1$.

When entrepreneurs enter the CM of period t they do not know if they will get investment opportunities in the DM of period $t+1$. They do know, however, that in period $t+1$ a fraction σ of them will get investment opportunities in the subsequent DM, where σ is independently and identically distributed over time with probability distribution function is $f(\sigma)$ and $\int_0^1 f(\sigma) = 1$. The σ realization is public information in the FM. The ex post probability that any particular entrepreneur gets an investment opportunity in the next DM is σ. Entrepreneurs can produce and consume in the CM. Let x_t represent the CM numeraire good CM and ℓ_t the amount labor used to produce it. One unit of labor produces one unit of the numeraire good. An entrepreneur's utility of consuming x_t units of the numeraire good is linear and equal to x_t; his disutility of labor is linear and equal to ℓ_t. Hence, the entrepreneur's preferences over the consumption and production of the numeraire good is $x_t - \ell_t$. The numeraire good is perishable.

The producer is only active in the DM and CM. In the CM the producer can consume but cannot produce. The utility associated with consuming x_t units of the CM good is linear and equal to x_t. Hence, the producer's preferences in period t are described by $x_t - q_t$, where q_t is the amount of the investment good produced in the DM. Given the DM bargaining protocol—the investor makes a take-it-or-leave-it offer to the producer—all producers, whether they are matched or unmatched, get utility equal to zero in each period t. (A matched producer receives money in the DM which he uses to buy the numeraire good in the CM.)

The dealer is only active in the FM. The dealer's preferences are linear in dividends and, hence, his objective is to maximize real asset holdings in each period.

Real assets "reside" in the FM in the sense that entrepreneurs do not carry the real assets into the DM and CM. All agents are anonymous in the DM and CM. In any subperiod, entrepreneurs cannot commit to promises made in previous subperiods. In the FM there exists a "book entry system" that verifies agents' identities and documents asset ownership—who owns how many assets. The book entry system can issue a physical claim regarding asset ownership to entrepreneurs, but these claims can be costlessly counterfeited. This, coupled with an absence of record keeping for agents in the DM and CM subperiods and entrepreneurs' inability to commit, implies that only money will be used as a medium of exchange in those subperiods. (Money cannot be counterfeited.) This is why entrepreneurs value and accumulate money balances in the CM, and why producers are willing to accept it in the DM. Since the book entry system can only record asset ownership,[5] money is also needed as a medium of exchange to buy assets in the FM.

3 Bargaining in the FM

Bargaining between the dealer, investors and liquidity providers occurs in two distinct stages. In the first stage, the dealer and σ investors are matched and bargain over the bid price and quantity, $\left(P^b, \tau^b\right)$; in the second stage, the dealer and $1-\sigma$ liquidity providers are matched and bargain over the ask price and ask quantity, (P^a, τ^a). The dealer has a "family" in $[0,1]$, where family members do all the bargaining and act to maximize the real asset holdings of the dealer. The family interpretation allows σ dealer family members to be matched one-on-one with the σ investors in the first stage and the remaining $1-\sigma$ family members to be matched one-on-one with the $1-\sigma$ liquidity providers in the second stage. When it leads to no confusion, we will refer to a dealer family member as simply a dealer (since, after all, the family member represents the interests of the dealer).

The bargaining outcome in each match—$\left(P^b, \tau^b\right)$ in the first stage investor-dealer match and (P^a, τ^a) in the second stage liquidity provider-dealer match—is determined by the (Kalai) proportional bargaining solution. In the first stage, an investor receives a θ share of the investor-dealer match surplus and the dealer receives a $1-\theta$ share. Similarly, in the second stage, the liquidity provider receives θ of the liquidity provider-dealer match surplus, while the dealer gets $1-\theta$.

The dealer or his family members cannot observe the money holdings of investors or liquidity providers prior to being matched. Once a dealer family member is matched, he can observe the real balances of the agent in his match—either an investor in the first stage or an liquidity provider in the second stage—but cannot observe real balances of investors or liquidity providers in other matches. Similarly, an investor or liquidity provider cannot observe the money holdings of other investors or liquidity providers.

In the bargaining game, it will be helpful to distinguish between real balances held by

[5] It cannot not, for example, document owners and issuers of IOU's.

an investor, z, and real balances held by a liquidity provider, $\hat{z}$. We assume that $z < q^*$. If $z \geq q^*$, then the bargaining problem is irrelevant: the investor has sufficient real balances to extract the maximum surplus in the DM if he is matched and, as a result, has no (strict) incentive to sell his assets in the FM.[6]

In a first-stage investor-family member dealer match, the investor and family dealer member must form expectations regarding the amount of real balances that will be supplied to their match by liquidity providers (at the second stage). The dealer allocates the same (expected) level of real balances from liquidity providers to each investor-family dealer match. Investors and family dealers member expect that each liquidity provider has $\hat{z}$ real balances. Hence, the dealer allocates $\hat{z}(1-\sigma)/\sigma$ real balances to each investor-family dealer match. Suppose that in a first stage investor-dealer match, the negotiated bid price and bid quantity is (P^b, τ^b). Notice that the family member dealer cannot *pay* for the τ^b assets in the first stage, as this is not feasible—the family member dealer does not have any real balances at this time. Instead, in the first stage the family member dealer buys the assets from the investor on consignment, as in Rubinstein and Wolinsky (1986, 589-591). The "contract" (P^b, τ^b) is financed in the second stage by the family member dealers that bargain with liquidity providers. Specifically, in the second stage the family member dealers will, in equilibrium, obtain $P^b\tau^b$ real balances from liquidity providers in exchange for assets. Upon receiving these real balances, the dealer transfers them to the investor and thus fulfils the first stage contract (P^b, τ^b). We assume that the dealer can commit to pay back investors and that the first stage bargain (P^b, τ^b) is not subject to renegotiation. If the dealer does not receive $P^b\tau^b$ real balances from liquidity providers—because, for example, liquidity providers that fund the bid contract (P^b, τ^b) accumulated less less real balances than was expected—then the bid contract (P^b, τ^b) specifies that the dealer buy as many of the investors assets that he can at the unit price P^b and return those assets that cannot be financed.[7]

In each of the σ first stage investor-dealer matches, a bid price and bid quantity (P^b, τ^b) is negotiated. At the beginning of the second stage, before liquidity providers and family member dealers are matched, these contracts, and assets associated with them, are divided among the $1-\sigma$ family members who will bargain with the liquidity providers. The contracts and assets are distributed among the second stage family dealers so that each bid contract

[6] If the growth rate of money is greater than that given by the Friedman Rule, i.e., $\mu > \beta$, then $z < q^*$.

[7] The dealer never has an incentive to "hold back" real balances for himself and return assets since he values the asset's dividends and does not value real balances.

will be allocated to $\hat{z}(1-\sigma)/\sigma$ real balances.[8] The surplus to be split in the second stage *per unit* of a bid contract, i.e., for contract $(P^b, 1)$, is $\beta\delta - P^b$. Since the dealer must pay P^b for each asset that he buys from the investor and the value of the asset is $\beta\delta$, the surplus per asset purchased by the dealer is $\beta\delta - P^b$. (Recall that an asset pays a dividend δ at the beginning of the next FM.) The equilibrium match surplus that a liquidity provider and dealer generate (per unit of a bid contract) can be expressed as

$$\theta\left(\beta\delta - P^b\right) = \left(\beta\delta - P^a\right)\tau_1^a \tag{1}$$

and

$$(1-\theta)\left(\beta\delta - P^b\right) = \beta\delta\left(1 - \frac{P^b}{P^a}\right), \tag{2}$$

respectively, where τ_1^a represents the amount of asset that the liquidity provider purchases per unit of a bid contract. The left side of (1) is a liquidity provider's share of the match surplus and the right side expresses this surplus in terms of an ask price and ask quantity. The left side of (2) is the dealer's share of the match surplus. The dealer keeps $1-\tau_1^a$ of the assets per unit that an investor sells, resulting in a payoff of $\beta\delta(1-\tau_1^a)$ to the dealer. Since real balances received by investors equals real balances given up by liquidity providers, i.e., $P^b = P^a\tau_1^a$, the dealer's payoff per asset purchased can also be expressed as the right side of (2).

The equilibrium ask price is obtained by rearranging (2),

$$P^a = \frac{P^b\beta\delta}{\theta\beta\delta + (1-\theta)P^b}, \tag{3}$$

and the equilibrium ask quantity per unit that the dealer buys is obtained by rearranging (1) so that we get

$$\tau_1^a = \frac{\theta\left(\beta\delta - P^b\right)}{\beta\delta - P^a}$$

or

$$\tau_1^a = \frac{\theta\beta\delta + (1-\theta)P^b}{\beta\delta}. \tag{4}$$

Notice that the ask price, P^a, does not depend on the amount of assets that the dealer purchased in the first stage bargaining.

Now let's move to the first stage bargain between the investor and family member dealer. The amount of real balances that an investor receives in the FM from selling assets, $P^b\tau^b$,

[8] For example, if $\sigma = 0.5$, then each second stage family dealer is given one bid contract; if $\sigma = 0.75$, then each second stage family dealer is given 3 bid contracts; and if $\sigma = 0.25$, each second stage family dealer is given 1/3 of a bid contract.

equals $\min\left\{\tilde{z}, \min\{P^b\bar{a}, q^* - z\}\right\}$, where $\tilde{z} = \hat{z}(1-\sigma)/\sigma$ is the total amount of liquidity providers' real balances per investor or the maximum amount of real balances that the dealer can transfer to the investor in exchange for his assets. If

$$\min\left\{\tilde{z}, \min\{P^b\bar{a}, q^* - z\}\right\} = q^* - z,$$

then we say that the investor is *unconstrained* in the FM because he is able to purchase q^* in the DM if he is matched. If the investor is unconstrained, then he has sufficient assets to obtain real balances equal to $q^* - z$, i.e., $P^b\bar{a} \geq q^* - z$, *and* the dealer has sufficient real balances—provided by $(1-\sigma)/\sigma$ liquidity providers—to purchase the assets, $\tilde{z} \geq q^* - z$. If

$$\min\left\{\tilde{z}, \min\{P^b\bar{a}, q^* - z\}\right\} = P^b\bar{a},$$

then we say the investor is *asset constrained* in the FM because his asset holdings, along with his own real balance holdings, are insufficient to purchase output q^* in the DM if he is matched. If the investor is asset constrained, then he sells all of his assets in the FM, which implies that $P^b\bar{a} < \tilde{z}$, but is unable to purchase q^* in the DM if he is matched because $P^b\bar{a} + z < q^*$. If

$$\min\left\{\tilde{z}, \min\{P^b\bar{a}, q^* - z\}\right\} = \tilde{z},$$

then we say the investor is *liquidity constrained* in the FM because he would like to sell more assets so the he is able to purchase more DM output $q < q^*$ if he is matched. If the investor is liquidity constrained, then he sells assets and gets all of the real balances available from the dealer, $\tilde{z} < \min\{P^b\bar{a}, q^* - z\}$, but is unable to purchase q^* in the DM if he is matched because $\tilde{z} + z < q^*$. For convenience we will say that $\bar{a}$ is "large" if $P^b\bar{a} \geq q^* - z$ and that $\bar{a}$ is "small" if $P^b\bar{a} < q^* - z$. Note that if $\bar{a}$ is large, then an investor can never be asset constrained and if $\bar{a}$ is small, then he can never be unconstrained.

The surplus the investor receives from selling τ^b assets at price P^b, where $P^b\tau^b = \min\left\{\tilde{z}, \min\{P^b\bar{a}, q^* - z\}\right\}$,[9] is

$$\sigma_D u\left(z + \min\left\{\tilde{z}, \min\{P^b\bar{a}, q^* - z\}\right\}\right) + (1-\sigma_D)\left(z + \min\left\{\tilde{z}, \min\{P^b\bar{a}, q^* - z\}\right\}\right) - \beta\delta\tau^b$$
$$- \left[\sigma_D u(z) + (1-\sigma_D) z\right].$$

The second line is (minus) the expected DM payoff that the investor receives if he does not sell any assets in the FM; the first line is the expected DM payoff if he sells assets in the FM

[9] It should be clear that if $P^b\tau^b < \min\left\{\tilde{z}, \min\{P^b\bar{a}, q^* - z\}\right\}$, then agents are leaving surplus on the table and this is not consistent with equilibrium.

minus the loss in the value of dividends associated with selling τ^b in the FM. The investor's surplus is, by construction, equal to θ times the match surplus. Recall that the surplus that the dealer obtains from this match is equal to $\left(\beta\delta - P^b\right)\tau^b$, which represents $1-\theta$ times the match surplus, i.e., match surplus is equal to $\left(\beta\delta - P^b\right)\tau^b/(1-\theta)$. Therefore, we have

$$\sigma_D[u\left(z+\min\left\{\tilde{z},\min\{P^b\bar{a},q^*-z\}\right\}\right)-u(z)]+(1-\sigma_D)\min\left\{\tilde{z},\min\{P^b\bar{a},q^*-z\}\right\}-\beta\delta\tau^b \\ =\frac{\theta}{1-\theta}\left(\beta\delta-P^b\right)\tau^b. \quad (5)$$

Suppose first that $\bar{a}$ is large, i.e., $P^b\bar{a} \geq q^*-z$. In this case, (5) is one equation in one unknown, τ^b, and, as a result, the first-stage bargaining solution (P^b,τ^b) is given by

$$P^b = \left(\tau^b\right)^{-1}\min\left\{q^*-z,\tilde{z}\right\} \quad (6)$$

and

$$\tau^b = \begin{cases} \{\theta\left(q^*-z\right)+(1-\theta)\left[\sigma_D\left[u\left(q^*\right)-u\left(z\right)\right]+\right. \\ \left.\left(1-\sigma_D\right)\left(q^*-z\right)\right]\}/\beta\delta & \text{if } q^*-z\leq\tilde{z} \\ \{\theta\tilde{z}+(1-\theta)\left[\sigma_D\left[u\left(z+\tilde{z}\right)-u\left(z\right)\right]+\right. \\ \left.\left(1-\sigma_D\right)\tilde{z}\right]\}/\beta\delta & \text{if } q^*-z>\tilde{z} \end{cases}. \quad (7)$$

Now suppose that $\bar{a}$ is small, i.e., $P^b\bar{a} < q^*-z$. If the investor turns out to be liquidity constrained, $P^b\bar{a} > \tilde{z}$, then the stage 1 bargaining solution, (P^b,τ^b), is $P^b=\tilde{z}/\tau^b$ and τ^b is given by the the lower branch of (7). If the investor turns out to be asset constrained, i.e., $\tilde{z} > P^b\bar{a}$, then, clearly, $\tau^b=\bar{a}$. To determine the bid price, we must describe the surplus the investor receives in the first stage bargain with the dealer, which is

$$\sigma_D u\left(z+P^b\bar{a}\right)+\left(1-\sigma_D\right)\left(z+P^b\bar{a}\right)-\beta\delta\bar{a} \\ -\left[\sigma_D u\left(z\right)+\left(1-\sigma_D\right)z\right].$$

The equilibrium bid price, which we denote by $\tilde{P}^b$, solves

$$\sigma_D u(z+\tilde{P}^b\bar{a})+\left(1-\sigma_D\right)\left(z+\tilde{P}^b\bar{a}\right)-\beta\delta\bar{a} \\ -\left[\sigma_D u\left(z\right)+\left(1-\sigma_D\right)z\right]=\frac{\theta}{1-\theta}(\beta\delta-\tilde{P}^b)\bar{a}. \quad (8)$$

The left side of (8) is the investor's surplus and the right side is the investor's share of the match surplus. Therefore, the bargaining solution to the first stage bargaining problem when the investor is asset constrained is given by $P^b=\tilde{P}^b$ and $\tau^b=\bar{a}$, where $\tilde{P}^b$ is given (implicitly) by (8).

In summary, *assuming that an investor holds* z *real balances, each liquidity provider holds* $\hat{z}$ *real balances and the fraction of entrepreneurs that are investors in the FM is* σ, the first-stage bargaining solution $(P^b(z,\hat{z},\sigma), \tau^b(z,\hat{z},\sigma))$ is given by

$$P^b(z,\hat{z},\sigma) = \begin{cases} (q^* - z)/\tau^b & \text{if } P^b\bar{a} \geq q^* - z \leq \tilde{z} \\ \tilde{z}/\tau^b & \text{if } \min\{P^b\bar{a}, q^* - z\} > \tilde{z} \\ \tilde{P}^b(z,\sigma) & \text{if } P^b\bar{a} < q^* - z \text{ and } P^b\bar{a} < \tilde{z} \end{cases}, \tag{9}$$

where $\tilde{P}^b(z,\sigma)$ solves (8) and

$$\tau^b(z,\hat{z},\sigma) = \begin{cases} \begin{array}{l} \{\theta\,(q^* - z) + (1-\theta)\,[\sigma_D\,[u\,(q^*) - u\,(z)] + \\ (1-\sigma_D)\,(q^* - z)\,]\}/\beta\delta \end{array} & \text{if } P^b\bar{a} \geq q^* - z \leq \tilde{z} \\ \begin{array}{l} \{\theta\tilde{z} + (1-\theta)\,[\sigma_D\,[u\,(z+\tilde{z}) - u\,(z)] + \\ (1-\sigma_D)\,\tilde{z}]\}/\beta\delta \end{array} & \text{if } \min\{P^b\bar{a}, q^* - z\} > \tilde{z} \\ \bar{a} & \text{if } P^b\bar{a} < q^* - z \text{ and } P^b\bar{a} < \tilde{z} \end{cases}. \tag{10}$$

In addition, assuming that each investor holds real balances equal to z, using (3) and (4), the second-stage bargaining solution $(P^a(z,\hat{z},\sigma), \tau^a(z,\hat{z},\sigma))$ is given by

$$P^a(z,\hat{z},\sigma) = \frac{P^b(z,\hat{z},\sigma)\beta\delta}{\theta\beta\delta + (1-\theta)\,P^b(z,\hat{z},\sigma)}, \tag{11}$$

and

$$\tau^a(z,\hat{z},\sigma) = \frac{\theta\beta\delta + (1-\theta)\,P^b(z,\hat{z},\sigma)}{\beta\delta}\tau^b(z,\hat{z},\sigma)\sigma/\,(1-\sigma)\,. \tag{12}$$

Notice that (4) describes the amount of assets that liquidity providers purchase *per unit* asset that the dealer buys. Since family member dealers brings in $\tau^b(z,\hat{z},\sigma)\sigma/\,(1-\sigma)$ assets into the second stage match, the amount of assets purchased by a liquidity provider is given by (12).[10]

[10] When the investor and dealer bargain they both believe that each liquidity provider is holding $\hat{z}$ real balances. Note that the ask price, P^a, is independent of the *actual* real balances that the dealer secures in the second stage bargain. This is because the ask price, which is determined in the second bargaining stage, is a function of the bid price—see (3)—and the bid price is determined in the first bargaining stage. Hence, the bid price is a "predetermined" variable in the second stage, and by (3), so is the ask price. Of course, the bid price is not independent of (some measure of) the real balances; it depends on the liquidity providers' *equilibrium* real balance holdings per consumer, $\tilde{z} = \hat{z}(1-\sigma)/\sigma$. That is, (to reemphasize) the dealer and investor bargain under the belief that liquidity providers are holding the equilibrium level of real balances, $\hat{z}$. If a liquidity provider accumulates more than the equilibrium real balances, then in the second stage liquidity provider-dealer bargaining match, the liquidity provider will purchase τ^a assets, given by (4), at the price P^a. That is, the liquidity provider does not purchase any additional assets with his "extra" real balances because the ask price is essentially predetermined and the dealer brings in $\tau^b\sigma/(1-\sigma)$ real assets to the match. If a liquidity provider accumulates less than the equilibrium real balances, say $\bar{z} < \hat{z}$, then in the stage 2 dealer-liquidity provider match the liquidity provider purchases $\bar{z}/P^a$ assets, which is less than the equilibrium amount.

4 Equilibrium

The analysis so far assumes that the entrepreneur's real balance holdings are z if he is an investor z and $\hat{z}$ if he is a liquidity provider. Real balances, however, are a choice variable for the entrepreneur in the CM. In this section, we first derive the entrepreneur's CM real money demand function. Then we characterize the symmetric steady-state equilibrium for the economy.

In the CM of period t, an entrepreneur decides on the amount of real balances that he will hold in period $t+1$, z_{t+1}. This decision depends on the level of real balances that the other entrepreneurs are accumulating. In his decision problem, the entrepreneur believes that all other entrepreneurs will accumulate real balances equal to $\hat{z}_{t+1}$.[11] The entrepreneur's real money demand function can be deduced from his value function in the CM. The value function for an entrepreneur in CM, W, of period t is,

$$W\left(z_t, a_t\right) = \max_{z_{t+1}} \{x_t - \ell_t + \beta \int_\sigma \left[\sigma Z_1\left(z_{t+1}, a_t\right) + (1-\sigma) Z_0\left(z_{t+1}, a_t\right)\right] f(\sigma) d\sigma\}$$
$$\text{s.t. } x_t + \mu z_{t+1} = \ell_t + z_t + T_t$$

or

$$\begin{aligned} W\left(z_t, a_t\right) &= z_t + T_t + \max_{z_{t+1}} \{-\mu z_{t+1} + \beta \int_\sigma [\sigma Z_1\left(z_{t+1}, a_t\right) + \\ &\quad (1-\sigma) Z_0\left(z_{t+1}, a_t\right)] f(\sigma) d\sigma\}, \end{aligned} \tag{13}$$

where Z_1 is the value function of a entrepreneur that becomes an investor in the DM and Z_0 is FM value function of a entrepreneur that becomes a liquidity provider.

The value functions in the FM for a given period, $t+1$, and realization, σ, are,

$$Z_1\left(z_{t+1}, a_t\right) = \delta a_t + V_1(z_{t+1} + P^b_{t+1}(z_{t+1}, \hat{z}_{t+1}, \sigma)\tau^b_{t+1}(z_{t+1}, \hat{z}_{t+1}, \sigma), \bar{a} - \tau^b_{t+1}(z_{t+1}, \hat{z}_{t+1}, \sigma)) \tag{14}$$

and

$$Z_0\left(z_{t+1}, a_t\right) = \delta a_t + V_0\left(z_{t+1} - P^a_{t+1}(\hat{z}_{t+1}, z_{t+1}, \sigma)\tau^a_{t+1}(\hat{z}_{t+1}, z_{t+1}, \sigma), \bar{a} + \tau^a_{t+1}(\hat{z}_{t+1}, z_{t+1}, \sigma)\right), \tag{15}$$

[11] Notice the slight change in notation. In the previous section $\hat{z}$ represented the real balances of liquidity providers. Here $\hat{z}$ represents the real balances held by *all other* entrepreneurs. When an entrepreneur makes his real balance decision, he must form expectations over what other entrepreneurs are doing. When we focus our analysis on symmetric steady state equilibrium, $\hat{z}$ will represent the proposed equilibrium real balances for entrepreneurs.

where V_1 is the value function of the investor in the DM and V_0 is the liquidity provider's DM value function. The first argument in the bid or ask variables represents the real balances of the investor and the second argument represents the real balances of the liquidity provider. Since the entrepreneur is an investor for bid variables and a liquidity provider for ask variables, the "ordering" of z_{t+1} and $\hat{z}_{t+1}$ will differ for bid and ask variables, see above. To reduce notational burden, we will define $P^{b,\sigma}_{t+1} \equiv P^{b}_{t+1}(z_{t+1}, \hat{z}_{t+1}, \sigma)$, $\tau^{b,\sigma}_{t+1} \equiv \tau^{b}_{t+1}(z_{t+1}, \hat{z}_{t+1}, \sigma)$, $P^{a,\sigma}_{t+1} \equiv P^{a}_{t+1}(\hat{z}_{t+1}, z_{t+1}, \sigma)$ and $\tau^{a,\sigma}_{t+1} \equiv \tau^{a}_{t+1}(\hat{z}_{t+1}, z_{t+1}, \sigma)$. In the FM, the investor sells $\tau^{b,\sigma}_{t+1}$ assets and receives $P^{b,\sigma}_{t+1}\tau^{b,\sigma}_{t+1}$ real balances; the liquidity provider buys $\tau^{a,\sigma}_{t+1}$ assets in exchange for $P^{a,\sigma}_{t+1}\tau^{a,\sigma}_{t+1}$ real balances. Plugging (14) and (15) into (13) we get,

$$\begin{aligned} W_t\left(z_t, a_t\right) &= z_t + \beta\delta a_t + T_t \\ &\max_{z_{t+1}}\{-\mu z_{t+1} + \beta\int_{\sigma}[\sigma V_1(z_{t+1} + P^{b,\sigma}_{t+1}\tau^{b,\sigma}_{t+1}, \bar{a} - \tau^{b,\sigma}_{t+1}) + \\ &(1-\sigma)\, V_0(z_{t+1} - P^{a,\sigma}_{t+1}\tau^{a,\sigma}_{t+1}, \bar{a} + \tau^{a,\sigma}_{t+1})]f(\sigma)d\sigma\}. \end{aligned} \tag{16}$$

If an investor is matched in the DM, then he purchases the DM good from a producer. The investor transfers τ^{m}_{t+1} units of real balances for q_{t+1} units of the investor good in the DM of period t. The value functions in the DM of period $t+1$ are

$$\begin{aligned} V_1\left(z_{t+1}, a_{t+1}\right) &= \sigma_D\left[u\left(q_{t+1}\right) + W\left(z_{t+1} - \tau^{m}_{t+1}, a_{t+1}\right)\right] + \\ &\quad (1-\sigma_D)\, W\left(z_{t+1}, a_{t+1}\right) \\ &= \sigma_D\left[u\left(q_{t+1}\right) - q_{t+1}\right] + W\left(z_{t+1}, a_{t+1}\right) \end{aligned} \tag{17}$$

and

$$V_0\left(z_{t+1}, a_{t+1}\right) = W\left(z_{t+1}, a_{t+1}\right). \tag{18}$$

Recall that $\tau^{m}_{t+1} = q_{t+1}$ since the investor makes a take-it-or-leave-it offer to the producer. Plugging (17) and (18) into (16), we get

$$\begin{aligned} W_t\left(z_t, a_t\right) &= z_t + \beta\delta a_t + T_t + \max_{z_{t+1}}\{-\mu z_{t+1} + \beta\int_{\sigma}[\sigma\sigma_D(u(q^{\sigma}_{t+1}) - q^{\sigma}_{t+1}) + \\ &\sigma W(z_{t+1} + P^{b,\sigma}_{t+1}\tau^{b,\sigma}_{t+1}, \bar{a} - \tau^{b,\sigma}_{t+1}) + \\ &(1-\sigma)\, W(z_{t+1} - P^{a,\sigma}_{t+1}\tau^{a,\sigma}_{t+1}, \bar{a} + \tau^{a,\sigma}_{t+1})]f(\sigma)d\sigma\}, \end{aligned} \tag{19}$$

where

$$q^{\sigma}_{t+1} = \min\{z_{t+1} + P^{b,\sigma}_{t}\tau^{b,\sigma}_{t+1}, q^{*}\}, \tag{20}$$

and q^* solves $u'(q^*) = 1$. Linearity of $W(z,a)$ implies that (19) can be simplified to,

$$\begin{aligned} W_t(z_t, a_t) &= z_t + T + \beta\delta a_t + \beta^2\delta\bar{a} + \beta W(0,0) + \\ &\max_{z_{t+1}}\{ -\mu z_{t+1} + \beta z_{t+1} + \beta\int_\sigma \{(\sigma P^{b,\sigma}_{t+1}\tau^{b,\sigma}_{t1} - (1-\sigma) P^{a,\sigma}_{t+1}\tau^{a,\sigma}_{t+1}) \\ &+\beta\delta[(1-\sigma)\tau^{a,\sigma}_{t+1} - \sigma\tau^{b,\sigma}_{t+1}] + \sigma\sigma_D[u(q^\sigma_{t+1}) - q^\sigma_{t+1}]f(\sigma)d\sigma\}\}. \end{aligned} \tag{21}$$

Notice that the entrepreneur's CM period t decision problem—the max term in (21)—does not depend on the amount of real balances, z_t, he brings into the CM. Hence, all entrepreneurs, independent of their history, face the identical CM decision problem. We focus our analysis on a symmetric steady-state equilibrium.

The entrepreneur's CM real balance decision is given by the solution to

$$\begin{aligned} &\max_{z_{t+1}}\{-\mu z_{t+1} + \beta z_{t+1} + \beta\int_\sigma [(\sigma P^{b,\sigma}_{t+1}\tau^{b,\sigma}_{t+1} - (1-\sigma) P^{a,\sigma}_{t+1}\tau^{a,\sigma}_{t+1}] \\ &+\beta\delta[(1-\sigma)\tau^{a,\sigma}_{t+1} - \sigma\tau^{b,\sigma}_{t+1}] + \sigma\sigma_D[u(q^\sigma_{t+1}) - q^\sigma_{t+1}]f(\sigma)\,d\sigma\}. \end{aligned} \tag{22}$$

The first two terms are standard and represent the date t cost and the date $t+1$ benefit of accumulating z_{t+1} real balances in the CM of period t. The first term of the integral expression of (22),

$$\int_\sigma [\sigma P^{b,\sigma}_{t+1}\tau^{b,\sigma}_{t+1} - (1-\sigma) P^{a,\sigma}_{t+1}\tau^{a,\sigma}_{t+1}]f(\sigma)\,d\sigma. \tag{23}$$

represents the expected transfer of real balances to the entrepreneur (when he is an investor) from the entrepreneur (when he is a liquidity provider) in the FM. The entrepreneur is interested in knowing how a change in his real balances affects the amount of real balances he can obtain in the FM, $P^b_t\tau^b_t$, at the margin when he is an investor and the amount of that he pays for assets in the FM, $P^a_t\tau^a_t$, at the margin when he is a liquidity provider. Although, in equilibrium, $\sigma P^b_t\tau^b_t = (1-\sigma)P^a_t\tau^a_t$, a change in the right side, brought about by a change in real balance holdings, need not equal the change in the left side.

Suppose the entrepreneur marginally increases his holdings of real balances in the CM of period t from the equilibrium level. If the entrepreneur turns out to be a investor, then he has more real balances of his own to spend in the DM. The higher real balances changes the solution to the bargaining problem since the bargaining surplus at the margin decreases and this will affect the cash, $P^b_{t+1}\tau^b_{t+1}$, that he receives and assets that he sells, τ^b, at the margin. Similarly, if the investor turns out to be a liquidity provider, a change in real balances *may* affect the amount of assets he buys, τ^a, and what he pays for them, $P^a\tau^a$, at the margin.

The second term on the second line of (22), i.e.,

$$\beta\delta\int_{\sigma}[(1-\sigma)\,\tau^{a,\sigma}_{t+1}-\sigma\tau^{b,\sigma}_{t+1}]f(\sigma)d\sigma$$

represents the expected value future dividends transferred to the dealer. Finally, the third line of (22), i.e.,

$$\int_{\sigma}\sigma\sigma_{M}[u\left(q^{\sigma}_{t+1}\right)-q^{\sigma}_{t+1}]f(\sigma)d\sigma$$

represents the investor's surplus in the DM. The quantity q^{σ}_{t+1} is a function of τ^{b}_{t+1} and P^{b}_{t+1}, i.e., $q^{\sigma}_{t+1}=z_{t+1}+\min\{q^{*}-\hat{z}_{t+1}(1-\sigma)/\sigma,P^{b,\sigma}_{t+1}\tau^{b,\sigma}_{t+1}\}$.

In the entrepreneur's money demand problem (22), the bargaining variables present themselves as τ^{a}_{t+1}, $P^{a}_{t+1}\tau^{a}_{t+1}$, τ^{b}_{t+1}, and $P^{b}_{t+1}\tau^{b}_{t+1}$ We first examine how these variables are affected by a change in the entrepreneur's real balances in the CM, and then we can characterize the solution to entrepreneur's money demand problem (22). The precise solution to the entrepreneur's CM money demand depends on whether $\bar{a}$ is large or small. We consider each case separately.

Asset endowment $\bar{a}$ is large

The asset endowment $\bar{a}$ is large when $P^{b}\bar{a}>q^{*}-z$. The effect that a change in real balances has on bid and ask variables depends on whether or not the entrepreneur is liquidity constrained in his FM match (if he turns out to be an investor).

Suppose first the entrepreneur turns out to be an investor who is unconstrained in his FM match: this implies that $P^{b}_{t+1}\tau^{b}_{t+1}=q^{*}-z_{t+1}$. Clearly,

$$\partial\left(P^{b}_{t+1}\tau^{b}_{t+1}\right)/\partial z_{t+1}=-1.$$

The effect that an increase in the investor's real balances, z_{t+1}, has on the bid quantity can be determined by using the upper expression in (10); that is,

$$\frac{\partial\tau^{b}_{t+1}}{\partial z_{t+1}}=-\frac{\theta+(1-\theta)[\sigma_{D}u'\left(q^{*}\right)+1-\sigma_{D}]}{\beta\delta}<0. \tag{24}$$

Now suppose that the investor is liquidity constrained: this implies that

$$P^{b}_{t+1}\tau^{b}_{t+1}=\tilde{z}_{t+1}, \tag{25}$$

where $\tilde{z}_{t+1} = \hat{z}_{t+1}(1-\sigma)/\sigma$. Equation (25) immediately implies that

$$\partial \left(P^b_{t+1}\tau^b_{t+1}\right)/\partial z_{t+1} = 0;$$

an increase in the investor's real balances has no effect on the total value of asset that he sells. The effect that an increase in the investor's real balances has on the bid quantity can be determined by using the middle expression in (10), that is

$$\frac{\partial \tau^b_{t+1}}{\partial z_{t+1}} = (1-\theta)\sigma_D \frac{u'(z_{t+1}+\tilde{z}_{t+1}) - u'\left(z_{t+1}\right)}{\beta\delta} < 0. \tag{26}$$

This result is somewhat interesting: if the investor brings in more real balances, he will sell less assets in the FM but at a higher price.

If the entrepreneur turns out to be a liquidity provider, then a marginal increase in the entrepreneur's real money holdings does not affect τ^b_{t+1} and $P^b_{t+1}\tau^b_{t+1}$. This is because τ^b_{t+1} and $P^b_{t+1}\tau^b_{t+1}$ are determined in the first stage when the dealer and investor bargain under the belief that liquidity providers are holding the equilibrium level of real balances, $\hat{z}$—see discussion in footnote 10. Therefore, if $z_{t+1} \geq \hat{z}_{t+1}$, then an increase in entrepreneur's real money holdings will not not affect τ^a_{t+1} and $P^a_{t+1}\tau^a_{t+1}$ when he turns out to be a liquidity provider—i.e.,

$$\partial(P^a_{t+1}(\hat{z}_{t+1}, z_{t+1}, \sigma)\tau^a_{t+1}(\hat{z}_{t+1}, z_{t+1}, \sigma))/\partial z_{t+1} = 0$$

and

$$\partial\tau^a_{t+1}(\hat{z}_{t+1}, z_{t+1}, \sigma)/\partial z_{t+1} = 0.$$

However, if $z_{t+1} < \hat{z}_{t+1}$, then

$$\partial(P^a_{t+1}(\hat{z}_{t+1}, z_{t+1}, \sigma)\tau^a_{t+1}(\hat{z}_{t+1}, z_{t+1}, \sigma))/\partial z_{t+1} = 1$$

and

$$\partial\tau^a_{t+1}(\hat{z}_{t+1}, z_{t+1}, \sigma)/\partial z_{t+1} = 1/P^a_{t+1},$$

(when $z_{t+1} < \hat{z}_{t+1}$, an additional unit of real balances can purchase $1/P^a_{t+1}$ units of the asset).

We can now solve the entrepreneur's CM maximization problem (22). The first-order condition to this problem is,

$$
\begin{aligned}
\frac{\mu}{\beta} - 1 = \int_0^{\sigma^*(z_{t+1})} & \{-1 \cdot \sigma - \boldsymbol{\delta} \cdot (1-\sigma) + 0 \cdot \beta\delta(1-\sigma) \\
& - \beta\delta \frac{\partial \tau^{b,\sigma}_{t+1}}{\partial z_{t+1}} \sigma + \sigma\sigma_D [u'(q^*) - 1]\} f(\sigma) d\sigma \\
& + \int_{\sigma^*(z_{t+1})}^1 \{0 \cdot \sigma + \boldsymbol{\delta} \cdot \frac{(1-\sigma)}{P^{a,\sigma}_{t+1}} + 0 \cdot \beta\delta(1-\sigma) \\
& - \beta\delta \frac{\partial \tau^{b,\sigma}_{t+1}}{\partial z_{t+1}} \sigma + \sigma\sigma_D [u'(z_{t+1} + \tilde{z}_{t+1}) - 1]\} f(\sigma) d\sigma,
\end{aligned}
$$

where

$$
\boldsymbol{\delta} = \begin{cases} 0 & \text{if } z_{t+1} \geq \hat{z}_{t+1} \\ 1 & \text{if } z_{t+1} < \hat{z}_{t+1} \end{cases},
$$

and $\sigma^*(z_{t+1}) = \hat{z}_{t+1}/(q^* + \hat{z}_{t+1} - z_{t+1})$. Using the Fisher equation $(1+i) = \mu(1+r)$ and $\beta = 1/(1+r)$, the nominal interest rate, i, can be expressed as

$$
i = \frac{\mu}{\beta} - 1. \tag{27}
$$

The above first-order condition can be simplified to

$$
\begin{aligned}
i \quad = \quad & \int_0^{\sigma^*(z_{t+1})} \sigma \left\{-\boldsymbol{\delta} \cdot (1-\sigma) - 1 + \theta + (1-\theta)\left[\sigma_D u'(q^*) + 1 - \sigma_D\right]\right\} f(\sigma) d\sigma \\
& + \int_{\sigma^*(z_{t+1})}^1 \{\boldsymbol{\delta} \cdot \frac{(1-\sigma)}{P^a_{t+1}} + \sigma\{-(1-\theta)\,\sigma_D\left[u'(z_{t+1} + \tilde{z}_{t+1}) - u'(z_{t+1})\right]\}\} f(\sigma) d\sigma \\
& + \int_{\sigma^*(z_{t+1})}^1 \sigma\sigma_D [u'(z_{t+1} + \tilde{z}_{t+1}) - 1] f(\sigma) d\sigma,
\end{aligned}
$$

or

$$
\begin{aligned}
i = -\int_0^{\sigma^*(z_{t+1})} \boldsymbol{\delta} \cdot (1-\sigma) f(\sigma) d\sigma + \int_{\sigma^*(z_{t+1})}^1 \boldsymbol{\delta} \cdot \frac{(1-\sigma)}{P^{a,\sigma}_{t+1}} f(\sigma) d\sigma + \\
\sigma_D \int_{\sigma^*(z_{t+1})}^1 \sigma\{(1-\theta)[u'(z_{t+1}) - 1] + \theta[u'(z_{t+1} + \hat{z}_{t+1}(1-\sigma)/\sigma) - 1]\} f(\sigma) d\sigma.
\end{aligned} \tag{28}
$$

Asset Endowment $\bar{a}$ is small

The asset endowment $\bar{a}$ is small when $P^b\bar{a} < q^* - z$. If the entrepreneur turns out to be an investor and is liquidity constrained, $P^b\bar{a} > \tilde{z}$, then the effect of an increase in real balances on $P^b\tau^b$ and τ^b is exactly the same as above, i.e., $\partial(P^b\tau^b)/\partial z = 0$ and $\partial\tau^b/\partial z$ is given by (26). If the investor is asset constrained, $P^b\bar{a} \leq \tilde{z}$, then, from (8) we get

$$\frac{\partial \tilde{P}^b}{\partial z} = -\sigma_D \frac{u'(z+\tilde{P}^b\bar{a}) - u'(z)}{\sigma_D u'(z+\tilde{P}^b\bar{a})\bar{a} + (1-\sigma_D)\,\bar{a} + \theta/(1-\theta)\bar{a}} > 0, \tag{29}$$

which implies that $\partial(P^b\tau^b)/\partial z = \bar{a}\partial\tilde{P}^b/\partial z > 0$. Since $\tau^b = \bar{a}$ when the investor is asset constrained, we get $\partial\tau^b/\partial z = 0$. If the entrepreneur turns out to be a liquidity provider, then the effect of an increase is his real balance holdings on τ^b_{t+1}, $P^b_{t+1}\tau^b_{t+1}$ τ^a_{t+1} and $P^a_{t+1}\tau^a_{t+1}$ are the same as described above when $\bar{a}$ is large.

The first order condition for the entrepreneur's CM maximization problem (22) is

$$\begin{aligned}
\frac{\mu}{\beta} - 1 \;\; = \;\; & \int_0^{\tilde{\sigma}(z_{t+1})} \{ \frac{\partial \tilde{P}^{b,\sigma}_{t+1}}{\partial z_{t+1}} \bar{a} \cdot \sigma - \boldsymbol{\delta} \cdot (1-\sigma) + 0 \cdot \beta\delta(1-\sigma) - 0 \cdot \beta\delta\sigma \\
& + \sigma\sigma_D[u'\left(z_{t+1} + \tilde{P}^b_{t+1}\bar{a}\right) - 1]\} f(\sigma)d\sigma \\
& + \int_{\tilde{\sigma}(z_{t+1})}^1 \{0 \cdot \sigma - \boldsymbol{\delta} \cdot \frac{(1-\sigma)}{P^{a,\sigma}_{t+1}} + 0 \cdot \beta\delta(1-\sigma) \\
& - \beta\delta \frac{\partial \tau^{b,\sigma}_{t+1}}{\partial z_{t+1}} \sigma + \sigma\sigma_D[u'\left(z_{t+1} + \tilde{z}_t\right) - 1]\} f(\sigma)d\sigma
\end{aligned}$$

or

$$\begin{aligned}
\frac{\mu}{\beta} - 1 \;\; = \;\; & -\int_0^{\tilde{\sigma}(z_{t+1})} \boldsymbol{\delta} \cdot (1-\sigma) f(\sigma)d\sigma + \int_0^{\tilde{\sigma}(z_{t+1})} \sigma\{ \frac{\partial \tilde{P}^{b,\sigma}_{t+1}}{\partial z_{t+1}} \bar{a} + \sigma_D[u'\left(z_{t+1} + \tilde{P}^{b,\sigma}_{t+1}\bar{a}\right) - 1]\} f(\sigma)d\sigma \\
& + \int_{\tilde{\sigma}(z_{t+1})}^1 \boldsymbol{\delta} \cdot \frac{(1-\sigma)}{P^{a,\sigma}_{t+1}} f(\sigma)d\sigma + \int_{\tilde{\sigma}(z_{t+1})}^1 \sigma\{-\beta\delta \frac{\partial \tau^{b,\sigma}_{t+1}}{\partial z_{t+1}} + \sigma_D[u'\left(z_{t+1} + \tilde{z}_{t+1}\right) - 1]\} f(\sigma)d\sigma,
\end{aligned}$$

where limit of integration is $\tilde{\sigma}(z_{t+1}) = \hat{z}_{t+1}/(\tilde{P}^b_{t+1}\bar{a} + \hat{z}_{t+1})$ and $\tilde{P}^b$ is given implicitly by (8). Substituting for the derivatives, this equation can be rewritten as

$$\begin{aligned}
i \;\; = \;\; & -\int_0^{\tilde{\sigma}(z_{t+1})} \boldsymbol{\delta} \cdot (1-\sigma) f(\sigma)d\sigma + \sigma_D \int_0^{\tilde{\sigma}(z_{t+1})} \sigma\Theta(z_{t+1},\sigma) f(\sigma)d\sigma \\
& + \int_{\tilde{\sigma}(z_{t+1})}^1 \boldsymbol{\delta} \cdot \frac{(1-\sigma)}{P^{a,\sigma}_{t+1}} f(\sigma)d\sigma + \sigma_D \int_{\tilde{\sigma}(z_{t+1})}^1 \sigma\Phi(z_{t+1},\sigma) f(\sigma)d\sigma,
\end{aligned} \tag{30}$$

where

$$\Theta(z_{t+1},\sigma) = -\frac{(1-\theta)[u'(z_{t+1} + \tilde{P}^{b,\sigma}_{t+1}\bar{a}) - u'(z_{t+1})]}{(1-\theta)\sigma_D[u'(z_{t+1} + \tilde{P}^{b,\sigma}_{t+1}\bar{a}) - 1] + 1} + u'\left(z_{t+1} + \tilde{P}^{b,\sigma}_{t+1}\bar{a}\right) - 1 > 0$$

and

$$\Phi(z_{t+1},\sigma) = (1-\theta)[u'(z_{t+1})-1] + \theta\{u'[z_{t+1}+\hat{z}_{t+1}(1-\sigma)/\sigma]-1\} > 0.$$

Equilibrium

In a steady state equilibrium, the real money supply is constant over time, $M_{t+1}p^m_{t+1} = M_t p^m_t$, which implies that

$$\frac{M_{t+1}}{M_t} = \frac{p^m_t}{p^m_{t+1}} = \mu. \tag{31}$$

As well, all entrepreneurs accumulate the same level of real balances in the CM of period t, $z_{t+1} = \hat{z}_{t+1}$. Since, the real money supply is constant over time, we have that, in a steady state equilibrium, $z_{t+1} = z_{s+1} = \hat{z}_{t+1} = \hat{z}_{s+1} = z$ for all s, t. An implication of constant and equal real balance holdings over entrepreneurs and time that is that, *for any given* σ, the values of bid and ask variables are time invariant, see (9), (10), (11) and (12).

For a given the money growth rate μ and, hence, nominal interest rate $i = \mu/\beta - 1$, when $\bar{a}$ is large, entrepreneurs choose real balances, z, to satisfy condition (28) where $z_{t+1} = \hat{z}_{t+1} = z$, i.e., they choose z to satisfy

$$i = \sigma_D \int_{z/q^*}^{1} \sigma\{(1-\theta)[u'(z)-1] + \theta[u'(z/\sigma)-1]\} f(\sigma) d\sigma. \tag{32}$$

Notice that when $z_{t+1} = \hat{z}_{t+1} = z$, the indicator function $\boldsymbol{\delta}$ is equal to zero. Denote the solution to (32) as $\bar{z}(\mu)$. When $\bar{a}$ is small, entrepreneurs choose real balances, z, to satisfy condition (30) where $z_{t+1} = \hat{z}_{t+1} = z$, i.e., they choose z to satisfy

$$\begin{aligned} i \;=\; & \sigma_D \int_0^{z/(\tilde{P}^b\bar{a}+z)} \sigma\Theta(z_{t+1},\sigma) f(\sigma) d\sigma \\ & + \sigma_D \int_{z/(\tilde{P}^b\bar{a}+z)}^{1} \sigma\Phi(z_{t+1},\sigma) f(\sigma) d\sigma. \end{aligned} \tag{33}$$

Denote the solution to (33) as $\underline{z}(\mu)$.

Up to this point we have characterized "$\bar{a}$ being large" as $P^b\bar{a} \geq q^* - z$ and "$\bar{a}$ being small" by $P^b\bar{a} < q^* - z$. This characterization is problematic because it depends on an endogenous variable, z. We now provide a characterization for "$\bar{a}$ being large" or "$\bar{a}$ being small" that relies only on fundamental (exogenous) parameters. Assume that $\bar{a}$ is large. The entrepreneur's real balance holdings in the CM is $\bar{z}(\mu)$. Since (by assumption), $\bar{a}$ is large, the investor will be unconstrained for all $\sigma \leq z^*(\mu)/q^*$. If $\sigma \in (0, z^*(\mu)/q^*)$, then the amount of assets that the entrepreneur sells in the FM is invariant to σ and given by the upper branch of (7), where we substitute $z = \bar{z}(\mu)$. Denote the amount of assets that the investor sells when he is unconstrained as $\bar{\tau}^b(\mu)$. Hence, $\bar{a}$ is large if $\bar{a} \geq \bar{\tau}^b(\mu)$ and is small if $\bar{a} < \bar{\tau}^b(\mu)$. If $\bar{a}$ is large, then the entrepreneur's real CM balance holdings will be given by the solution to (32), which is $\bar{z}(\mu)$; and if it is small, then his real CM balance holdings will be given by the solution to (33), which is $\underline{z}(\mu)$.

Since $z_t = \hat{z}_t = z$, we can compactly write the bid price and bid quantity as $P^b(z,\sigma)$ and $\tau^b(z,\sigma)$, respectively. Using (9) and (10), if $\bar{a} \geq \bar{\tau}^b(\mu)$, then the bid price and bid quantity are given by, respectively,

$$P^b(z,\sigma) = \left(\tau^b\right)^{-1} \min\left\{z(1-\sigma)/\sigma, q^* - z\right\} \tag{34}$$

and

$$\tau^b(z,\sigma) = \begin{cases} \begin{array}{l} \{\theta\left(q^* - z\right) + (1-\theta)\left[\sigma_D\left[u\left(q^*\right) - u\left(z\right)\right] + \right. \\ \left. (1-\sigma_D)\left(q^* - z\right)\right]\}/\beta\delta \end{array} & \text{if } q^* \leq z/\sigma \\ \begin{array}{l} \{\theta z(1-\sigma)/\sigma + (1-\theta)\left[\sigma_D\left[u\left(z/\sigma\right) - u\left(z\right)\right] + \right. \\ \left. (1-\sigma_D)\, z(1-\sigma)/\sigma\right]\}/\beta\delta \end{array} & \text{if } q^* > z/\sigma \end{cases}, \tag{35}$$

where $z = \bar{z}(\mu)$. If $\bar{a} < \bar{\tau}^b(\mu)$, then the bid price and bid quantity are given by, respectively,

$$P^b(z,\sigma) = \begin{cases} z(1-\sigma)/[\sigma\tau^b(z,\sigma)] & \tilde{P}^b(z,\sigma)\bar{a} \geq z(1-\sigma)/\sigma \\ \tilde{P}^b(z,\sigma) & \tilde{P}^b\bar{a}(z,\sigma) < z(1-\sigma)/\sigma \end{cases}, \tag{36}$$

and

$$\tau^b(z,\sigma) = \begin{cases} \begin{array}{l} \{\theta z(1-\sigma)/\sigma + (1-\theta)\left[\sigma_D\left[u\left(z/\sigma\right) - u\left(z\right)\right] + \right. \\ \left. (1-\sigma_D)\, z(1-\sigma)/\sigma\right]\}/\beta\delta \end{array} & \tilde{P}^b(z,\sigma)\bar{a} \geq z(1-\sigma)/\sigma \\ \bar{a} & \tilde{P}^b(z,\sigma)\bar{a}(z,\sigma) < z(1-\sigma)/\sigma \end{cases}, \tag{37}$$

where $z = \underline{z}(\mu)$ and $\tilde{P}^b(z,\sigma)$ solves (8). From (11) and (12), the ask price, $P^a(z,\sigma)$, and ask quantity, $\tau^a(z,\sigma)$, are given by

$$P^a(z,\sigma) = \frac{P^b(z,\sigma)\beta\delta}{\theta\beta\delta + (1-\theta)\,P^b(z,\sigma)}, \tag{38}$$

and

$$\tau^a(z,\sigma) = \frac{\theta\beta\delta + (1-\theta)\,P^b(z,\sigma)}{\beta\delta}\tau^b(z,\sigma)\sigma/\left(1-\sigma\right), \tag{39}$$

respectively.

The equilibrium DM output, (20), can be written as

$$q(z,\sigma) = \min\{z + P^b(z,\sigma)\tau^b(z,\sigma), q^*\}. \tag{40}$$

In period t, the aggregate demand for real balances *evaluated in period t prices* is $\mu z_{t+1} = \mu z$ and the total supply of real balances is $p_t^m M_t$. Since $z_t = z$ for t, the market clearing price of money is

$$p_t^m = \frac{\mu z}{M_t}. \tag{41}$$

We can now provide a definition of equilibrium for our economy.

Definition 1 *Given a money growth rate μ, an initial money supply, M_0, and a per period asset endowment $\bar{a}$, a steady-state equilibrium is characterized by: (i) real balances z, given by (32) if $\bar{a} \geq \bar{z}(\mu)$ or (33) if $\bar{a} < \bar{z}(\mu)$; (ii) a set of CM prices $\{p_t^m\}_{t=0}^{\infty}$, given by (41); and for each realization of $\sigma \in [0, 1]$, (iii) a bid price $P^b(z, \sigma)$, given by (34) if $\bar{a} \geq \bar{z}(\mu)$ or (36) if $\bar{a} < \bar{z}(\mu)$; (iv) a bid quantity $\tau^b(z, \sigma)$, given by (35) if $\bar{a} \geq \bar{z}(\mu)$ or (37) if $\bar{a} < \bar{z}(\mu)$; (v) an ask price $P^a(z, \sigma)$, given by (38); (vi) an ask quantity $\tau^a(z, \sigma)$, given by (39); and (vii) a DM output level $q(z, \sigma)$, given by (40).*

We provide a proof for existence and unique of equilibrium in the Appendix. The proof requires the properties of the comparative static $dz/d\mu$, which are derived in Section 6.

5 Returns, Spreads and Cash-in-the-Market Pricing of Assets

We are interested in understanding how supply and demand for liquidity in the FM affects asset prices, returns and bid-ask spreads. Although liquidity is a nebulous concept, there are a number of ways one can attempt to measure or characterize it. One measure of liquidity is the fraction of entrepreneurs that turn out to be liquidity providers in the FM, $1 - \sigma$. Once entrepreneurs exit the CM and enter the FM, their liquidity needs may be revised: An entrepreneur who turns out to be an investor in the subsequent DM may desire additional real balances, and an entrepreneur who turns out to be a liquidity provider is willing to supply them. The total amount of real balances that liquidity providers can supply to investors in the FM is $(1 - \sigma)z$. When σ is "large," there may be a shortage of FM liquidity since the measure of liquidity providers in the FM is "small." And, when σ is "small," liquidity may be plentiful in the FM.

Another, rather obvious, measure of the supply of liquidity is the amount of real balances, z, that entrepreneurs accumulate in the CM. A change in various model parameters may cause investors to change their real balance holdings. For example, a change in the money growth rate changes the cost of holding real balances and, thus the amount of real balances that entrepreneurs accumulate in the CM. We defer this discussion to the subsequent section.

The equilibrium ask and bid prices are given by (3) and (34) or (36), respectively. There are a couple of ways to measure asset returns, e.g., a gross bid return, r^b, and a gross ask return, r^a. These returns can be measured as

$$r^b = \delta / P^b \tag{42}$$

and $r^a = \delta / P^a$. Using (3), we can define the equilibrium bid-ask spread as

$$\frac{P^a}{P^b} = \frac{\beta\delta}{\theta\beta\delta + (1-\theta)\, P^b}. \tag{43}$$

Notice that the wedge between the bid and ask price is created by the bargaining friction. When the bargaining friction disappears, i.e., $\theta = 1$, the dealer receives no surplus and $P^a = P^b$.

The effect that a change in the liquidity measure σ has on the bid price is simply $\partial P^b / \partial \sigma$, and on the (bid) asset return is,

$$\frac{\partial r^b}{\partial \sigma} = -\frac{\delta}{\left(P^b\right)^2} \frac{\partial P^b}{\partial \sigma}. \tag{44}$$

Note that, not surprisingly, bid prices and asset returns are negatively correlated.[12] The effect that a change in the liquidity measure σ has on the bid-ask spread is

$$\frac{\partial \left(P^a / P^b\right)}{\partial \sigma} = -\frac{\beta\delta\,(1-\theta)}{\left[\theta\beta\delta + (1-\theta)\, P^b\right]^2} \frac{\partial P^b}{\partial \sigma}. \tag{45}$$

The finance literature has consistently documented a positive correlation between changes in asset returns and bid-ask spreads. Our bargaining environment predicts that such a relation-

[12] If one is interested in the "ask return," then, using (43), we have

$$r^a = \theta r^b + (1-\theta)\,,$$

and

$$\frac{\partial r_t^a}{\partial \sigma} = \theta \frac{dr^b}{d\sigma}.$$

ship will prevail (at least for the liquidity measure σ),[13] i.e., simply compare (44) and (45),

$$\frac{\partial \left(P^a/P^b\right)}{\partial \sigma} = \frac{\beta (1-\theta) \left(P^b\right)^2}{\left[\theta \beta \delta + (1-\theta) P^b\right]^2} \frac{\partial r_t^b}{\partial \sigma}.$$

Let's examine the effect that a change in the composition of investors and liquidity providers in the FM has on assets prices. In particular, what is $\partial P^b / \partial \sigma$? Recall that the value of σ is drawn from the probability distribution $f(\sigma)$, where $\sigma \in [0,1]$. Hence, in a steady-state equilibrium, σ will take on different values over time. If the investor is not liquidity constrained in the FM, i.e., either $P^b \bar{a} < \tilde{z} = z(1-\sigma)/\sigma$ or $q^* - z < \tilde{z}$, then it is obvious from the equilibrium conditions—(34) and (35) if $\bar{a}$ is large or (36) and (37) if $\bar{a}$ is small—that $\partial P^b / \partial \sigma = 0$. Intuitively, since the amount of assets that investors sell in the FM is unaffected by an increase in FM liquidity when the investor is not liquidity constrained, then so too is the (bid) price.

Now suppose that the investor is liquidity constrained in the FM, i.e., $\tilde{z} < \min\{P^b \bar{a}, q^* - z\}$. Then $P^b \tau^b = \tilde{z}$. If $P^b \tau^b = \tilde{z}$, then conditions (34) and (35) when $\bar{a}$ is big or (36) and (35) when $\bar{a}$ is small both imply that

$$\frac{\partial P^b}{\partial \tilde{z}} \frac{\partial \tilde{z}}{\partial \sigma} = -\frac{P^{b^2} (1-\theta) \sigma_D}{\beta \delta \tilde{z}^2} \left[u(z+\tilde{z}) - u(z) - \tilde{z} u'(z+\tilde{z})\right] \frac{z}{\sigma^2} < 0. \tag{46}$$

If the fraction of liquidity providers in the FM decreases, then the amount of real balances available in a bargaining match also falls. As a result, the marginal value of the DM expected surplus increases, which makes real balances more valuable to the investor. Hence, the investor is willing to sell the real asset at a lower bid price, i.e., $\partial P^b / \partial \sigma < 0$. This result is reminiscent of cash-in-the-market pricing, see Allen and Gale (2005, 2008). Cash-in-the-market pricing has the basic flavor that asset prices fall below their fundamental values because, relative to the amount of assets that are liquidated, cash is in scarce supply. If, however, cash is "plentiful," then asset prices will be at their fundamental values. In our environment, asset prices in the FM are always below their fundamental values because asset markets are not competitive. But, the amount of cash (real balances) available in the (FM) market, $(1-\sigma) z$, can directly influence asset prices: Higher cash holdings can lead to higher asset prices. This result may seem anomalous from a classic asset pricing theory perspective, where the value of an asset is equal to the discounted stream of its dividends.

[13] In fact, as long as the change in asset prices are brought about by a change in a variable or parameter that is *not* equal to β, δ, or θ, we can conclude there is a positive correlation between changes in asset returns and the bid-ask spread.

Over time, asset prices will "fluctuate" in the steady state equilibrium. More specifically, when $\bar{a}$ is large investors are liquidity constrained for all $\sigma > q^*/\bar{z}(\mu)$ and when it is small, they are liquidity constrained for all $\sigma > \underline{z}(\mu)/[P^b\bar{a} + \underline{z}(\mu)]$. As a result, when $\bar{a}$ is large asset prices are "constant" over $\sigma \in (0, q^*/\bar{z}(\mu))$ and declining over $\sigma \in (q^*/\bar{z}(\mu), 1)$; when $\bar{a}$ is small, asset prices are constant over $\sigma \in (0, \underline{z}(\mu)/[P^b\bar{a} + \underline{z}(\mu)])$ and declining over $\sigma \in (\underline{z}(\mu)/[P^b\bar{a} + \underline{z}(\mu)], 1)$. Therefore, overtime, asset prices will "move around" since the asset price will reflect, among other things, the amount of liquidity available in the FM, $(1-\sigma)\,z$, and the amount of liquidity varies over time because σ is a random variable. Once again, from classic asset pricing theory perspective, this result appears to be anomalous because asset price changes are not accompanied by new information regarding asset fundamentals.

6 Inflation, Asset Prices and Returns

We now examine the effect that changes in liquidity measure z has on asset prices and returns. Since real balances, z, is an endogenous choice variable, something (exogenous) must change in order to induce a change in z. We focus our attention on a change in the money growth rate, μ, which also changes inflation one-to-one, and the nominal interest rate, i, by the factor $1/\beta$. We first determine how a change in money growth or inflation affects real balances; we are then in a position to examine how inflation, by changing real balances, affects asset prices and returns.

We first suppose that the asset endowment, $\bar{a}$, is large. From the equilibrium condition (32), the relationship between equilibrium real balances and the (gross) inflation rate is given by

$$\frac{dz}{d\mu} = \{\beta\sigma_D \int_{z/q^*}^{1} \sigma[(1-\theta)u''(z) + \theta u''(z/\sigma)/\sigma] f(\sigma) d\sigma \\ - \beta/q^*[(1-\theta)[u'(z)-1] f(z/q^*)\}^{-1} < 0, \quad (47)$$

which means that an increase in inflation, decreases entrepreneurs' real balance holdings.

Now suppose that the asset endowment, $\bar{a}$, is small. From the equilibrium condition (33), the relationship between equilibrium real balances and the (gross) inflation rate in the steady state is given by

$$
\begin{aligned}
\frac{dz}{d\mu} &= \beta\sigma_D\{\int_0^{\tilde{\sigma}(z)} \sigma\frac{d\Theta(z,\sigma)}{dz}f(\sigma)d\sigma \\
&\quad + \int_{\tilde{\sigma}(z)}^{1} \sigma\frac{d\Phi(z,\sigma)}{dz}f(\sigma)d\sigma \qquad (48)\\
&\quad + \frac{d\tilde{\sigma}\,(z)}{dz}\hat{\sigma}[\Theta(z,\tilde{\sigma})-\Phi(z,\tilde{\sigma})]f(\tilde{\sigma})\}^{-1},
\end{aligned}
$$

where

$$\tilde{\sigma}(\hat{z}) = \frac{z}{\tilde{P}^b\bar{a}+z},$$

$$
\begin{aligned}
\frac{d\Theta(z,\sigma)}{dz} &= \frac{u''(z+\tilde{P}^b_{t+1}\bar{a})\left[1+\bar{a}d\tilde{P}^b/dz\right]\left[\theta+(1-\theta)\sigma_D(u'(z+\tilde{P}^b_{t+1}\bar{a})-1)\,(u'\,(z)-1)\right]}{(1-\theta)\sigma_D[u'\left(z+\tilde{P}^b_{t+1}\bar{a}\right)-1]+1} \\
&+\frac{(1-\theta)u''\,(z)}{(1-\theta)\sigma_D[u'\left(z+\tilde{P}^b_{t+1}\bar{a}\right)-1]+1}+\frac{(1-\theta)[u'(z+\tilde{P}^b_{t+1}\bar{a})-u'(z)]}{(1-\theta)u''(z+\tilde{P}^b_{t+1}\bar{a})\{\sigma_D[u'\left(z+\tilde{P}^b_{t+1}\bar{a}\right)-1]+1\}^2}<0,
\end{aligned}
$$

$$\frac{d\Phi(z,\sigma)}{dz} = (1-\theta)u''(z)+\theta u''\,(z/\sigma)\,/\sigma<0,$$

and

$$\frac{d\tilde{\sigma}}{dz} = \bar{a}[\tilde{P}^b - z\frac{d\tilde{P}^b}{dz}]. \qquad (49)$$

By construction, the investor is both asset constrained and money constrained at $\tilde{\sigma}$, i.e., $\tilde{P}^b\bar{a} = z(1-\tilde{\sigma})/\tilde{\sigma}$. It is straightforward to show that $\Theta(z,\tilde{\sigma})-\Phi(z,\tilde{\sigma})<0$. Intuitively, $\Theta(z,\sigma)$ represents the marginal benefit to an entrepreneur of having an additional unit of real balances when investors turn out to be asset constrained and $\Phi(z,\sigma)$ represents the marginal benefit when investors turn out to be money constrained. Since this difference is negative, the marginal benefit is higher when investors are money constrained as opposed to asset constrained.

If the asset price elasticity, $(d\tilde{P}^b/\tilde{P}^b)(dz/z)$, is less than one or equal to one, then, from (49), that $d\tilde{\sigma}/dz>0$ and, therefore, $dz/d\mu<0$. Unfortunately, the size of the asset price price elasticity depends on preferences. For example, if the utility function u is described by

$$u(c) = \frac{c^{1-\alpha}}{1-\alpha}, \qquad (50)$$

then for $0 < \alpha < 1$, we have $d\tilde{\sigma}/dz > 0$. However, when $\alpha > 1$, it is not possible to analytically sign $d\tilde{\sigma}/dz$ or, more importantly, $dz/d\mu$. We will deal with the small $\bar{a}$ case by numerically evaluating the derivative (48). Common to all of our examples are the utility function, given by (50), a uniform distribution for $f(\sigma)$, and parameter values $\sigma_D = 0.8$, $\beta = 0.98$, and $\delta = 0.5$. We consider various values for the preference parameter α, the nominal interest rate, i, asset endowment, $\bar{a}$, and bargaining parameter, θ; in particular $\alpha \in \{2.5, 3, 4, 5, 6, 7, 10, 15, 20, 40\}$, $i \in \{0.01, 0.02, 0.03, \ldots, 0.24, 0.25\}$, $\bar{a} \in \{0.01, 0.05, 0.1\}$, and $\theta = \{0.1, 0.5, 0.9\}$. For the sets of parameters that imply that $\bar{a}$ is small,[14] we find that every numerical example is characterized by $dz/d\mu < 0$. Hence, we will assume that when asset holdings, $\bar{a}$, are small, an increase in the money growth rate decreases equilibrium real balance holdings. This, of course, is a standard and robust result in monetary economics.

We can now investigate the effect that inflation—or, equivalently, a change in real balances—has on asset prices, asset real returns and the bid-ask spread. The effect that a change in real balances (for *all* entrepreneurs) has on the bid asset price depends on whether the investor is unconstrained, asset constrained or liquidity constrained, asset constraint in the FM. If investors are not liquidity constrained in their bargaining matches[15], i.e., $P^b\tau^b = q^* - z$, then using (35), (34) can be written as

$$\beta\delta\frac{q^* - z}{P^b} = \theta\left(q^* - z\right) + (1-\theta)\left\{\sigma_D\left[u\left(q^*\right) - u\left(z\right)\right] + \left(1-\sigma_D\right)\left(q^* - z\right)\right\}$$

or

$$\frac{\beta\delta}{P^b} = \theta + (1-\theta)\,\sigma_D\frac{u\left(q^*\right) - u\left(z\right)}{q^* - z} + (1-\theta)\left(1-\sigma_D\right). \tag{51}$$

Therefore,

$$\frac{dP^b}{dz} = \frac{P^{b^2}\left(1-\theta\right)\sigma_D}{\beta\delta\left(q^* - z\right)^2}\left[u'\left(z\right)\left(q^* - z\right) - u\left(q^*\right) + u\left(z\right)\right] > 0.$$

If investors are asset constrained,[16] i.e., $P^b\bar{a} < \tilde{z}$, where $z(1-\sigma)/\sigma$ and $P^b\bar{a} + z < q^*$, then using (8), we get

$$\frac{dP^b}{dz} = \frac{(1-\theta)[\sigma_D u(z + P^b\bar{a}) - 1]}{(1-\theta)[\sigma_D u(z) - 1] + 1} > 0.$$

Finally, if investors are liquidity constrained,[17] i.e., $P^b\tau^b = \tilde{z}$, where $\tilde{z} = z(1-\sigma)/\sigma$, then

[14] Some parameter configurations do not have $\bar{a}$ small. For example, when $\bar{a} = 0.1$ and $\alpha \in \{20, 40\}$, $\bar{a}$ is never "small" for any $01 \leq i \leq 0.25$.

[15] This necessarily implies that the asset endowment, $\bar{a}$, is large.

[16] This necessarily implies that the asset endowment, $\bar{a}$, is small.

[17] Investors can be liquidity constrained when $\bar{a}$ is large and when it is small.

using (35), (34) can be written as

$$\beta\delta\frac{z(1-\sigma)/\sigma}{P^b}=\theta z(1-\sigma)/\sigma+(1-\theta)\left[\sigma_D\left[u\left(z/\sigma\right)-u\left(z\right)\right]+\left(1-\sigma_D\right)z(1-\sigma)/\sigma\right.$$

or

$$\frac{\beta\delta}{P^b}=\theta+(1-\theta)\,\sigma_D\frac{u(z/\sigma)-u\left(z\right)}{z(1-\sigma)/\sigma}+(1-\theta)\left(1-\sigma_D\right). \tag{52}$$

Therefore,

$$\frac{dP^b}{dz}=\frac{P^{b^2}\left(1-\theta\right)\sigma_D}{\beta\delta z^2(1-\sigma)/\sigma}\{u'\left(z\right)z-u'(z/\sigma)z/\sigma+u(z/\sigma)-u\left(z\right)\}>0.$$

An increase in the real balances of all entrepreneurs—and hence, an increase in the total real money supply—increases asset prices, and this is independent of whether the investor is liquidity constrained, asset constrained or unconstrained. The intuition behind this result is straightforward. There are two effects associated with an increase in real balances that work in the same direction. If an *investor's* real balances increase (holding the liquidity provider's balances constant), then the marginal value of the bargaining match surplus falls. If a *liquidity provider's* real balances increase (holding the investor's balances constant), then although the match surplus increases, the marginal value of the surplus falls. (This latter effect is not operative if the investor is not liquidity constrained.) In both cases, the investor's value for an additional unit of real balances falls, which implies that bid asset price increases, i.e., the investor gives up a smaller quantity of the asset for an additional unit of real balances. Interestingly, higher levels of liquidity, as measured by z, are associated with higher asset prices: this is the cash-in-the-market effect at work. Since an increase in inflation decreases real balance holdings, an increase in inflation actually *decreases* asset prices. This is consistent with the observation that periods of low inflation are usually associated with periods of high asset prices. This is in contrast to the standard so-called Mundell-Tobin effect, which is present in many models of money, that posits a positive relationship between inflation and asset prices.

The effect that a change in real balances has on asset returns and the bid-ask spread is similar to the effect of a change in σ, (44) and (45); in particular,

$$\frac{dr^b}{dz}=-\frac{\delta}{\left(P^b\right)^2}\frac{dP^b}{dz}.$$

and

$$\frac{d\left(P^a/P^b\right)}{dz}=-\frac{\beta\delta\left(1-\theta\right)}{\left[\theta\beta\delta+\left(1-\theta\right)P^b\right]^2}\frac{dP^b}{dz}.$$

Hence, asset returns and the bid-ask spreads are positively correlated. An increase in inflation *increases* asset returns, as well as the bid-ask spread. The latter result reinforces a standard view of liquidity and bid-ask spreads: An increase in inflation reduces real balances (liquidity) in the economy which results in higher bid-ask spreads.

We conclude this section by pointing out that there is a subtle but important difference in interpreting the results for liquidity measure z and liquidity measure σ. In the steady-state equilibrium, entrepreneurs accumulate μz real balances in the CM in period t measured in period t prices, and then enter the FM. The amount of liquidity available to investors, $z(1-\sigma)$, will fluctuate over time, as will asset prices, asset returns and bid-ask spreads. So we can sensibly ask, *in the equilibrium*, how does an increase or decrease in liquidity affect asset prices, asset returns and bid-ask spreads? In contrast, in equilibrium, there is a unique value of real balances, z, given by the solution to (32) or (33). We can compare differences in asset prices, asset returns and bid-ask spreads associated with different values of real balances—that result from different money growth rates—but it should be recognized that this is a comparison *across equilibria* associated with different economies.

7 Conclusions

We examined a world where money is needed to facilitate investments. Since money is costly to hold, entrepreneurs tend to underinvest. A financial market that allows investors to trade their less liquid assets for more liquid ones can improve matters. When financial markets are over-the-counter, we find that asset prices depend on the amount of liquidity that is available in the market. Interestingly, the amount of liquidity that is available depends on the ex post distribution of investment opportunities: individuals who do not find opportunities are willing to provide liquidity to those that do. If few investment opportunities are available, then the supply of liquidity will be high and so will asset prices. If money becomes more costly to hold because inflation increases, agents will hold less real balances. This implies higher inflation is associated with in lower levels of liquidity available in the OTC financial market, and will result in *lower* asset prices. This is in contrast to the standard Mundell-Tobin effect, where an increase in inflation implies that asset prices will *increase*. Since our financial markets are over-the-counter, a bid-ask spread emerge as long as the agent who operates the market, the dealer, has some bargaining power. If liquidity in financial market changes because of changes in inflation or entrepreneurs' investment opportunities, bid-ask spreads are negatively correlated with asset returns. A change in bargaining power has a direct liquidity effect—entrepreneurs will accumulate lower real balances—and an indirect effect—for a given z, investors are able to extract more liquidity per asset sold. Since these effects work in opposite directions, a change in bargaining power has ambiguous effects on asset prices, returns and bid-ask spreads.

8 References

Allen, Franklin and Douglas Gale (2005) "From Cash-in-the-Market Pricing to Financial Fragility," Journal of the European Economic Association, April-May, 3(2-3), 535-546.

Allen, Franklin and Douglas Gale (2007) Understanding Financial Crises, Oxford University Press.

Amihud, Yakov and Haim Mendelson, 1986, Asset pricing and the bid-ask spread, Journal of Financial Economics 17, 223–249.

Amihud, Yacov, Haim Mendelson, and Lasse Pedersen, 2005, Liquidity and asset pricing, Foundations and Trends in Finance 1, 269–364.

Christiano Lawrence, Cosmin L. Ilut, Roberto Motto & Massimo Rostagno (2010) "Monetary Policy and Stock Market Booms," NBER Working Papers 16402, National Bureau of Economic Research, Inc.

Duffie, Darrell, Nicolae Garleanu, and Lasse Pedersen, 2005, Over-the-counter markets, Econometrica 73, 1815–1847.

Geromichalos, Athanasios and Lucas Herrenbrueck (2012) "Monetary Policy, Asset Prices, and Liquidity in Over-the-Counter Markets," manuscript.

Kiyotaki, Nobuhiro and John Moore (2001) "Liquidity, Business Cycles, and Monetary Policy," manuscript (new 2012 version).

Kocherlakota, Narayana R. (1998) "Money Is Memory," *Journal of Economic Theory* 81, 232-251.

Lagos, Ricardo and Randall Wright (2005) "A Unified Framework for Monetary Theory and Policy Analysis," *Journal of Political Economy* 113, 463-484

Lagos, Ricardo, and Guillaume Rocheteau, 2009, Liquidity in asset markets with search frictions, Econometrica 77, 403–426.

Lagos, Ricardo and Shenxging Zhang (2013) "A Model of Monetary Exchange in Over-the-Counter Markets," in progress.

Myers, Stewart C., and Nicholas S. Majluf (1984) "Corporate financing and investment decisions when firms have information that investors do not have," *Journal of Financial Economics* 13, 187–221.

Appendix

In this appendix we provide a proof for the existence and uniqueness of equilibrium.

Proposition 2 *(i) In the large asset case, $\bar{a} \geq \bar{\tau}^b(\mu)$, an equilibrium exists and is unique. (ii) In the small asset case, $\bar{a} < \bar{\tau}^b(\mu)$, an equilibrium exists. The equilibrium is unique if $dz/d\mu < 0$; a sufficient condition for uniqueness is $(d\tilde{P}^b/\tilde{P}^b)(dz/z) \leq 1$.*

Proof. (i) Denote right side of (32) by $\Lambda(z)$. The Inada conditions imply that $\lim_{z\to 0}\Lambda(z) = \infty$ and $\lim_{z\to\infty}\Lambda(z) = -\sigma_D\int_0^1 \sigma f(\sigma)d\sigma < 0$. From (47) we have $d\Lambda/dz < 0$. Since $\Lambda(z)$ is continuous and everywhere differentiable, there is a unique $z > 0$ that solves $\Lambda(z) = i = \mu/\beta - 1$. Given such z, (34) and (35) uniquely determine τ^b and P^b, (40) uniquely determines the level of DM output q, and (41) uniquely determines the market clearing price of money, p^m.

(ii) Denote the right side of (33) by $\Gamma(z)$. The Inada conditions imply that $\lim_{z\to 0}\Gamma(z) = \infty$ and $\lim_{z\to\infty}\Gamma(z) = -\sigma_D\int_0^1 \sigma f(\sigma)d\sigma - \sigma_D\int_1^1 \sigma f(\sigma)d\sigma = -\sigma_D\int_0^1 \sigma f(\sigma)d\sigma < 0$. Since $\Gamma(z)$ is continuous and everywhere differentiable, there exists at least one $z > 0$ that solves $\Gamma(z) = i = \mu/\beta - 1$. Given such a z, (36) and (37) uniquely determine τ^b and P^b, (40) uniquely determines the level of DM output q, and (41) uniquely determines the market clearing price of money, p^m. If $dz/d\mu < 0$, then $\Gamma'(z) < 0$, and there exists a unique z that solves $\Gamma(z) = i = \mu/\beta - 1$. A sufficient condition for uniqueness is that $(d\tilde{P}^b/\tilde{P}^b)(dz/z) \leq 1$, i.e., if $(d\tilde{P}^b/\tilde{P}^b)(dz/z) \leq 1$, then $dz/d\mu < 0$. ■

Remark 3 *For utility function (50), we were unable to find a set of parameters for which*

$dz/d\mu > 0$. *From this, we conclude that there exists a large of set parameters for which the equilibrium is unique for the small asset case.*