Sorting in the Interbank Money Market*

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Abstract

We present a model of monetary policy implementation through a corridor system. Banks bid for reserves before trading on an OTC market, while they have access to the central bank standing facilities. The model provides insight on price dynamics, the nature of market volume, liquidity and volatility which features largely absent from the canonical models of monetary policy implementation in the tradition of Poole (1968). As there is an active role for interbank trading in the model, it provides a framework for discussing a number of money market features highlighted by the financial crisis and the recent period of unconventional monetary policies, including counterparty risk. Moreover, we find that the model fits well with a number of stylized facts with respect to money market dynamics during normal times, as well as during the recent period of unconventional monetary policies. We consider different matching protocols in the OTC market, and we find that the data is best explained by a matching protocol which is not very efficient. We conclude that there is room to improve the efficiency of the interbank money market, by, e.g., using electronic trading platform.

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1 Introduction

We present a model of the interbank money market that is able to match four stylized facts of the interbank market in a corridor system. Using data from six jurisdictions that use a corridor system broadly defined, Bech and Monnet (2013) draw the following stylized facts:

1. In neutral liquidity conditions,\(^1\) the overnight rate level tends to hover around the mid-point of the corridor, while large liquidity surplus drives the rate to the floor of the corridor.

2. The overnight rate volatility is a decreasing function of liquidity surplus.

3. The aggregate market volume is a decreasing function of excess reserves.

4. The overnight rate is an increasing function of counterparty risk.\(^2\)

In addition, using data from the Eurosystem, we show that banks access both the lending and the deposit facilities, even though there is a liquidity surplus. The access to the lending facility decreases, while the access to the deposit facility increases with excess reserves. However, when the liquidity surplus is small but positive, deposits increase less than one-to-one with excess reserves, while they increase almost one-to-one when the liquidity surplus is large.

Standard economic models used to study the interbank market and monetary policy implementation have difficulties in matching those facts. In particular, models of banks’ reserve management in the tradition of Poole (1968), e.g., Woodford (2002), Bindseil (2004), Whitesell (2006) and Ennis and Keister (2011), focus solely on price (i.e. overnight rate) dynamics and consequently, have little to say about quantity and liquidity dynamics. In these models, a representative bank trades reserves in a frictionless market before it receives a liquidity shock. If following the shock, the bank ends up with a negative reserves balance, then it accesses the lending facility. Otherwise, it can earn the interest paid on excess reserves. The interbank rate is then equal to an average of the facilities rates, weighted by the probability to access each facility. In the basic

\(^1\)Neutral liquidity conditions means that there is no liquidity surplus or deficit.
\(^2\)In Bech and Monnet (2013), the spread between the EONIA and the EURONIA captures the effect of counterparty risk. However, the EONIA panel is dominated by large banks from the core countries which limits the effect of counterparty risk. Casual conversations with ECB representatives indicates that the EONIA for PIGS countries displays large volatility that is not decreasing with excess reserves.
model, there is just one representative bank, so there is no trading in the interbank market in equilibrium.³

We modify the basic model in two important ways: First, we assume that banks receive a liquidity shock before the interbank market opens. Second, we assume that banks trade bilaterally in an over-the-counter (OTC) market and bargain over the lending rates.

We consider three matching functions in the bilateral trading stage: Random matching, random matching with directed search, and perfect matching. We see these three matching functions as proxies for different levels of money market efficiency. Under perfect pairing, banks are matched in a way that allows them to perfectly insure against the liquidity shock. In this sense, the matching process is the most efficient. However, this model fails along several dimensions to explain the stylized facts, and we can rule out this level of efficiency. The model with directed search comes closest to explaining the stylized facts.

In the directed search model banks can choose whether they want to borrow or lend reserves. Then banks on either side of the market are matched randomly. The equilibrium decision to become a borrower or a lender implies an interbank market tightness: if there are very few lenders, the market for reserves will be very tight and some borrowers will have to access the lending facility. Remarkably, the unique equilibrium is one where banks that are short of reserves, relative to the reserve requirement, become borrowers while banks that are long become lenders. Hence, excess reserves have a direct impact on the interbank market tightness. We then enrich this model by introducing counterparty risk. We show that counterparty risk can decrease the volume of the interbank market but increases the overnight rate. So, this model is consistent with all the stylized facts.

The model is remarkably tractable and we are also able to compute banks’ willingness to pay for liquidity at a central bank liquidity auction. While we do not model the details of the auction, to our knowledge, this is the first model that offers a notion of willingness to bid for reserves, while taking into account the option to trade these reserves on the interbank market. In addition, we evaluate that banks would be willing to pay xyz% of their profit from their liquidity management operations to introduce a more efficient matching technology. Finally, our model can be used to analyze market

³It is of course possible to extend Poole’s model to assume banks are heterogeneous. However, this would imply that the interbank market trading is only due to shocks, and not to the level of aggregate excess reserves.
volume and other statistics as a function of the width of the corridor. In particular, we find that the volume of trades on the interbank market is not a function of the size of the corridor, except when there is counterparty risk.

In terms of the literature on monetary policy implementation, we already mentioned Poole (1968). The corridor system has also been studied in Berentsen and Monnet (2008) although from the point of view of the transmission mechanism of monetary policy rather than on the functioning of the interbank market. Afonso and Lagos (2012) may be the paper closest to ours, and we review it in depth in Bech and Monnet (2013). Here, let us just mention that they present a continuous time model to explain the intraday pattern of reserves holdings in the federal funds market, which shows that reserve holdings across banks tend to narrow as the day advances (see Ashcraft and Duffie, 2007). In their model, banks also trade in an OTC market and bargain over the terms of trade. However, their OTC market consists of several rounds of random bilateral meetings. In their model, the aggregate market volume is a function of the volatility of the payment shock and the number of trading rounds but does not depend on the amount of excess reserves. Finally, they do not model default risk.

We structure the paper as follows. Section 2 presents the four stylized facts. The basic environment is in Section 3. We then study the equilibrium for the two matching functions in Section 4 and we introduce counterparty risk in Section 5. We conclude in Section 6.

2 The Environment

We consider the following model of banks’ reserves management within a day. There is a unit measure of risk neutral banks indexed by \( k \in \mathcal{K} \). Banks hold no reserves at the beginning of the day, but they are required to hold an amount \( \bar{m} \in \mathbb{R} \) of reserves at the end of the day. If they hold more than \( \bar{m} \), banks earn \( i_d \) per unit of excess reserves by depositing at the central bank. However, if they hold less than \( \bar{m} \), banks borrow the shortfall at the central bank at the penalty rate \( i_p \) where \( i_p \geq i_d \). Banks are trying to maximize their profit given they are required to hold reserves \( \bar{m} \).

At the beginning of the day, the central bank auctions \( M \) units of reserves in a way we specify below. Doing so, the central bank decides on the aggregate amount of excess reserves maintenance period can be several weeks long. Our model is a proxy for banks’ behavior over the course of the maintenance period which we take as being just one day long.
reserves \( x = M - \int_k \tilde{m}dk \) (also known as aggregate liquidity surplus). Bank \( k \) exits the auction with reserves \( m_k \).

Following the auction, banks receive an idiosyncratic shock to their reserve holdings, which is symmetrically distributed over the support \([-\bar{\nu}, \bar{\nu}]\) according to a cumulative distribution function \( F \).\(^5\) Next, banks can access a bilateral over-the-counter (OTC) market for reserves. In this decentralized market, banks first decide whether they want to lend or borrow reserves. Second, borrowers and lenders are randomly matched with each other in a way we describe precisely below.

Banks still holding too little or too much reserves relative to \( \bar{m} \) will access the central bank’s facilities where they can borrow or deposit reserves as described before. Finally, we assume for simplicity that required reserves are remunerated at the same rate as excess reserves, \( i_d \).

3 Equilibrium

We solve the model backward. First we compute the payoff of each bank at the end of the economy depending on their history of trade. Second, given those payoff we can compute the rates and volumes in the OTC interbank market. given those payoffs, we can solve for the decision of banks to become borrowers or lenders, and therefore, the interbank market tightness. Finally, we can then find a bank’s willingness to pay for reserves at the time of the central bank’s reserves auction.

3.1 End of day payoff

The Bernoulli payoff of a bank holding \( m \) at the end of the OTC market that borrowed \( q \geq 0 \) (\( q < 0 \) if lent) at rate \( i \) is

\[
P(i, q, m) = \bar{m}i_d + (m - \bar{m}) \left[ I_{\{m \leq \bar{m}\}} i_p + I_{\{m > \bar{m}\}} i_d \right] - qi\tag{1}
\]

where \( I_A \) is an indicator function that equals 1 if \( A \) is true and zero otherwise. The first term is the remuneration the bank obtains on its required reserves \( \bar{m} \). The second term is the penalty the bank pays on its shortfall, or the remuneration it obtains on its excess reserves. The last term is the income from trading in the OTC market.

\(^5\)This shock capture the net inflow of reserves as a result of interbank payments made on behalf of customers.
3.2 OTC market rates and volumes

Once two banks are matched, they bargain over the terms of trade (the quantity borrowed $q$ and the interest rate $i$). We assume that banks use Nash bargaining and have the same bargaining power. This implies that they split the gains from trade equally. However, as the banks’ payoffs are linear, multiple combinations of $q$ and $i$ achieve the same outcome. To circumvent this indeterminacy, we assume banks equate their reserves holdings so that the amount borrowed is\footnote{Afonso and Lagos (2012) uses the same assumption.}

$$ q(m_{\ell}, m_b) = \frac{m_{\ell} - m_b}{2}, \tag{2} $$

where the borrower holds $m_b$ reserves and the lender holds $m_{\ell} > m_b$ reserves prior to trading. Given (2), banks bargain over the interest rate $i(m_{\ell}, m_b)$. In the Appendix, we show that the OTC rates as a function of the reserve holdings of the lender and the borrower are defined as follows

$$ i(m_{\ell}, m_b) = \begin{cases} 
  i_d & \text{if } m_{\ell} > m_b > \bar{m} \\
  \frac{m_{\ell} - \bar{m}}{m_{\ell} - m_b} i_d + \frac{\bar{m} - m_b}{m_{\ell} - m_b} i_p & \text{if } m_b < \bar{m} < m_{\ell} \\
  i_p & \text{if } m_b < m_{\ell} < \bar{m} 
\end{cases} \tag{3} $$

In words, when both banks have enough reserves to satisfy the reserves requirement, they trade at the deposit rate. On the other hand, if both banks have too little reserves to satisfy their requirement, then they trade at the lending rate. Finally, if one of them can satisfy the reserve requirement and the other cannot, then they trade at a rate which is a weighted average of the corridor rates, where the weights are given by the relative reserves position of each bank: if the lender has a lot of excess reserves, then the rate will tend to the deposit rate, and otherwise, the rate will tend to the lending rate. Notice that we can compute the rates and trades in each match irrespectively of how banks are matched.

Substituting (2) and (3) in (1), we obtain the end of day payoff of the borrower and the lender banks. Denoting the payoff of a borrower bank holding $m_b$ and meeting a
lender bank holding \( m_\ell \) by \( P^b(m_\ell, m_b) \), we have

\[
P^b(m_\ell, m_b) = \begin{cases} 
  m_b i_d & \text{if } m_\ell > m_b > \bar{m} \\
  \bar{m} i_d + (m_b - \bar{m}) i_p + \frac{m_b - m_\ell}{2} (i_p - i_d) & \text{if } m_b < \bar{m} < m_\ell \text{ and } m_b + m_\ell > 2\bar{m} \\
  \bar{m} i_d + (m_b - \bar{m}) i_p + \frac{m_\ell - \bar{m}}{2} (i_p - i_d) & \text{if } m_b < \bar{m} < m_\ell \text{ and } m_b + m_\ell < 2\bar{m} \\
  \bar{m} i_d + (m_\ell - \bar{m}) i_p & \text{if } m_b < m_\ell < \bar{m} 
\end{cases}
\]

and the payoff of the lender bank is

\[
P^f(m_\ell, m_b) = \begin{cases} 
  m_\ell i_d & \text{if } m_\ell > m_b > \bar{m} \\
  m_\ell i_d + \frac{m_b - m_\ell}{2} (i_p - i_d) & \text{if } m_b < \bar{m} < m_\ell \text{ and } m_b + m_\ell > 2\bar{m} \\
  m_\ell i_d + \frac{m_\ell - \bar{m}}{2} (i_p - i_d) & \text{if } m_b < \bar{m} < m_\ell \text{ and } m_b + m_\ell < 2\bar{m} \\
  \bar{m} i_d + (m_\ell - \bar{m}) i_p & \text{if } m_b < m_\ell < \bar{m} 
\end{cases}
\]

Both payoffs are quite intuitive. It is the no-trade payoff increased by the surplus from trade which banks split equally. If both banks in the pair have enough reserves to satisfy the reserve requirement, then they both get the interest on reserves. When the lender can compensate the borrower’s shortfall, there are gains from trading the first \( \bar{m} - m_b \) units of reserves. The gains are measured by the width of the corridor, as the borrower saves \( i_p - i(m_\ell, m_p) \) on these units while the lender obtains \( i(m_\ell, m_p) - i_d \) per unit. Then we obtain the width of the corridor by adding both surpluses. Similarly, when the lender cannot compensate the borrower’s shortfall, there are gains from trading the first \( m_\ell - \bar{m} \) units of reserves, with the gain being again measured by the width of the corridor. Finally, when the pair does not have enough reserves to satisfy either of the reserve requirements, then there is no surplus from trade and both banks access the lending facility to borrow reserves, while they receive the deposit rate on their required reserves.

### 3.3 Search in the OTC market

We consider two search protocols. We use random search as our benchmark protocol according to which banks search randomly for a trading partner in the OTC market. Under the second protocol, banks direct their search in a way that resembles a brokered market.
3.3.1 Random search

Let $G$ be the distribution of reserve holdings across banks following the idiosyncratic shock. Anticipating that all banks will exit the auction with the same amount of reserves $m$, we get $G(\tilde{m}) = F(\tilde{m} - m)$ for any $\tilde{m} \in \mathbb{R}$. With random search, a bank selects a trading partner at random, so that the reserve holdings of the selected banks is distributed according to $G$. Therefore the value of entering the OTC market with $\tilde{m}$ units of reserves is simply

$$V^s(\tilde{m}) = \int_{\tilde{m}}^{\tilde{m} + m} P^b(m', \tilde{m}) dG(m') + \int_{\tilde{m} - m}^{\tilde{m}} P^l(\tilde{m}, m_b) dG(m_b),$$

as a bank with $\tilde{m}$ reserves becomes a borrower to a bank holding more than $\tilde{m}$, and it is a lender otherwise.

3.3.2 Directed (brokered) search

Let $G$ be the distribution of reserve holdings across banks following the idiosyncratic shock. Anticipating that all banks will exit the auction with the same amount of reserves $m$, we get $G(\tilde{m}) = F(\tilde{m} - m)$ for any $\tilde{m} \in \mathbb{R}$. Banks first decide whether they prefer to lend or to borrow. We denote by $\mathcal{B} \subset \mathbb{R}$ the set of reserves holdings of banks who choose to become borrowers and by $\mathcal{L} = \mathbb{R}/\mathcal{B}$ the set of reserve holdings of banks who choose to become lenders. Precisely, a strategy for bank $i$ are two sets $\mathcal{B}_i$ and $\mathcal{L}_i = \mathbb{R}/\mathcal{B}_i$ such that if $\tilde{m}_i \in \mathcal{B}_i$ then bank $i$ chooses to become a borrower and otherwise a lender. We consider symmetric equilibrium, so that $\mathcal{B}_i = \mathcal{B}$ for all banks $i$. Then the strategies and the realized shocks give us $\mathcal{B}$. The measure of borrowers is then $n \equiv \int_{m \in \mathcal{B}} dG(m) \in [0, 1]$. As a consequence, the measure of lenders is $1 - n$. We assume that the matching technology forms pairs of one lender and one borrower. If there are more borrowers than lenders (i.e. $n > 1 - n$) then not all borrowers can be matched. Otherwise, we assume that borrowers are matched for sure. Precisely, we assume a Leontief matching function so that the probability that a borrower meets a lender is given by $\theta(n) = \min\{1; \frac{1 - n}{n}\}$. Similarly, the probability that a lender meets a borrower is $\frac{n}{1 - n} \theta(n)$.\(^7\) The ratio $(1 - n)/n$ measures how tight the interbank market

\(^7\)In the appendix, we consider an alternative matching function where lenders and borrowers are “perfectly” paired rather than randomly paired. Different matching functions could be interpreted as different brokerage arrangements: In the absence of brokers, banks meet at random. Otherwise, banks can contact their broker either as a lender or a borrower and depending on the quality of their broker, they can be matched more or less well.
is. Notice that the matching is not purely random, in the sense that if a bank wants to borrow, then it meets a lender bank; however, which lender bank is random.

Now, we characterize the choice of each bank to become a borrower or a lender given the above equilibrium OTC rates. We denote by $V^b(\tilde{m})$ the expected value of being a borrower for a bank holding $\tilde{m}$ units of reserves when all banks with reserves $m_{\ell} \in \mathcal{L}$ choose to be lenders and all banks with reserves $m_b \in \mathcal{B}$ choose to be borrowers. Similarly, $V^l(\tilde{m})$ is the expected value of being a lender for the same bank, as it enters the OTC market for reserves. Therefore,

$$V^b(\tilde{m}) = \theta(n) \left[ \int_{\tilde{m} \leq m_{\ell} \in \mathcal{L}} P^b(m_{\ell}, \tilde{m})dG(m_{\ell}) + \int_{\tilde{m} > m_{\ell} \in \mathcal{L}} P^l(\tilde{m}, m_{\ell})dG(m_{\ell}) \right] + (1 - \theta(n))P(0, 0, \tilde{m})$$

(6)

Notice that if the bank chose the set $\mathcal{B}$ it could still lend if it is matched with a bank in $\mathcal{L}$ with a lower reserve holding than his. Similarly, we obtain,

$$V^l(\tilde{m}) = \frac{n}{1-n} \theta(n) \left[ \int_{\tilde{m} < m_b \in \mathcal{B}} P^b(m_b, \tilde{m})dG(m_b) + \int_{\tilde{m} > m_b \in \mathcal{B}} P^l(\tilde{m}, m_b)dG(m_b) \right] + \left( 1 - \frac{n}{1-n} \theta(n) \right) P(0, 0, \tilde{m})$$

(7)

Then the decision of the banks is

$$V^b(\tilde{m}) \geq V^l(\tilde{m}) \iff \tilde{m} \in \mathcal{B},$$

(8)

and

$$V^l(\tilde{m}) \geq V^b(\tilde{m}) \iff \tilde{m} \in \mathcal{L}.$$  

(9)

Once we will have defined those values we will show that, when we restrict strategies (or equilibrium) to convex $\mathcal{B}$ and $\mathcal{L}$, then there is a unique threshold $\hat{\tilde{m}}$ such that $V^b(\hat{\tilde{m}}) = V^l(\hat{\tilde{m}})$ and for all $\tilde{m} < \hat{\tilde{m}}, V^b(\tilde{m}) > V^l(\tilde{m})$. Therefore all banks with reserves $\tilde{m} < \hat{\tilde{m}}$ will choose to become borrowers and all banks with reserves $\tilde{m} > \hat{\tilde{m}}$ will prefer to become lenders. To simplify matter further, we will assume (and later verify) that all banks exit the (“auction”-like) centralized reserves market with the same amount of reserves balances, equal to the aggregate supply of reserves: $m$. Therefore, the position of each bank can be summarized by $m$ as well as its idiosyncratic shock $v$. 

9
3.4 Central banks auction

In the very first period of the economy, banks have the opportunity to purchase reserves at the central bank. We assume that the central bank conducts a full allotment auction at a fixed interest rate $r$. Therefore, and as all banks are identical at the time of the auction, they will all demand an amount $m$ of reserves to equate the marginal benefit of reserves to $r$. If $r$ is such that $m = \bar{m}$ then we say that the central bank has a neutral liquidity policy, while if $m > \bar{m}$ the central bank has an aggregate liquidity surplus policy. Given our previous results we can compute the willingness to pay of each bank when the central bank supplies $m$ units of reserves. In particular, to find a bank’s willingness to pay, we need to define the value of exiting this centralized stage with reserves $m$. With random search this is

$$W_s(m) = \int_{-\bar{v}}^{\bar{v}} V^s(m + v)dF(v),$$

and banks demand $m$ such that

$$W'_s(m) = r. \quad (10)$$

With directed search this is

$$W_d(m) = \int_{-\bar{v}}^{\bar{v}} \max[V^b(m + v), V^\ell(m + v)]dF(v)$$

$$= \int_{m + v \in B} V^b(m + v)dF(v) + \int_{m + v \in L} V^\ell(m + v)dF(v)$$

and banks demand $m$ such that

$$W'_d(m) = r. \quad (11)$$

3.5 Definition

**Definition 1.** Given the policy variable $r, i_d$ and $i_p$, an equilibrium with random search is a number $m$ that solves (10).

Notice that, with random search, banks decide only on their reserves at the auction, taking into account the bargaining protocol in the OTC market.

**Definition 2.** Given the policy variable $r, i_d$ and $i_p$, an equilibrium with directed search a list $\{m, L\}$ such that (9) defines the set $L$ and $m$ solves (11).
Notice that given $m$ and $L$, the equilibrium is well defined as $B = [m - \bar{v}, m + \bar{v}] / L$, and then the value functions $W_d$, $V^b$ and $V^\ell$ are all well defined given $i_d$ and $i_{\ell}$. The rest of the paper shows that an equilibrium exists and characterizes its properties.

4 Characterization: random search benchmark

In this section, we characterize the equilibrium with random search. Given (10) and the definition of the equilibrium with random search, it should be obvious that an equilibrium with random search exists and is unique. We now characterize the equilibrium in more details.

4.1 Bidding behavior

To determine banks’ bidding behavior we need to compute $W'_s(m)$, or the derivative of $\int_{-\bar{v}}^{\bar{v}} \int_{m+v}^{m+v} \frac{\partial P^b(m_{\ell}, m + v)}{\partial m} dG(m_{\ell}) dF(v) + \int_{-\bar{v}}^{\bar{v}} \int_{m-v}^{m-v} \frac{\partial P^\ell(m + v, m_b)}{\partial m} dG(m_b) dF(v)$.

Using Leibniz’s rule as well as (4) and (5), we obtain after some manipulations (details are in the Appendix):

$$W'_s(m) = F(\bar{m} - m) i_p + (1 - F(\bar{m} - m)) i_d - \left( \frac{i_p - i_d}{2} \right) \left[ F(\bar{m} - m) - \int_{-\bar{v}}^{\bar{v}} F(2(\bar{m} - m) - v) dF(v) \right]$$

The first two terms are the same as in Poole (1968). The last term, however, is new and refers to banks’ ability to improve their lot relative to accessing the standing facilities by sharing their balances on the OTC market. The term in bracket is the expected probability that the bank is in a match where there are positive gains from trade: To see this, rewrite it as

$$F(\bar{m} - m) \left[ 1 - \int_{-\bar{v}}^{\bar{v}} F(2(\bar{m} - m) - v) \frac{dF(v)}{F(\bar{m} - m)} \right].$$

This is the probability that a bank cannot meet its reserve requirement but meets another bank that can cover its shortfall.\(^8\) When the monetary authority has a neutral

\[^8\int_{-\bar{v}}^{\bar{v}} F(2(\bar{m} - m) - v) \frac{dF(v)}{F(\bar{m} - m)}\] is the expected probability that two banks cannot both meet their reserves requirement, conditional on one of them being short of reserves.
stance, \( \bar{m} - m = 0 \), the term in bracket vanishes\(^9\) and banks are only willing to pay the rate corresponding to the mid-point of the corridor, \( (i_p + i_d) / 2 \).

Banks are willing to pay less than the Poole rate whenever there is a liquidity surplus \( \bar{m} - m < 0 \), as there is a positive probability of covering one’s shortfall in the OTC market. Otherwise, banks are willing to pay a premium for reserves relative to the Poole rate when there is a liquidity shortage, \( \bar{m} - m > 0 \), as there is a probability that they can lend their excess reserves in the OTC market.

### 4.2 Random search: OTC volume, weighted average rate

We now compute the aggregate volume in the OTC market as well as the average interest rate as a function of the excess reserves \( x \) when banks search randomly in the OTC market. The details of the calculations are in the Appendix. The aggregate volume \( Q^s(x) \) is given by

\[
Q^s(x) = \int_{m-\bar{m}}^{m+\bar{m}} \int_{m-\bar{v}}^{m+\bar{v}} \frac{|m_l - m_b|}{2} dG(m_b) dG(m_l)
\]

\[
= \int_{-\bar{v}}^{\bar{v}} \left( v_l - v_b \right) dF(v_b) dF(v_l)
\]

which is constant in the amount of excess reserves \( x \).

We define the average interest rate as

\[
\mu^s_i(x) = \int_{m-\bar{m}}^{m+\bar{m}} \int_{m-\bar{v}}^{m+\bar{v}} \frac{q(m_l, m_b)}{Q^s} i(m_l, m_b) dG(m_l) dG(m_b)
\]

and using the interest rate function, we obtain (using \( m - \bar{m} = x \)) after some manipulations,

\(^9\)Using symmetry, \( \int_{-\bar{v}}^{\bar{v}} F(-v) dF(v) = \int_{-\bar{v}}^{\bar{v}} 1 - F(v) dF(v) = 1 - \int_{-\bar{v}}^{0} F(v) dF(v) - \int_{0}^{\bar{v}} F(v) dF(v) \) and changing the sign of the integrands, \( \int_{-\bar{v}}^{\bar{v}} F(-v) dF(v) = 1 - \int_{0}^{\bar{v}} F(-v) dF(v) - \int_{0}^{\bar{v}} F(v) dF(v) = 1 - \int_{0}^{\bar{v}} (1 - F(v)) dF(v) - \int_{0}^{\bar{v}} F(v) dF(v) = 1/2 \).
\[ \mu^s_i(x)Q^s = \int_{-x}^{x} \int_{v_b}^{0} (v_\ell - v_b)i_d dF(v_\ell) dF(v_b) + \int_{-x}^{x} \int_{v_b}^{0} (v_\ell - v_b)i_p dF(v_\ell) dF(v_b) \\
\quad + \int_{-x}^{x} \int_{-x}^{0} \frac{1}{2} [(v_\ell + x)i_d - (x + v_b)i_p] dF(v_\ell) dF(v_b) \\
\quad + \int_{-x}^{0} \int_{-x}^{x} \frac{1}{2} [(v_b + x)i_d - (x + v_\ell)i_p] dF(v_\ell) dF(v_b) \]

Then notice that, the mean rate is equal to the deposit rate when reserves are plentiful, i.e. \( \mu^s_i(\infty) = i_d \), to the penalty rate when excess reserves are extremely scarce \( \mu^s_i(-\infty) = i_p \), and

\[ \frac{\partial \mu^s_i(x)}{\partial x} = -\frac{(i_p - i_d)}{2Q^s} F(x) [1 - F(x)] \]

Finally, the mean rate is in the mid-point of the corridor defined by the deposit and the penalty rates when \( x = 0 \), i.e., \( \mu^s_i(0) = (i_d + i_p)/2 \). We summarize these results as

**Proposition 1.** With random search, the aggregate volume traded on the OTC market is independent of excess reserves, while the average interest rate is decreasing with excess reserves with a maximum of \( i_p \) and a minimum of \( i_d \).

### 4.3 Random search: Access to lending/deposit facilities

With random search, all banks that cannot satisfy their reserve requirement will access the lending facility to borrow the shortfall from the central bank. Who borrows then? Any two banks that, jointly, do not have enough reserves to satisfy the reserve requirements. That is to say, any two banks \( \ell \) and \( b \), such that \( m_\ell + m_b < 2\bar{m} \). Hence, total borrowing at the lending facility when search is random \( L_R \) is

\[ L_R(\bar{m} - m) = \frac{1}{2} \int_{-\bar{m}}^{\bar{m}} \int_{-\bar{m}}^{\bar{m}} \max\{0, 2(\bar{m} - m) - v_\ell - v_b\} dF(v_\ell) dF(v_b) \]
\[ = \frac{1}{2} \int_{-\bar{m}}^{\bar{m}} \int_{-\bar{m}}^{\bar{m}} 2(\bar{m} - m) - v_\ell - v_b dF(v_\ell) dF(v_b) \]
In particular, under neutral liquidity conditions \( \bar{m} = m \) we have

\[
L_R(0) = \frac{1}{2} \int_{-\bar{v}}^{\bar{v}} \int_{-\bar{v}}^{\bar{v}} (-v_{\ell} - v_{b}) dF(v_{\ell}) dF(v_{b}) = \frac{1}{2} \int_{-\bar{v}}^{\bar{v}} \int_{-\bar{v}}^{\bar{v}} (v_{\ell} + v_{b}) dF(v_{\ell}) dF(v_{b})
\]

Similarly, banks use the deposit facility whenever they jointly have enough to satisfy their reserve requirements, or

\[
D_R(\bar{m} - m) = \int_{-\bar{v}}^{\bar{v}} \int_{-\bar{v}}^{\bar{v}} \max\{0; m - \bar{m} + \frac{v_{\ell} + v_{b}}{2}\} dF(v_{\ell}) dF(v_{b}) = \int_{-\bar{v}}^{\bar{v}} \int_{2(\bar{m} - m) - \bar{v}}^{\bar{v}} (m - \bar{m} + \frac{v_{\ell} + v_{b}}{2}) dF(v_{\ell}) dF(v_{b})
\]

Later, we compare these statistics with the one of the model with directed search.

5 Characterization: directed search

In this section, we compute the same statistics for the model where banks use directed search, as above. In the Appendix, we show that an equilibrium with directed search exists, or more precisely:

**Proposition 2.** Let \( \mathcal{B} \) be convex, then a unique equilibrium exists where all banks with reserves below \( \bar{m} \) choose to become borrowers, while banks with reserves above \( \bar{m} \) choose to become lenders. The number of borrowers is \( n = F(\bar{m} - m) \).

Notice that choosing to become a “borrower” or a “lender” here is similar to choosing a “bin” with a particular color (say blue and red). The proposition says that banks with reserves below \( \bar{m} \) will choose the same bin (whether blue or red) while banks with reserves above \( \bar{m} \) will all choose the bin with the other color. Therefore, while the proposition shows that all banks who need to borrow will be in the same bin (named the “borrowers” bin), we cannot say if banks will coordinate on the blue or the red bin. However, this is somewhat inconsequential for the rest of the paper and we ignore this detail here.

If we allow \( \mathcal{B} \) to be non-convex\(^{10}\) then there might be other equilibria (although

\(^{10}\)For example let \( \bar{m} = 0 \) then banks could choose to be borrowers for all \( m \in [-\infty, -10] \cup [0, 10] \) and lenders for all \( m \in [-10, 0] \cup [10, \infty] \).
we could not find one). For example, consider the sets $B = \bigcup_{n \in \mathbb{Z}}[n - \frac{1}{2}, n]$ and $L = \bigcup_{n \in \mathbb{Z}}(n, n + \frac{1}{2})$. Then borrowers and lenders are spread all over the real line and it is not obvious why deviating would be beneficial then. However, this type of equilibrium has the feature that some banks in $B$ will end up lending, while they were in to borrow. In the Appendix we refine our concept of equilibrium, by further imposing that a bank’s ex-ante choice has to be consistent with its ex-post action. In our context, this means that a bank who chose the borrower’s bin ex-ante cannot lend ex-post (it does not trade). Then the unique equilibrium for any strategy space is the one of Proposition 2 and we can dispense with the requirement that $B$ is convex.

When $B$ is convex, its supremum defines a threshold of reserves below which banks will choose to become borrowers. It is rather intuitive that this threshold be $\bar{m}$, the amount of required reserves. We sketch part of the argument below and we relegate the cumbersome steps of the proof to the Appendix.

Proof. Suppose that all banks but one (bank $j$, say) choose the threshold $\hat{m}$. Suppose bank $j$ has more reserves than $\hat{m}$ ($m_j > \hat{m}$, i.e. it receives a shock $v_j > \hat{m} - m$). If bank $i$ chooses to become borrower, then it will meet a lender bank that has more that $\hat{m}$ (as all banks, but bank $j$ chose to be lenders only if they have more than $\hat{m}$). We know these two banks will trade at $i_d$. In this case the payoff of bank $j$ is just $i_dm_j$, so that bank $j$ makes no gain out of choosing to become borrower. If bank $j$ chooses to become a lender, then it will meet a borrower with some reserves $m_b < \hat{m}$. Looking at (5), the payoff of bank $j$ is always higher than if it chooses to become a borrower. Therefore, given all banks choose a threshold $\hat{m}$, bank $j$ will also choose the threshold $\hat{m}$. The same argument applies when bank $j$ has few reserves, or $m_j < \hat{m}$. Hence, we have shown that the threshold $\hat{m}$ is an equilibrium. However, is there another equilibrium threshold?

Suppose that all banks, but bank $j$, choose a threshold $\hat{m} > \hat{m}$. The same argument as above shows that if bank $j$ has $m_j > \hat{m}$ then it will choose to be a lender. Now, suppose $\hat{m} > m_j > \hat{m}$. In this position, other banks choose to become borrowers. What should bank $j$ do? If it chooses to become a borrower, then it will get payoff $i_dm_j$. If it chooses to become a lender, then bank $j$ will meet a borrower with $m_b < \hat{m} < m_j$ with probability $F(\hat{m} - m)/F(\hat{m} - m)$. In this case bank $j$ gets a higher payoff than $i_dm_j$, as shown by (5). In case bank $j$ meets a borrower with $\hat{m} < m_b < \hat{m}$ then bank $j$’s payoff is again $i_dm_j$. Therefore, bank $j$ has a higher payoff from becoming a lender whenever $\hat{m} > m_j > \hat{m}$, although all other banks in this position choose to be borrower.
Therefore, $\hat{m} > \bar{m}$ is not an equilibrium. A similar argument applies when $\hat{m} < \bar{m}$.

Hence the unique equilibrium is one where the threshold is $\bar{m}$.

It then easily follows that the measure of borrowers is given by the measure of banks receiving a low shock, i.e. $v$ such that $m + v < \bar{m}$. Hence, the measure of borrowers is $n = F(\bar{m} - m)$.

Given the simple structure of the equilibrium, we can easily compute the equilibrium of borrowers and lenders (in the Appendix). Then the marginal values of reserves are easily interpreted. The marginal value of holding an additional unit of reserves for a borrower is the rate that it won’t have to pay at the lending facility $1 + i_p$, minus the expected benefit of acquiring this unit in the OTC market. The expected benefits naturally depends on the market tightness $\theta(n)$, as well as the probability to meet a lender who has enough reserves so that both banks can satisfy their reserve requirements. This is similar for lenders.

In this equilibrium, the marginal value of reserves is

$$\frac{\partial V^b(\bar{m})}{\partial \bar{m}} = i_p - \theta(n) \frac{(i_p - i_d) 1 - F(2\bar{m} - m - \bar{m})}{1 - F(\bar{m} - m)}$$

for any $\hat{m} < \bar{m}$

$$\frac{\partial V^l(\bar{m})}{\partial \bar{m}} = i_d + \frac{n}{1 - n} \theta(n) \frac{(i_p - i_d) F(2\bar{m} - m - \bar{m})}{F(\bar{m} - m)}$$

for any $\bar{m} \geq \hat{m}$

Relative to the Poole (1968) model where banks cannot access any OTC market, it is useful to notice that the OTC market is making reserves less valuable for borrowers and more valuable for lenders. The reason is that the OTC market is one way for banks to smooth their reserves holdings which is not present in the standard Poole model.

Proposition 2 is crucial in understanding the bidding behavior of banks in the centralized auction like market. In particular, banks will tend to be more responsive to a rate change as they can “insure” themselves somewhat on the OTC market. We study this next.

### 5.1 Bidding behavior: directed search

To determine banks’ bidding behavior we need to define the value of reserves at the end of the “auction”. This is $W_d(m)$:
\[ W_a(m) = \int_{-\bar{v}}^{\bar{v}} \max \left[ V^b(m + v), V^\ell(m + v) \right] dF(v) \]
\[ = \int_{-\bar{v}}^{\bar{v}} V^b(m + v)dF(v) + \int_{m-m}^{\bar{v}} V^\ell(m + v)dF(v) \]
as banks will choose to become borrowers when they receive a sufficiently low shock and lenders when they receive a high enough shock. The interest rate on the auction market will be given by the marginal value of reserves, or \( W_a' (m) \). Using Leibniz’ rule as well as (34) and (37) we find that \( W_a' (m) \) is

\[ W_a'(m) = F(\bar{m} - m)i_p + (1 - F(\bar{m} - m))i_d \]
\[ -\theta(n)(i_p - i_d) \left[ \int_{-\bar{v}}^{m-m} \left[ 1 - F(2(\bar{m} - m) - v) \right] \frac{dF(v)}{1 - F(\bar{m} - m)} \right] \]
\[ + \frac{n}{1 - n} \theta(n)(i_p - i_d) \int_{m-m}^{\bar{v}} F(2(\bar{m} - m) - v) \frac{dF(v)}{F(\bar{m} - m)} \]

(12)

Again, the first two terms are the same as in Poole (1968), while the last term refers to the likelihood of being able to share one’s balances on the OTC market. To understand the second term, notice that \( 1 - F(2(\bar{m} - m) - v) \) is the probability that, when a bank holds \( m + v \) reserves, this bank meets another bank holding enough reserves for them both to meet the reserve requirements. Therefore, \( 1 - F(2(\bar{m} - m) - v) \) is the probability that a borrower bank holding \( m + v < \bar{m} \) reserves and in an OTC meeting meets the reserve requirements at the end of the day. So the first term in bracket is the probability that a bank gets a sufficiently high shock \( v \) that, by borrowing on the interbank market, would still meet its reserve requirements: This reduces the bank’s willingness to pay for extra reserves by the cost of reserves in such a meeting, or \( i_p - i_d \).

Similarly, when a bank turns out to be a lender, it might have to lend the extra reserves to the borrower making a gain of \( i_p - i_d \) on it then. This is the third term in the above expression. Notice that in the case of neutral liquidity stance (and symmetric liquidity shock), where \( \bar{m} = m \), these two extra terms (due to the OTC market) vanish, and we are back to the basic Poole’s model. Using \( n = F(\bar{m} - m) \), we can simplify (12) as

\[ W_a'(m) = ni_p + (1 - n)i_d - \theta(n) \left( \frac{i_p - i_d}{1 - n} \right) \left[ n - \int_{-\bar{v}}^{m-m} F(2(\bar{m} - m) - v)dF(v) \right] \]

Figure 1 below plots the rates for the Poole’s model in green and the one for our
benchmark random search model in blue, and the one for the model with directed search in red, for required reserves of $\bar{m} = 1$ and aggregate reserves of $m$ varying from $-5$ to $+5$, where $v$ is distributed according to a Normal with mean 0 and variance 2. We used $i_p = 1.025$ and $i_d = 1.015$.

![Figure 1: Willingness to pay for reserves. Poole 1968’s model (green); Random search (blue); Directed search (red).](image)

Notice that directed search offers more insurance against liquidity shock relative to random search or no search (as in the Poole model), so that banks may appear to shade their bids more with directed search when there is an aggregate excess liquidity, or bid more aggressively when there is an aggregate liquidity deficit.

5.2 Directed search: OTC volume, weighted average rate, and rate volatility

**Market volume** Using the model, we can now describe how the aggregate liquidity deficit $\bar{m} - m > 0$ impacts the volume of trade on the OTC interbank market. The aggregate trade volume is given by the total size of trades $\tilde{Q}$ times the number of matches. Given our matching function we obtain that the aggregate trade volume is $Q(\bar{m} - m) = \min\{n, 1 - n\} \tilde{Q}(\bar{m} - m)$ where $\tilde{Q}$ is the average trade size when a trade occurs,

$$\tilde{Q}(\bar{m} - m) = \int_{m-\tilde{v}}^{\bar{m}} \int_{m}^{m+\tilde{v}} q(m_b, m_{\ell}) dF_{\ell}(m_{\ell}) dF_b(m_b)$$
where $F_\ell$ is the distribution of the lenders’ reserves in the OTC market and, similarly, $F_b$ is the distribution of the borrowers’ reserves on the OTC market. Defining $Q$, we have already taken into account that, in equilibrium, borrowers always hold less than $\bar{m}$ units of reserves, while lenders always hold more than $\bar{m}$. Also, we know that in equilibrium borrowers hold $m_b = m + v_b$ for some $v_b \leq \bar{m} - m$ while lenders hold $m_\ell = m + v_\ell$ for some $v_\ell \geq m - \bar{m}$. Using (2) we have the following result,

**Proposition 3.** Suppose $v$ has a zero mean and is distributed symmetrically around its mean. Then $Q(\bar{m} - m) \geq 0$ for all values of $\bar{m} - m$. Also the aggregate trading volume $Q(\bar{m} - m)$ is single-picked and attain its unique maximum at $\bar{m} = m$. Also $\lim_{\bar{m} - m \to 0} Q(\bar{m} - m) = \lim_{\bar{m} - m \to -\bar{m}} Q(\bar{m} - m) = 0$, i.e. aggregate volume decreases to zero as the liquidity deficit or surplus become large.

Under the assumption of Proposition , we can simplify $Q$ as

$$Q(x) = \frac{\min\{F(x), 1 - F(x)\}}{2F(x)[1 - F(x)]} \int_{-\bar{v}}^{x} (-v) dF(v),$$

(13)

where $x$ measures the liquidity deficit. In general, we can show that $\lim_{\bar{m} - m \to 0} Q(\bar{m} - m) = \lim_{\bar{m} - m \to -\bar{m}} Q(\bar{m} - m) = 0$. Figure 2 plots the aggregate volume on the interbank market as a function of $x$, for the same parameters as before.

Figure 2: Volume on the interbank market as a function of $\bar{m} - m$. 

19
Average rate Once we have the interbank market volume, it is straightforward to compute the weighted average rate in the OTC market. This is $\mu_i$ which satisfies

$$\mu_i = \min\{F(\bar{m} - m), 1 - F(\bar{m} - m)\} \int_{m-b}^{\bar{m}} \int_{m}^{m+b} \frac{q(m_b, m_l)}{Q(\bar{m} - m)} i_m(m_b, m_l) dF_l(m_l) dF_b(m_b)$$

which we can simplify as

$$\mu_i = \int_{m-b}^{\bar{m}} \int_{m}^{m+b} \frac{q(m_b, m_l)}{Q(\bar{m} - m)} i_m(m_b, m_l) dF_l(m_l) dF_b(m_b)$$

and using (2), (13) combined with (3) as well as $E[v] = 0$, we can simplify this expression further to obtain (the details are in the Appendix):

$$\mu_i = (1 - F(\bar{m} - m)) i_p + F(\bar{m} - m) i_d + \frac{(\bar{m} - m)}{Q(\bar{m} - m)} \frac{(i_p - i_d)}{2}$$

The last component of $\mu_i$ is the average gains from trading in the OTC market. The first and second components do not have the “usual” Poole’s weights. To understand why, notice that the weights have a natural interpretation. The first weight $1 - F(\bar{m} - m)$ is half of the average liquidity shock of a borrower relative to the aggregate trading volume, and the reservation value on these shocks is $i_p$. Also, $F(\bar{m} - m)$ is half of the average liquidity shock of a lender relative to the aggregate trading volume, and the reservation value on these trades is $i_d$. Therefore $\mu_i$ is the average of the facility rates weighted by the relative size of the borrowers’ and lenders’ shocks and adjusted for the gains from trade. With neutral liquidity conditions, $\bar{m} = m$, the interbank rate is at the mid-corridor point. As a function of the liquidity deficit $x$, the interbank rate is

$$\mu_i(x) = (1 - F(x)) i_p + F(x) i_d + \frac{x}{Q(x)} \frac{(i_p - i_d)}{2}$$

That is,

$$1 - F(\bar{m} - m) = \frac{1}{2Q(\bar{m} - m)} \int_{-\bar{v}}^{\bar{m} - m} -v \cdot dF(v)$$

while

$$F(\bar{m} - m) = \frac{1}{2Q(\bar{m} - m)} \int_{-\bar{m} - m}^{\bar{v}} v \cdot dF(v) = \frac{1}{2Q(\bar{m} - m)} \left\{ \frac{E[v]}{F(\bar{m} - m)} - \int_{-\bar{v}}^{\bar{m} - m} v \cdot dF(v) \right\}$$
and it’s derivative

\[
\mu_i'(x) = f(x)(i_d - i_p) + \frac{(i_p - i_d) \tilde{Q}(x) - x\tilde{Q}'(x)}{2 \tilde{Q}(x)^2}
\]

We cannot sign \( \mu_i'(x) \) as the first term is always negative. On the one hand, increasing the liquidity deficit will tend to increase the interbank market rate, as the average liquidity shock of a borrower relative to the aggregate volume of trade decreases. On the other, this effect is tampered by the change in the surplus from trade. For a normal distribution we obtain \( \mu_i'(x) > 0 \) for all \( x \) as shown in the graph below (same parameters as before, magenta curve shows \( \mu_i(\bar{m} - m) \) where \( m \) in on the x-axis relative to the usual Poole’s rate - green - and the auction rate - red).

The figure below shows the average interbank rate in the Poole model (green), the random search model (blue) and the directed search model (red). Clearly, the average rate with directed search is much less sensitive to a change in excess reserves. The reason being that with random search, many of the trades are conducted between banks who are both long or short of reserves, while this never happens with directed search. Therefore when there is a liquidity deficit, the rate with random search will exceed the rate with directed search and will be below otherwise.

**Volatility** We now compute the rate volatility, using the weighted variance,

\[
\sigma^2_w(i) = \int_{\bar{v}}^{\bar{m} - m} \int_{-\bar{v}}^{\bar{v}} \frac{q(v_\ell, v_b)}{Q(\bar{m}, m)} [i(v_\ell, v_b) - \mu_i]^2 \frac{dF(v_\ell)}{1 - F(\bar{m} - m)} \frac{dF(v_b)}{F(\bar{m} - m)},
\]
and in the Appendix, we show that whenever $c = F(\bar{m} - m) - \frac{(\bar{m} - m)}{2Q(\bar{m} - m)} \geq 0$ (which is always the case when $m > \bar{m}$), then

$$\sigma_w^2(i_m) \leq (i_p - i_d)^2 c(1 + c).$$

In particular, with neutral provision of liquidity, the volatility is less than $\frac{3}{4}(i_p - i_d)^2$. Therefore, if the corridor width is 100 basis points, then the volatility is less than 0.75 basis point.

To find a lower bound for the OTC rate volatility, we restrict our attention to the case where there is a liquidity surplus, or $m > \bar{m}$. In this case, we show in the Appendix that

$$\sigma_w^2(i) \geq (i_p - i_d)^2 \left[\int_{\bar{v}}^{\bar{m} - m} \int_{\bar{m} - m}^{\bar{v}} \frac{(v_t - v_b)}{2Q} \left(\frac{v_t - \bar{m} - m}{v_t - v_b}\right)^2 \frac{dF(v_t)}{1 - F(\bar{m} - m)} \frac{dF(v_b)}{F(\bar{m} - m)} - 2c + c^2\right]$$

Notice that the first term in bracket is positive but less than unity.\(^\text{12}\) From there it follows that $\sigma_w^2(i_m) \geq (i_p - i_d)^2 c(c - 2)$. However, this may be a loose bound if $c < 2$. Figure 5 plots the standard deviation of OTC rates whenever the shocks are normally distributed (left panel) or uniformly distributed (right panel).

\(^{\text{12}}\) i.e. $\int_{\bar{v}}^{\bar{m} - m} \int_{\bar{m} - m}^{\bar{v}} \frac{(v_t - v_b)}{2Q} \left(\frac{v_t - \bar{m} - m}{v_t - v_b}\right)^2 \frac{dF(v_t)}{1 - F(\bar{m} - m)} \frac{dF(v_b)}{F(\bar{m} - m)} \leq 1.$
5.3 Access to lending/deposit facilities

In this section, we describe the access to the lending and deposit facilities as a function of excess reserves. We present the absolute values as well as the relative values. To some extent, this is more telling of the success of each model (random or directed search) as the absolute access to facilities do not depend on the trading protocol (i.e. it does not depend on (2)) but rather on whether both banks in a match have enough joint reserves to satisfy their reserves requirements. The figure below shows the average access to the ECB deposit facility from 1999 to 2014 over one month (in millions EUR) as a function of excess liquidity at the end of the same month (in millions EUR).

The following graphs show the recourse to the deposit facility between January 1999 and March 2014, splitting the sample in three periods: The green triangles show the pre-crisis period of relatively small liquidity surplus. There the correlation between liquidity surplus and access to the deposit facility is around 0.72 (for one additional EUR of excess reserves, 0.72 cents are deposited with the ECB). The red squares show the post-Lehman crisis period with rapidly increasing and high liquidity surplus and falling interest rates. In this period, the correlation edged up to 0.99. Finally, the blue diamonds show the period with high liquidity surplus and zero interest rate at the deposit facility. As banks are indifferent between leaving reserves on their account or

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13In its May 2002 Monthly Bulletin, the ECB defines “excess reserves” as the current account holdings (in excess of the reserve requirements) of banks which have already fulfill their reserve requirements. Those excess reserves are not deposited at the deposit facility and so do not earn interests. Therefore in the Eurosystem, the excess liquidity is the amount of excess reserves plus the amount at the deposit facility.
using the deposit facility, it is not surprising that the correlation in this period falls to below 0.5.

Figure 6: Average amount deposited with the ECB during one month (in millions EUR), as a function of excess reserves at the end of the same month (in millions EUR). Source: ECB.

We conclude from the ECB experience that the correlation between positive liquidity surplus and the access to the deposit facility is large and increasing with the magnitude of the liquidity surplus, although it always stays below unity.

5.3.1 Directed search: Access to lending/deposit facilities

When search is directed, total borrowing at the lending facility consists of those borrowing banks – a measure $F(\bar{m} - m)$ – who did not find a partner – a measure $1 - \theta(n)$ of them – as well as those who found a lender that could not cover their shortfalls. Finally, a measure of lenders will also borrow: those who found a borrower, but who could not cover their shortfall. Hence, total borrowing at the lending facility when search is directed $L_D$ is

$$L_D(\bar{m} - m) = (1 - \theta(n)) F(\bar{m} - m) \int_{-\bar{v}}^{\bar{m} - m} (\bar{m} - m - v) \frac{dF(v)}{F(\bar{m} - m)}$$

$$+ \theta(n) F(\bar{m} - m) \int_{-\bar{v}}^{\bar{m} - m} \int_{m - m}^{2(\bar{m} - m) - v_b} (\bar{m} - m - \frac{v_t + v_b}{2}) \frac{dF(v_t)}{1 - F(\bar{m} - m)} \frac{dF(v_b)}{F(\bar{m} - m)}$$

$$+ \frac{n}{1 - n} \theta(n) (1 - F(\bar{m} - m)) \int_{-\bar{v}}^{\bar{m} - m} \int_{m - m}^{2(\bar{m} - m) - v_b} (\bar{m} - m - \frac{v_t + v_b}{2}) \frac{dF(v_b)}{F(\bar{m} - m)} dF$$
As we imposed pairwise matching between borrowers and lenders, the measure of the last two groups has to be the same. Hence, we obtain

\[\mathcal{L}_D(\bar{m} - m) = (1 - \theta(n)) \int_{-\bar{v}}^{\bar{m} - m} (\bar{m} - m - v)dF(v) + \theta(n) \int_{-\bar{v}}^{\bar{m} - m} \int_{\bar{m} - m}^{2(\bar{m} - m) - v_b} [2(\bar{m} - m) - (v_\ell + v_b)] \frac{dF(v_\ell)}{F(\bar{m} - m)} \frac{dF(v_b)}{F(\bar{m} - m)} \]

and using \(\theta(n) = \min \left\{1, \frac{1 - F(\bar{m} - m)}{F(\bar{m} - m)} \right\}\) we obtain the following figure for \(\mathcal{L}_R(\bar{m} - m)\) (red) and \(\mathcal{L}_D(\bar{m} - m)\) (green) - left.

![Figure 7: Borrowing at the CB lending facility as a function of \(m\) (\(\bar{m} = 0\))](image)

Also, we have

\[\mathcal{L}_D(0) = 2 \int_{-\bar{v}}^{0} \int_{0}^{v_b} (-v_\ell - v_b)dF(v_\ell)dF(v_b) = 2 \int_{0}^{v_b} \int_{0}^{v_b} (v_\ell + v_b)dF(v_\ell)dF(v_b)\]

Similarly, with directed search, total deposits \(\mathcal{D}_D\) is

\[\mathcal{D}_D(\bar{m} - m) = (1 - \frac{n}{1 - n} \theta(n)) \int_{\bar{m} - m}^{\bar{m} - m - v_\ell} (m + v - \bar{m})dF(v) + \frac{n}{1 - n} \theta(n) \int_{\bar{m} - m}^{\bar{m} - m - v_\ell} \int_{2(\bar{m} - m) - v_b}^{\bar{m} - m} [2(m - \bar{m}) + v_\ell + v_b] \frac{dF(v_\ell)}{F(\bar{m} - m)} \frac{dF(v_b)}{F(\bar{m} - m)} \]

\(\mathcal{D}_R(\bar{m} - m)\) and \(\mathcal{D}_D(\bar{m} - m)\) are shown below.

The slope of the deposit curve with directed search (green line) is around 0.81 for relatively low excess reserves \(\bar{m} - m \in [0, 2]\) - and around 0.98 for relatively high excess reserves \(\bar{m} - m \in [2, 15]\). Hence, the directed search model has the ability to replicate the Eurosystem data. The difference in the slopes is coming from the matching protocol: When reserves are plenty, there are a lot of matches where \(m_b + m_\ell > 2\bar{m}\)
so that both banks will end up depositing some reserves. In this context, additional reserves are very likely to end up at the deposit facility. Conversely, when reserves are relatively scarcer, many matches will have $m_b + m_\ell < 2\bar{m}$ so that neither banks will end up have reserves in excess of the requirement. Then, additional reserves are used to satisfy the reserves requirement and it is unlikely that they will end up at the deposit facility. Notice that a less efficient match increases the number of matches where $m_b + m_\ell < 2\bar{m}$, so that the correlation between deposit facility access and excess reserves is lower as the matching protocol is less efficient. For example, with perfect pairing, the correlation is always 1 as $m_b + m_\ell = m$ for all matches.

The difference in the red and green curve is a measure of the efficiency loss due to the matching protocol. With the perfect matching protocol that we study in the Appendix, the deposit curve is the purple straight line going through the origin. However, when banks are not allowed to trade once they receive their shock, the access to the deposit facility is given by the blue curve. Naturally all the lines converge when excess liquidity grows. So, Figure 9 says that the matching protocols matters most for efficiency when the liquidity conditions are neutral.
6 Counterparty risk

We now introduce the probability that banks default, but only in the case with directed search. To introduce counterparty risk, we assume that each bank knows that it can disappear with probability \( \delta \in (0, 1) \), whether they are buyers or sellers. We assume that the central bank still charge \( i_p \) when a bank accesses its lending facility, irrespective of \( \delta \).

Again, as banks are risk neutral, we assume that, if they trade, banks equate their reserves holdings so that (2) will hold, even with default. Lenders will adjust the rate they charge in order to compensate for the expected loss from a default and we use \( i^\delta \) to denote the rate they charge when there is counterparty risk. In the Appendix, we show that the OTC rate as a function of \((m_\ell, m_b)\) are defined as follows

\[
i^\delta(m_\ell, m_b) = i(m_\ell, m_b) + \frac{\delta}{(2 - \delta)} i(m_\ell, m_b)
\]

where we have defined \( i(m_\ell, m_b) \) in (3). Hence, the probability of default introduces a risk premium that is proportional to the risk-free rate.\(^{14}\) Notice that when \( \delta \) is high, we will get \( i^\delta(m_\ell, m_b) > i_p \) in which case the borrower will prefer to borrow from the lending facility. In other words, banks will not trade – i.e. \( q^\delta(m_\ell, m_b) = 0 \) – whenever \( i^\delta \geq i_p \) or, equivalently, whenever\(^{15}\)

\[
i(m_\ell, m_b) \geq \left(1 - \frac{\delta}{2}\right) i_p.
\]

\(^{14}\)The reason why the risk premium is not increasing to infinity as \( \delta \to 1 \) is that lenders are also disappearing with almost certainty in that case, so they don’t really care to be reimbursed or not.

\(^{15}\)We have used (2) to compute the rates (14). Technically, there could still be some trade \( q^\delta < \frac{m_\ell - m_b}{2} \) at the rate \( i^\delta = i_p \), when (15) is satisfied. However, there is no surplus from trade in this case, as the borrower is indifferent between borrowing \( q^\delta < q \) from the lender at \( i^\delta = i_p \) and accessing the lending facility. Therefore, the borrower’s surplus is nil. Since the borrower and the lender equate their surplus, the lender’s payoff is also nil. Therefore, we chose to ignore those trades by setting \( q^\delta = 0 \) in those cases.
Using (3) and (14), \( q^\delta(m_\ell, m_b) \) is defined as

\[
q^\delta(m_\ell, m_b) = \begin{cases} 
0 & \text{if } m_b < m_\ell < \bar{m} \\
0 & \text{if } m_b < \bar{m} < m_\ell \text{ and } i_m \geq \left(1 - \frac{\delta}{2}\right)i_p \\
0 & \text{if } \bar{m} < m_b < m_\ell \\
\frac{m_\ell - m_b}{2} & \text{otherwise}
\end{cases}
\]

Notice that trade collapses when both banks have enough reserves to satisfy their reserves requirement: in this case the borrower would have to pay the risk premium while only getting a benefit of \( i_d \) on the amount borrowed. Therefore the potential borrower would not want to borrow in this case. Using (3) and arranging, we can rewrite the no trade condition (15) when \( m_b < \bar{m} < m_\ell \) as

\[
(\Delta - 1)(m_\ell - \bar{m}) \leq \bar{m} - m_b,
\]

where

\[
\Delta = \frac{2(i_p - i_d)}{\delta i_p}.
\]

Therefore, trading will collapse when the corridor shrinks to zero, as we have assumed \( m_\ell > \bar{m} > m_b \). Now, when \( \bar{m} < m_b < m_\ell \), we can rewrite (15) as \( \delta i_p > 2(i_p - i_d) \). Since \( i_m \geq i_d \), notice that if this last inequality holds then banks will never trade, under no circumstances, as (16) is also trivially satisfied. Therefore, banks never trade if the corridor is small relative to \( \delta \). This is our first result regarding counterparty risk.

**Lemma 1.** There is no trade on the interbank market if \( \delta \geq \frac{2(i_p - i_d)}{i_p} \).

Therefore, if the corridor becomes very small while the default rate increases (a situation that occurred during the recent crisis), then the interbank market may well cease to function altogether.

To make the analysis a little more interesting, we will assume from now that \( \Delta > 1 \), so that trade would occur if the “conditions are right,” (i.e. if (16) does not hold). When there is a risk of default and when \( \bar{m} > m_b \), the trading condition is

\[
m_\ell - \bar{m} > \frac{\bar{m} - m_b}{\Delta - 1}.
\]

In particular, if \( \Delta \leq 2 \) then the right hand side of (18) is always greater than \( \bar{m} - m_b \). Therefore, if there is sufficient counterparty risk (i.e. \( \Delta \leq 2 \)) and \( \bar{m} > m_b \) and when
there is trade, then we are necessarily in the case where $m_\ell + m_b > 2\bar{m}$. In words, with high counterparty risk, lenders only lend when they are sure to have enough reserves, even after they extend a loan. Figure 10 shows the regions of pairs $(m_\ell, m_b)$ where there is trade in the case of high counterparty risk. The “no trade” regions are the ones where banks do not trade because of counterparty risk. As the figure illustrates, with a high risk of default, banks will not trade, even though there is sufficient reserves within the pair to cover the reserves requirements of both banks. As $\Delta$ decreases to 1 the red curve becomes a straight vertical line at $m_b = \bar{m}$ and trade collapses.

With little counterparty risk, i.e. $\delta < \frac{i_p - i_d}{i_p}$, Figure 11 shows that banks do not trade only if the lender bank cannot sufficiently cover the reserves needs of the borrower. In this case, when $\bar{m} > m_b$ and when there is trade, then we are necessarily in the case where $m^*(\Delta) < m_\ell + m_b \leq 2\bar{m}$, for some $m^*(\Delta)$. Figure 11 shows the regions of pairs $(m_\ell, m_b)$ where there is trade. As $\Delta$ increases to infinity (or $\delta$ decreases to zero), the red curve becomes a straight horizontal line at $m_\ell = \bar{m}$ and trade converges to the no-risk case.

We can use (2) and (3) to obtain the payoff of the borrower and the lender banks. Denoting the payoff of a borrower bank holding $m_b$ and meeting a lender bank holding
Figure 11: Trade region with low counterparty risk $\delta < \frac{i_p - i_d}{i_p}$

$m_\ell$ by $P_b(m_\ell, m_b; \delta)$, we have (using the fact that there is no trade when $m_b + m_\ell < 2\bar{m}$),

$$
P_b(m_\ell, m_b; \delta) \frac{i_d m_b}{1 - \delta} = \begin{cases} 
    i_d m_b & \text{if } m_\ell > m_b > \bar{m} \\
    P_b(m_\ell, m_b; 0) - \frac{\delta}{2 - \delta} i q & \text{if } \begin{cases} 
        m_b < \bar{m} < m_\ell \\
        m_\ell - \bar{m} > \frac{\bar{m} - m_b}{\Delta - 1}
    \end{cases} \text{ and } \\
    i_p (m_b - \bar{m}) + i_d \bar{n} & \text{if } \begin{cases} 
        m_b < \bar{m} < m_\ell \\
        m_\ell - \bar{m} \leq \frac{\bar{m} - m_b}{\Delta - 1}
    \end{cases} \text{ and } \\
    i_p (m_b - \bar{m}) + i_d \bar{n} & \text{if } m_b < m_\ell < \bar{m}
\end{cases}
$$

(19)
and the payoff of the lender bank is

\[
P^\ell(m_\ell, m_b; \delta) = \begin{cases} 
  i_d m_\ell & \text{if } m_\ell > m_b > \bar{m} \\
  P^\ell(m_\ell, m_b; 0) + \frac{\delta}{2 - \delta} i q & \text{if } m_b < \bar{m} < m_\ell \text{ and } m_\ell - \bar{m} > \frac{m - m_b}{\Delta - 1} \\
  i_d m_\ell & \text{if } m_b < \bar{m} < m_\ell \text{ and } m_\ell - \bar{m} \leq \frac{m - m_b}{\Delta - 1} \\
  i_p(m_\ell - \bar{m}) + i_d \bar{m} & \text{if } m_b < m_\ell < \bar{m}
\end{cases}
\]

where \( i_m \) is given by (3) in the specified region and \( q \) is defined in (2). Notice that, when there trade, counterparty risk increases (resp. decreases) the payoff of the lender (resp. borrower) by the expected loss in case default occurs.

Assuming that banks choose to become lenders whenever \( m + v \geq \bar{m} \) for some \( \bar{m} \) (and borrowers otherwise), we can now compute the value of becoming a borrower (resp. a lender). For a borrower, if \( \hat{m} > \bar{m} > \bar{m} \) (i.e. the threshold is above \( \bar{m} \) and the borrower has more reserves than necessary to achieve the requirement) then

\[
V^b(\bar{m}; \delta) = (1 - \delta)i_d \bar{m}. \quad \text{If } \bar{m} < \bar{m} \text{ then}
\]

\[
\frac{V^b(\bar{m}; \delta)}{1 - \delta} = V^b(\bar{m}; 0) - \theta(n) \int_{\bar{m} + \frac{\mu - \bar{m}}{\Delta - 1}}^{\bar{m} + \bar{v}} \frac{\delta}{2 - \delta} i(m_\ell, \bar{m}) q^\delta(m_\ell, \bar{m}) dF^\ell(m_\ell)
\]

\[
- \theta(n) \int_{\bar{m}}^{\bar{m} + \frac{\mu - \bar{m}}{\Delta - 1}} \left\{ P^b(m_\ell, \bar{m}; 0) - [i_* (\bar{m} - \bar{m}) + i_d \bar{m}] \right\} dF^\ell(m_\ell)
\]

where \( F^\ell(m) \) is the distribution of lenders’ money holdings. There are two effects of counterparty risk for the value of becoming a borrower: First the borrower has to pay a risk premium whenever he can borrow, and second he will not be able to borrow from some lenders (while he would have borrowed in the absence of counterparty risk). Therefore, the value of holding some reserves is the value without counterparty risk, minus the risk premium when the borrower trades, and minus the loss of payoff when the borrower rejects a trade that is too expensive. Notice that \( V^b(\bar{m}; \delta) \) when \( \bar{m} = \bar{m} \) is simply equal to

\[
V^b(\bar{m}; \delta)|_{\bar{m} = \bar{m}} = i_d \bar{m}
\]

as we use the fact that \( q(m_\ell, \bar{m}; \delta) = 0 \) in this case, and from (4) we have \( P^b(m_\ell, \bar{m}; 0) = \)
Where \( V^*_{\ell}(\bar{m}; \delta) \) is the distribution of borrowers' money holdings. Counterparty risk is increasing the value of reserves for a lender by the risk premium, but it also makes it smaller as lenders cannot trade as often. Again notice that \( V^*_{\ell}(\bar{m}; \delta) \) when \( \hat{m} = \bar{m} \) is simply equal to

\[
V^*_{\ell}(\bar{m}; \delta) \big|_{\hat{m} = \bar{m}} = i_d \bar{m}
\]

as we use the fact \( F_b(\hat{m}) = \frac{F(\bar{m} - m)}{F(\bar{m} - m)} \) when the threshold is \( \hat{m} \). Therefore, we again have the following result regarding the choice to become a borrower or a lender,

**Proposition 4.** With counterparty risk, all banks with reserves below \( m \) choose to become borrowers, while banks with reserves above \( m \) choose to become lenders. The number of borrowers is \( n = F(\bar{m} - m) \).

Notice that counterparty risk increases the value of cash for the borrower, because (1) the borrower pays a higher rate on every borrowed reserves, and (2) it increases the likelihood to get a loan (i.e., to fall in the region where the lender is willing to extend a loan at a rate lower than \( i_p \)). Relative to the case with no risk, the effect of counterparty risk on lenders is unclear: While they would extract more resources from borrowers if they can lend, lending becomes more difficult, as some borrowers will not borrow at a rate above \( i_p \). For the sake of completeness, we compute the marginal value of reserves with counterparty risk in the Appendix. We find that for all \( \delta > 0 \),

\[
\frac{\partial V^b(\bar{m}; \delta)}{\partial \bar{m}} > (1 - \delta) \frac{\partial V^b(\bar{m}; 0)}{\partial \bar{m}}
\]

while it is not possible to say if \( \frac{\partial V^*_{\ell}(\bar{m}; \delta)}{\partial \bar{m}} \) is larger or smaller than \( (1 - \delta) \frac{\partial V^*_{\ell}(\bar{m}; 0)}{\partial \bar{m}} \).
6.1 Market volume with counterparty risk

Using the model, we can now describe how the aggregate liquidity deficit $\bar{m} - m > 0$ impacts the volume of trade on the OTC interbank market. The aggregate trade volume is given by the total size of trades $\bar{Q}$ times the number of matches. Given our matching function we obtain that the aggregate trade volume is $Q(\bar{m} - m; \delta) = \min\{n, 1 - n\} \bar{Q}(\bar{m} - m; \delta)$ where $\bar{Q}$ is the average trade size when a trade occurs. Since there is only trading when (18) is satisfied, $\bar{Q}$ is

$$\bar{Q}(\bar{m} - m; \delta) = \int_{m-\bar{v}}^{\bar{m}} \int_{\bar{m}+\bar{v}}^{m+\bar{v}} q(m_b, m_t) dF_t(m_t) dF_b(m_b)$$

and since $\bar{m} \geq m_b$ and $\Delta'(\delta) < 0$ we obtain naturally that counterparty risk decreases the market volume for all liquidity conditions,

$$\frac{\partial \bar{Q}(\bar{m} - m; \delta)}{\partial \delta} < 0.$$

Also, notice that the decline in market volume is even more pronounced as liquidity is scarce, i.e. $\bar{m} - m_b = \bar{m} - m - v_b > 0$ is large.

The following figure shows the aggregate trading volume with no risk ($\delta = 0$ in blue) and with $\delta = 1\%$ (red) as shown a little counterparty risk has a large impact on the trading volume when the liquidity conditions are neutral, while the impact is diminished when there is ample liquidity (recall that $m$ is on the $x-$axis).
6.2 Weighted average OTC rates with counterparty risk

\[ i^\delta(m, m_b) = i(m, m_b) + \frac{\delta}{2-\delta} i(m, m_b) \]

The weighted average of the OTC rates in the presence of counterparty risk is \( \mu(i^\delta) \) which satisfies

\[ \mu(i^\delta) = \min\{F(\bar{m}-m), 1-F(\bar{m}-m)\} \int_{m-\theta}^{m+\theta} \frac{\bar{m} - m_b}{2Q(\bar{m}, m; \delta)} i^\delta(m, m_b) dF(m_b) dF_b(m_b). \]

Since there is only trading when (18) is satisfied, we can simplify the weighted mean as

\[ \mu(i^\delta) = \frac{\delta}{2-\delta} \int_{m-\theta}^{m+\theta} \frac{\bar{m} - m_b}{2Q(\bar{m}, m; \delta)} i(m, m_b) dF(m_b) dF_b(m_b) \]

This expression highlights two the counteracting effects of counterparty risk on the weighted average OTC rates: The direct effect of counterparty risk is to increase all the rates as we have seen. The indirect (negative) effect works through the weights: All trades where \( m \) is relatively small are not taking place. As a result, the aggregate volume is decreasing, but also more weight is given to those trade with relative large \( m \). However, these are the trades with relatively lower rates. As a result, the integral is smaller with counterparty risk than without. The overall effect of counterparty risk on the weighted rate therefore depends (once again) on the distribution of shocks. With relatively large shocks, the average weighted OTC rate will be larger with counterparty risk.

6.3 Volatility of OTC rates with counterparty risk

We now compute the rate volatility, using the weighted variance,

\[ \sigma^2_w(i^\delta) = \int_{m-\theta}^{m} \int_{m}^{m+\theta} \frac{q^\delta(m, m_b)}{Q(\bar{m}, m; \delta)} [i^\delta(m, m_b) - \mu(i^\delta)]^2 dF(m_b) dF_b(m_b), \]

which can be simplified as

\[ \sigma^2_w(i^\delta) = \int_{m-\theta}^{m} \int_{m}^{m+\theta} \frac{q^\delta(m, m_b)}{Q(\bar{m}, m; \delta)} [i^\delta(m, m_b) - \mu(i^\delta)]^2 dF(m_b) dF_b(m_b), \]

\[ = \left( \frac{2}{2-\delta} \right)^2 \int_{m-\theta}^{m} \int_{m}^{m+\theta} \frac{m - m_b}{Q(\bar{m}, m; \delta)} \left[ \frac{\delta}{2-\delta} + i(m, m_b) - \mu(i^\delta) \right]^2 dF(m_b) dF_b(m_b) \]
Can we say $\sigma_w^2(i_m)$ increases with $\delta$?

7 Conclusion

We presented a model of the interbank market with directed search. We showed that this framework is able to replicate basic statistics of the interbank market, both in normal and exceptional circumstances. It has the ability to match the data on volume, rates and volatility, even in the presence of counterparty risk. We showed that counterparty risk decreases aggregate trading volume and increases rates, but also that a little risk can go a long way in paralyzing the interbank market. Our approach also allows us to contrast our result from the workhorse model of Poole (1968) where volume, volatility, and counterparty risk are absent.

Of course we made some strong assumptions....

A main contribution of our paper is to show that the best model of the interbank market is something like this: banks choose whether they want to borrow or lend, and then they pick a borrower/lender at random. Purely random search or perfect pairing would not match the data. In this sense, there seems to be some leeway to improve the functioning of the money market. Then, an important question is why do banks choose not to use brokers, or in the case they do, why are brokers not more efficient in matching banks?

Appendix

OTC rates

In this section we show that the OTC rates are given by (3).

There are four types of trades that we have to consider: (1) meetings where $m_b + m_\ell \geq 2\bar{m}$, $m_b < \bar{m}$, and $m_\ell > \bar{m}$, (2) meetings where $m_b + m_\ell \geq 2\bar{m}$, $m_b > \bar{m}$, and $m_\ell > \bar{m}$, (3) meetings where $m_b + m_\ell < 2\bar{m}$, $m_b < \bar{m}$, and $m_\ell > \bar{m}$, and (4) meetings where $m_b + m_\ell < 2\bar{m}$, $m_b < \bar{m}$, and $m_\ell < \bar{m}$. In meetings (1)-(2) the pair of banks has enough reserves for each bank to satisfy their reserve requirements $\bar{m}$. In meetings (3)-(4) however, there is not enough reserves in the pair for both banks to satisfy their reserve requirements.

We now solve the rate in meeting (1). Since the rates are given by equalizing surplus
from trade, rates in meetings (1) will have to satisfy

$$(m_b + q - \bar{m})i_d + \bar{m}i_d - qi_m - [(\bar{m} - m_b)i_p + \bar{m}i_d] =$$

$$(m_\ell - q - \bar{m})i_d + \bar{m}i_d + qi - [(m_\ell - \bar{m})i_d + \bar{m}i_d]$$

Since there are enough reserves for both banks in the pair to satisfy their reserve requirements, both banks exit the interbank market with excess reserves. The borrower exits with $m_b + q - \bar{m}$ excess reserves and the lender exits with $m_\ell - q - \bar{m}$. Since excess reserves are remunerated at the rate $i_d$ we get the first term in both surplus. The other terms read in the same way. Notice the difference between the outside option of the borrower and the lender: while the borrower would have to borrow $\bar{m} - m_b$, the lender could deposit $m_\ell - \bar{m}$. Simplifying using (2) we obtain

$$i = \frac{m_\ell - \bar{m}}{m_\ell - m_b}i_d + \frac{\bar{m} - m_b}{m_\ell - m_b}i_p.$$  

We now solve for the rate in meeting (2). Since the rates are given by equalizing surplus from trade, rates in meetings (2) will have to satisfy

$$(m_b + q - \bar{m})i_d + \bar{m}i_d - qi - [(m_b - \bar{m})i_d + \bar{m}i_d] =$$

$$(m_\ell - q - \bar{m})i_d + \bar{m}i_d + qi - [(m_\ell - \bar{m})i_d + \bar{m}i_d]$$

The only difference with case (1) is in the outside option of the borrower: in case (2), even if the borrower does not borrow, he still has excess reserves. Simplifying, we obtain

$$1 + i_m = 1 + i_d.$$  

We now solve for the rate in meeting (3). In this case, both the borrower and the lender have to borrow at the lending facility after they exit the interbank market. Since the rates are given by equalizing surplus from trade, rates in meetings (3) will have to satisfy

$$-(\bar{m} - (m_b + q))i_p + \bar{m}i_d - qi - [-(\bar{m} - m_b)i_p + \bar{m}i_d] =$$

$$-(\bar{m} - (m_\ell - q))i_p + \bar{m}i_d + qi - [(m_\ell - \bar{m})i_d + \bar{m}i_d]$$
Simplifying, we obtain
\[ i = \frac{m_\ell - \bar{m}}{m_\ell - m_b} i_d + \frac{\bar{m} - m_b}{m_\ell - m_b} i_p. \]

Finally, we now solve for the rate in meeting (4). In this case, both the borrower and the lender have to borrow at the lending facility after they exit the interbank market, whether or not they traded on the interbank market. Since the rates are given by equalizing surplus from trade, rates in meetings (4) will have to satisfy
\[ -(\bar{m} - (m_b + q)) i_p + \bar{m} i_d - qi - [-(\bar{m} - m_b) i_p + \bar{m} i_d] = -(\bar{m} - (m_\ell - q)) i_p + \bar{m} i_d + qi - [-(\bar{m} - m_\ell) i_p + \bar{m} i_d] \]

Simplifying, we obtain
\[ i = i_p. \]

**Random matching: Details of the calculations**

Then the aggregate volume \( Q^*(x) \) is given by
\[
Q^*(x) = \int_{m_\ell - \bar{v}}^{m_\ell + \bar{v}} \int_{m_b - \bar{v}}^{m_b + \bar{v}} \frac{|m_\ell - m_b|}{2} dG(m_b) dG(m_\ell) \\
= \int_{m_\ell - \bar{v}}^{m_\ell + \bar{v}} \int_{m_b - \bar{v}}^{m_b + \bar{v}} \frac{m_\ell - m_b}{2} dG(m_b) dG(m_\ell) + \int_{m_\ell - \bar{v}}^{m_\ell + \bar{v}} \int_{m_b - \bar{v}}^{m_b + \bar{v}} \frac{m_b - m_\ell}{2} dG(m_\ell) dG(m_b) \\
= 2 \int_{-\bar{v}}^{\bar{v}} \int_{v_b}^{v_\ell} \frac{v_\ell - v_b}{2} dF(v_b) dF(v_\ell)
\]

which is constant in the amount of excess reserves \( x \).

The average rate is
\[
\mu_i^*(x) = \int_{m_\ell - \bar{v}}^{m_\ell + \bar{v}} \int_{m_b - \bar{v}}^{m_b + \bar{v}} \frac{q(m_\ell, m_b)}{Q^*} i(m_\ell, m_b) dG(m_\ell) dG(m_b) \\
= \int_{m_\ell - \bar{v}}^{m_\ell + \bar{v}} \int_{m_b - \bar{v}}^{m_b + \bar{v}} \frac{m_\ell - m_b}{2Q^*} i(m_\ell, m_b) dG(m_\ell) dG(m_b) + \int_{m_\ell - \bar{v}}^{m_\ell + \bar{v}} \int_{m_b - \bar{v}}^{m_b + \bar{v}} \frac{m_b - m_\ell}{2Q^*} i(m_\ell, m_b) dG(m_\ell) dG(m_b) \\
= \int_{-\bar{v}}^{\bar{v}} \int_{v_b}^{v_\ell} \frac{v_\ell - v_b}{2Q^*} i(v_\ell, v_b) dF(v_\ell) dF(v_b) + \int_{-\bar{v}}^{\bar{v}} \int_{v_b}^{v_\ell} \frac{v_b - v_\ell}{2Q^*} i(v_b, v_\ell) dF(v_b) dF(v_\ell)
\]
and using the interest rate function, we obtain (using \( m = \bar{m} = x \)),

\[
\mu_i^x(x) Q^x = \int_{-\infty}^0 \int_{-\infty}^0 \frac{v_i - v_b}{2} i_d dF(v_t) dF(v_b) + \int_{-\infty}^0 \int_{-\infty}^x \frac{1}{2} [(v_e + x) i_d - (x + v_b) i_p] dF(v_t) dF(v_b)
+ \int_{-\infty}^0 \int_{-\infty}^v \frac{v_i - v_b}{2} i_d dF(v_t) dF(v_b)
+ \int_{-\infty}^v \int_{-\infty}^x \frac{v_i - v_b}{2} i_d dF(v_t) dF(v_b) + \int_{-\infty}^v \int_{-\infty}^x \frac{v_i - v_b}{2} i_d dF(v_t) dF(v_b)
+ \int_{-\infty}^x \int_{-\infty}^v \frac{v_i - v_b}{2} i_d dF(v_t) dF(v_b)
\]

and using the equality

\[
0 = \int_{-\infty}^0 \int_{-\infty}^0 \frac{v_i - v_b}{2} dF(v_t) dF(v_b) = \int_{-\infty}^0 \int_{-\infty}^v \frac{v_i - v_b}{2} i_d dF(v_t) dF(v_b) + \int_{-\infty}^0 \int_{-\infty}^v \frac{v_i - v_b}{2} i_d dF(v_t) dF(v_b)
\]

we obtain

\[
\int_{-\infty}^0 \int_{-\infty}^v \frac{v_i - v_b}{2} dF(v_t) dF(v_b) = \int_{-\infty}^0 \int_{-\infty}^v \frac{v_i - v_b}{2} dF(v_t) dF(v_b)
\] (21)

Similarly, as

\[
0 = \int_{-\infty}^0 \int_{-\infty}^v \frac{v_i - v_b}{2} dF(v_t) dF(v_b) = \int_{-\infty}^0 \int_{-\infty}^x \frac{v_i - v_b}{2} i_d dF(v_t) dF(v_b) + \int_{-\infty}^0 \int_{-\infty}^x \frac{v_i - v_b}{2} i_d dF(v_t) dF(v_b)
\]

we obtain

\[
\int_{-\infty}^0 \int_{-\infty}^x \frac{v_i - v_b}{2} dF(v_t) dF(v_b) = \int_{-\infty}^0 \int_{-\infty}^x \frac{v_i - v_b}{2} dF(v_t) dF(v_b)
\] (22)

so that

\[
\mu_i^x(x) Q^x = \int_{-\infty}^x \int_{-\infty}^0 \frac{v_i - v_b}{2} i_d dF(v_t) dF(v_b) + \int_{-\infty}^x \int_{-\infty}^x \frac{1}{2} [(v_e + x) i_d - (x + v_b) i_p] dF(v_t) dF(v_b)
+ \int_{-\infty}^x \int_{-\infty}^v \frac{v_i - v_b}{2} i_d dF(v_t) dF(v_b)
+ \int_{-\infty}^v \int_{-\infty}^x \frac{v_i - v_b}{2} i_d dF(v_t) dF(v_b)
+ \int_{-\infty}^x \int_{-\infty}^v \frac{v_i - v_b}{2} i_d dF(v_t) dF(v_b)
\]

38
Then

\[ \mu_i^s(\infty) = i_d, \]

\[ \mu_i^s(-\infty) = i_p \]

and

\[ \frac{\partial \mu_i^s(x)}{\partial x} = -\frac{(i_p - i_d)}{2Q^s} F(x) [1 - F(x)] \]

and when \( x = 0 \), we can simplify the expression for the means as follows,

\[ \mu_i^s(0)^* = \int_0^\infty \int_{v_b}^\infty (v_\ell - v_b)i_d dF(v_\ell)dF(v_b) + \int_0^0 \int_{v_b}^0 (v_\ell - v_b)i_p dF(v_\ell)dF(v_b) \]

\[ + \int_0^0 \int_{v_b}^0 \frac{1}{2} (v_\ell i_d - v_b i_p) dF(v_\ell)dF(v_b) + \int_0^\infty \int_{v_b}^0 \frac{1}{2} (v_b i_d - v_\ell i_p) dF(v_\ell)dF(v_b) \]

Now, focusing on the first line, notice that (pay attention to the change in signs)

\[ \int_{-\infty}^0 \int_{v_b}^0 (v_\ell - v_b)i_p dF(v_\ell)dF(v_b) = \int_{-\infty}^0 \int_{0}^{-v_b} (-v_\ell - v_b)i_d dF(v_\ell)dF(v_b) \]

\[ = \int_{0}^{v_b} \int_{0}^{v_b} (v_b - v_\ell)i_p dF(v_\ell)dF(v_b) \]

\[ = \int_{0}^{v_b} \int_{v_b}^\infty (v_\ell - v_b)i_d dF(v_\ell)dF(v_b) \]

where the last equality follows from (21). Hence,

\[ \mu_i^s(0)^* = \int_0^\infty \int_{v_b}^\infty (v_\ell - v_b)(i_d + i_p)dF(v_\ell)dF(v_b) \]

\[ + \int_0^0 \int_{v_b}^0 \frac{1}{2} (v_\ell i_d - v_b i_p) dF(v_\ell)dF(v_b) + \int_0^\infty \int_{v_b}^0 \frac{1}{2} (v_b i_d - v_\ell i_p) dF(v_\ell)dF(v_b) \]

Now, rearranging the last line we obtain

\[ \mu_i^s(0)^* = \int_0^\infty \int_{v_b}^\infty (v_\ell - v_b)(i_d + i_p)dF(v_\ell)dF(v_b) + \int_{-\infty}^0 \int_{v_b}^0 \frac{1}{2} (v_\ell - v_b)(i_d + i_p)dF(v_\ell)dF(v_b) \]
and decomposing the last term, we obtain
\[ \int_{-\theta}^{0} \int_{0}^{\theta} \frac{1}{2} (v_\ell - v_b) dF(v_\ell) dF(v_b) = \int_{-\theta}^{0} \int_{v_b}^{\theta} \frac{1}{2} (v_\ell - v_b) dF(v_\ell) dF(v_b) - \int_{0}^{\theta} \int_{v_b}^{\theta} \frac{1}{2} (v_\ell - v_b) dF(v_\ell) dF(v_b) \]
and using the same steps as for (23),
\[ \int_{-\theta}^{0} \int_{0}^{\theta} \frac{1}{2} (v_\ell - v_b) dF(v_\ell) dF(v_b) = \int_{-\theta}^{0} \int_{v_b}^{\theta} \frac{1}{2} (v_\ell - v_b) dF(v_\ell) dF(v_b) - \int_{0}^{\theta} \int_{v_b}^{\theta} \frac{1}{2} (v_\ell - v_b) dF(v_\ell) dF(v_b) \]
and replacing into the expression for \( \mu^*_i(0) \) we get
\[
\mu^*_i(0) Q^* = (i_d + i_p) \left[ \int_{-\theta}^{0} \int_{v_b}^{\theta} (v_\ell - v_b) dF(v_\ell) dF(v_b) + \int_{-\theta}^{0} \int_{v_b}^{\theta} \frac{1}{2} (v_\ell - v_b) dF(v_\ell) dF(v_b) - \int_{0}^{\theta} \int_{v_b}^{\theta} \frac{1}{2} (v_\ell - v_b) dF(v_\ell) dF(v_b) \right] \\
= (i_d + i_p) \left[ \int_{-\theta}^{0} \int_{v_b}^{\theta} \frac{1}{2} (v_\ell - v_b) dF(v_\ell) dF(v_b) \right] \\
= \frac{(i_d + i_p)}{2} Q^*
\]
Therefore \( \mu^*_i(0) = (i_d + i_p)/2 \).

**Willingness to pay in the auction.**

To compute the willingness to pay, we need to find the derivative of \( \int_{-\theta}^{\theta} V^*(m + v) dF(v) \) which is
\[
\int_{-\theta}^{\theta} \int_{m+v}^{m+\theta} \frac{\partial P_b(m_\ell, m + v)}{\partial m} dG(m_\ell) dF(v) + \int_{-\theta}^{\theta} \int_{m-\theta}^{m-v} \frac{\partial P^c(m + v, m_b)}{\partial m} dG(m_b) dF(v).
\]
Using (4) and (5), we have
\[
\frac{\partial P_b(m_\ell, m_b)}{\partial m_b} = \begin{cases} 
  i_d & \text{if } m_\ell > m_b > m \\
  \frac{1}{2} (i_p + i_d) & \text{if } m_b < m < m_\ell \text{ and } v_\ell > 2m - 2m - v_b \\
  i_p & \text{if } m_b < m < m_\ell \text{ and } v_\ell < 2m - 2m - v_b \\
  i_p & \text{if } m_b < m_\ell < m
\end{cases}\] 
(24)

40
and

\[
\frac{\partial P}{\partial m} (m, m_b) = \begin{cases} 
  i_d & \text{if } m_b > m > \bar{m} \\
  i_d & \text{if } m_b > \bar{m} < m_b + m > 2\bar{m} \\
  \frac{1}{2}(i_p + i_d) & \text{if } m_b < \bar{m} < m_b + m < 2\bar{m} \\
  i_p & \text{if } m_b < m < \bar{m}
\end{cases} 
\]  

(25)

Replacing these equalities in the corresponding expression, we obtain\textsuperscript{16}

\[
\int_{-\bar{m}}^{\bar{m}} \int_{m+\bar{v}}^{m+\bar{v}} \frac{\partial P}{\partial m} (m, m + v) dG(m) dF(v) = \\
\int_{-\bar{m}}^{\bar{m}} \int_{m+\bar{v}}^{m+\bar{v}} \frac{\partial P}{\partial m} (m, m + v) dG(m) dF(v) + \int_{-\bar{m}}^{\bar{m}} \int_{m+\bar{v}}^{m+\bar{v}} \frac{\partial P}{\partial m} (m, m + v) dG(m) dF(v) \\
= \int_{-\bar{m}}^{\bar{m}} \int_{m+\bar{v}}^{m+\bar{v}} i_d dF(v) dF(v) + \int_{-\bar{m}}^{\bar{m}} \int_{m+\bar{v}}^{m+\bar{v}} i_p dF(v) dF(v) \\
\quad + \int_{-\bar{m}}^{\bar{m}} \int_{m+\bar{v}}^{m+\bar{v}} (i_p + i_d) dF(v) dF(v) \\
= \int_{-\bar{m}}^{\bar{m}} \int_{m+\bar{v}}^{m+\bar{v}} i_d dF(v) dF(v) + \int_{-\bar{m}}^{\bar{m}} \int_{m+\bar{v}}^{m+\bar{v}} i_p dF(v) dF(v) \\
\quad + \int_{-\bar{m}}^{\bar{m}} \int_{m+\bar{v}}^{m+\bar{v}} \frac{(i_p + i_d)}{2} dF(v) dF(v) \\
= i_d \int_{-\bar{m}}^{\bar{m}} (1 - F(v)) dF(v) + i_p \int_{-\bar{m}}^{\bar{m}} [F(2(\bar{m} - m) - v) - F(v)] dF(v) \\
\quad + \frac{(i_p + i_d)}{2} \int_{-\bar{m}}^{\bar{m}} [1 - F(2(\bar{m} - m) - v)] dF(v)
\]

And

\textsuperscript{16}The derivatives with respect to the limits cancel out.
Adding both terms, and using the fact that

\[ \frac{\partial P^\ell(m + v, m + v_b)}{\partial m} = \frac{\partial P^\ell(m + v, m + v_b)}{\partial m} = \frac{\partial P^\ell(m + v, m + v_b)}{\partial m} = \frac{\partial P^\ell(m + v, m + v_b)}{\partial m} \]

we obtain the desired expression:

\[
\int_{-\theta}^{\theta} \int_{-\theta}^{\theta} \frac{\partial P^\ell(m + v, m + v_b)}{\partial m} dF(v_b) dF(v) = \\
\int_{-\theta}^{\theta} \int_{-\theta}^{\theta} \frac{\partial P^\ell(m + v, m + v_b)}{\partial m} dF(v_b) dF(v) + \int_{-\theta}^{\theta} \int_{-\theta}^{\theta} \frac{\partial P^\ell(m + v, m + v_b)}{\partial m} dF(v_b) dF(v) \\
+ \int_{-\theta}^{\theta} \int_{-\theta}^{\theta} \frac{\partial P^\ell(m + v, m + v_b)}{\partial m} dF(v_b) dF(v) \\
= \int_{-\theta}^{\theta} \int_{-\theta}^{\theta} i_d dF(v_b) dF(v) + \int_{-\theta}^{\theta} \int_{-\theta}^{\theta} \left[ \int_{-\theta}^{\theta} \frac{\partial P^\ell(m + v, m + v_b)}{\partial m} dF(v) + \int_{-\theta}^{\theta} \frac{i_p + i_d}{2} dF(v_b) + \int_{-\theta}^{\theta} i_d dF(v_b) \right] dF(v) \\
= i_d \int_{-\theta}^{\theta} \left[ F(v) - F(m - m) \right] dF(v) \\
+ \frac{i_p + i_d}{2} \int_{-\theta}^{\theta} F(2(m - m) - v) dF(v) + i_d \int_{-\theta}^{\theta} \left[ F(m - m) - F(2(m - m) - v) \right] dF(v) \\
+ i_p \int_{-\theta}^{\theta} F(v) dF(v)
\]

Adding both terms, and using the fact that

\[
\int_{-\theta}^{\theta} 1 - F(2(m - m) - v) dF(v) = \int_{-\theta}^{\theta} 1 - F(2(m - m) - v) dF(v) - \int_{-\theta}^{\theta} 1 - F(2(m - m) - v) dF(v)
\]

we obtain the desired expression:

\[
W_s'(m) = i_p F(m - m) + i_d \left[ 1 - F(m - m) \right] - \left( \frac{i_p - i_d}{2} \right) \left[ F(m - m) - \int_{-\theta}^{\theta} F(2(m - m) - v) dF(v) \right]
\]

42
7.1 Directed search with perfect pairing

As a useful benchmark, we first consider the case where banks are perfectly paired. By perfect pairing, we mean the following. Suppose the monetary policy stance is to supply $m$ units of reserves in the central market. In equilibrium, all banks exit the central stage with $m$ units of reserves. Then they receive some liquidity shocks. With perfect pairing, the matching technology will pair a bank holding $m + v$ with a bank holding $m_2 = m - v$ such that both banks exit the OTC market with exactly $m$ units of reserves, as if they did not receive any shocks.

Notice a very important assumption regarding the matching technology: it is not a function of a bank’s shock but only of its reserves holdings. Therefore if a bank holds $\tilde{m} \neq m$ at the start of the OTC market and receives no shock, then it is matched with a bank that, after the shock, holds $m - \tilde{m}$. This is crucial to understand the marginal value of reserves in the central market.

Why is this a useful benchmark? With no further shock, all banks exit the central market with $m$ units of reserves, and they either all access the lending facility if $m < \bar{m}$, or the deposit facility if $m > \bar{m}$. Therefore the case of perfect pairing mimics a central market where banks could just trade their shocks away. However, as we will see below, there are important differences between perfect pairing and a central market. Most importantly, while there would be a unique rate on the market, there will be some rate dispersion with perfect pairing that is originating from the bargaining protocol.

7.1.1 Equilibrium rates

For now, we will assume and later verify that all banks enter the OTC market with the same amount of reserves, $m$. A bank who receives a positive shock $v > 0$ is matched with the bank who received shock $-v$. Therefore the bank with the positive shock lends $v$ to the other at the rates described in (3). Following trade, all banks exit the OTC market with $m$ units of balances. We can then rewrite the rates and the payoff of the lender and borrower banks, making use of the fact that $m_b = m - v$ while $m_l = m + v$. Arranging the expression, banks trade reserves at the rate

$$i(v) = \begin{cases} 
  i_d & \text{if } v < m - \bar{m} \\
  \frac{i_p + i_d}{2} + \frac{m - m}{2v}(i_p - i_d) & \text{if } v > m - \bar{m} > -v \\
  i_p & \text{if } v < \bar{m} - m
\end{cases}$$

(26)
where \( v > 0 \) is the lender’s shock. This function has a natural interpretation, e.g., when \( v < \bar{m} - m \) the borrower has enough to cover the reserve requirement \( \bar{m} \) and he does not need to borrow, which forces the OTC rate to \( i_d \).

In particular, it is clear that all rates will be equal to the mid-point of the corridor if the central bank has a neutral stance, i.e. \( m = \bar{m} \). Also, notice that perfect pairing implies that there are only two possible rates depending on the aggregate liquidity supply. In a liquidity surplus, \( m > \bar{m} \) and

\[
i(v) = \begin{cases} 
  i_d & \text{if } v < m - \bar{m} \\
  \frac{\bar{m} - m}{2v} (i_p - i_d) & \text{if } v > m - \bar{m}
\end{cases}
\]

while in a liquidity deficit, \( m < \bar{m} \) and

\[
i(v) = \begin{cases} 
  \frac{\bar{m} + m}{2v} (i_p - i_d) & \text{if } v > \bar{m} - m \\
  i_p & \text{if } v < \bar{m} - m
\end{cases}
\]

In particular, if \( v < \bar{m} - m \) then the lender has too few reserves to satisfy his requirement. As a result he will have to borrow at the lending facility and he will only be willing to lend to the borrower at rate \( i_p \). Figure 13 illustrates \( i_m(v) \) as a function of the liquidity surplus/deficit. Banks have the same amount of reserves \( m \) before they are hit by the liquidity shock. Since they are perfectly paired, a bank with the shock \( -v < 0 \) (on the \( x \)-axis) is matched with a bank with shock \( v > 0 \) (on the \( y \)-axis). If the resulting reserves for the pair \( (m - v, m + v) \) falls in the green region, then the rate is \( i_d \) if \( m - \bar{m} > 0 \) and \( i_p \) otherwise, as both banks have enough to satisfy their reserves requirement in the first case, while neither does in the second. If the reserves for the pair falls in the red region, then banks trade at the midpoint of the corridor plus a term to share the gains from trade.

We can write the payoff of a bank with liquidity shock \( v \in [-\bar{v}, \bar{v}] \), when the central bank policy stance is \( m \), as

\[
P(m, v) = i_d \bar{m} + (m - \bar{m})[\mathbb{1}_{m > \bar{m}} i_d + \mathbb{1}_{m < \bar{m}} i_p] + i(|v|)v
\]

The first term is the interest rate paid on banks’ required reserves. Absent any OTC market, the second term captures the benefits/cost of accessing the central bank facilities to cover excess reserves \( m - \bar{m} \). Finally, the last term captures the gain from
trading the liquidity shock $v$ on the OTC market.

We can now compute the willingness to pay for reserves in the central market. All banks face the same uncertainty and they will therefore behave in the same way in this market. Since the supply of reserves is $m$, they will all exit the central market with $m$ units of reserves. However, to compute their marginal value at $m$, we need to compute their marginal value at any other level $\tilde{m}$. Notice that a bank who exits the central market with $\tilde{m} \neq m$ (before the shock hits) and is then hit by a shock of size $v$ has a payoff

$$P(\tilde{m}, v) = P(m, v + \tilde{m} - m).$$

Indeed, it is as if the bank had exited the central stage with holding $m$ and received a shock $v + \tilde{m} - m$. Therefore, the value of exiting the centralized stage with $\tilde{m}$ units of reserves when the central bank supplies $m$ is simply

$$W(\tilde{m}) = \int_{m}^{\tilde{m}} P(m, v + \tilde{m} - m) dF(v)$$

Equipped with this expression, we can then compute $W'(\tilde{m})$, which we do in the Appendix. In a symmetric equilibrium, all banks exit the central stage with $m$ units of reserves and their willingness to pay for reserves at $m > \tilde{m}$ is simply

$$W'(m) = i_d [F(m - \tilde{m}) - F(\tilde{m} - m)] + \frac{i_p + i_d}{2} [1 - F(\tilde{m} - m) + F(\tilde{m} - m)]$$

This is rather intuitive: Starting from a situation of liquidity surplus, banks exit the

\[17\] Our assumption regarding the matching technology plays a very important role here.
central market with more reserves than they need to satisfy their reserves requirements \( \tilde{m} \). If banks receive a relatively small shock \( v \in [\tilde{m} - m; m - \tilde{m}] \) they know they will be matched with a bank who also received a shock of the same magnitude but of opposite sign and they still both have enough reserves to satisfy their reserves requirements. Hence, they will trade at the floor rate. If however, banks receive a large shock \( v \in [-\tilde{v}, \tilde{m} - m] \cup [m - \tilde{m}, \tilde{v}] \) they are paired with a bank who also received a large shock and one of them cannot satisfy the reserves requirement. In this case they trade at the mid-point of the corridor, sharing the surplus from trade. A similar expression results in the case with liquidity deficit, \( m < \tilde{m} \). Using symmetry of the shock distribution, we can simplify both expressions for the rates to

\[
W'(m) = i_d F(m - \tilde{m}) + i_p [1 - F(m - \tilde{m})].
\]

Therefore, and maybe surprisingly, the banks’ willingness to pay for reserves in the “auction” market in the model with perfect pairing is the same as the banks’ willingness to pay in the Poole model. This may be surprising as there is no market in the Poole’s model where banks can trade their shocks away. However, looking at (29) notice that banks do not expect any gains on average from participating in the OTC market as the shock averages to zero. Therefore, they value reserves in the auction market as if there was no OTC market.

### 7.1.2 OTC volume, weighted average rate, and volatility

We can now compute some OTC market statistics, when the matching functions pairs banks perfectly. We compute the market volume, average weighted rates and the rate volatility.

**OTC market volume** Ignoring those trades with relatively low shocks, that do not generate any surplus, the market volume is simply\(^{18}\)

\[
Q(\tilde{m} - m) = \int_{|m - \tilde{m}|}^{\tilde{v}} vdF(v)
\]

which is a decreasing function of the central bank’s liquidity stance. Figure (14) plots the aggregate volume as a function of excess reserves, \( m - \tilde{m} \) on the \( x \)-axis for normally

\(^{18}\)If we were to consider those trades, the market volume would be \( \int_{0}^{\tilde{v}} vdF(v) \), a constant independent of the size of the liquidity surplus.
distributed shocks with standard deviation of 2.

**OTC rate** Then the weighted average rate is simply

\[ \hat{i}_m(\bar{m} - m) = \frac{1}{Q(\bar{m} - m)} \int_{|\bar{m}-m|}^{\bar{m}} i_m(v)vdF(v) \]

and using (27)-(28) we obtain

\[ \hat{i}_m(\bar{m} - m) = \frac{i_p + i_d}{2} + \frac{\bar{m} - m}{Q(\bar{m} - m)} \frac{(i_p - i_d)}{2} [1 - F(|\bar{m} - m|)]. \]

Hence, the average rate is the mid-point of the corridor, plus the average gains of trading in the OTC market. In particular, under a neutral liquidity provision \( m = \bar{m} \) the average rate is at the mid-point of the corridor, it is below whenever there is an aggregate liquidity surplus \( m > \bar{m} \), and it is above if there is a liquidity deficit. Figure 15 plots the weighted rate as a function of the liquidity surplus, \( m - \bar{m} \). The OTC rate is steeper than the Poole rate (or the auction rate) for a simple reason: the OTC rate gives more weight to the corridor mid-point as this is the prevalent rate when there is relatively large shock. For example, the Poole rate gives more weight to \( i_d \) as \( m \) increases, while the OTC rates gives more weight to \((i_p + i_d)/2\), which explains why the red curve is above the green curve when \( m \) is large. Symmetrically, this explains why the order is reversed when \( m \) is small, as the Poole rate now gives more weight to \( i_p \) while the OTC rates gives more weight to \((i_p + i_d)/2\).
Volatility

Finally, we compute the rate volatility, as

\[
\sigma_w^2(i_m) = \int_{|\bar{m} - m|}^{\bar{v}} \frac{v}{Q(\bar{m} - m)} [i_m(v) - \bar{i}_m]^2 dF(v)
\]

and using the expression for the rates and the weighted rate, we obtain

\[
\sigma_w^2(i_m) = \frac{(\bar{m} - m)^2 (i_p - i_d)^2}{4} \left[ \int_{|\bar{m} - m|}^{\bar{v}} \frac{1}{v} dF(v) - \frac{[1 - F(|\bar{m} - m|)]^2}{Q(\bar{m} - m)} \right]
\]

Notice that the variability is M-shaped, as it is zero whenever \(\bar{m} = m\), or when \(|\bar{m} - m|\) is large. Also, it is symmetric around \(\bar{m} - m = 0\), as the expression only depends on the absolute value \(|\bar{m} - m|\). Furthermore, as

\[
Q(\bar{m} - m) \geq |\bar{m} - m| (1 - F(|\bar{m} - m|))
\]

we can also bound \(\sigma_w^2(i_m)\) as follows

\[
\sigma_w^2(i_m) \leq \frac{(i_p - i_d)^2}{16}.
\]

Hence the rates are not going to vary much in this model, even if the shocks are very volatile. For example, if the corridor width is 100 basis points, then the volatility of the rate is at most \(\frac{1}{16}\) basis points. Figure shows the volatility of the OTC market rate.

Clearly, the volatility of the model with perfect pairing is at odd with the data. We now move to analyzing the model with directed search and imperfect pairing.
Marginal value of reserves perfect pairing

Using (29) we obtain

\[ W(\tilde{m}) = I + \int_{-\theta}^{\theta} (1 + i_m(v + \tilde{m} - m))(v + \tilde{m} - m)dF(v) \]

where \( I = i_d\tilde{m} + (m - \tilde{m})[\mathbb{1}_{\{m > \tilde{m}\}}i_d - \mathbb{1}_{\{m < \tilde{m}\}}i_p] \) is the payoff when there is no liquidity shock. Using (26) we obtain again four cases for \( i_m(v + \tilde{m} - m) \):

1. If \( m - \tilde{m} > \tilde{m} - m > 0 \) then

\[
i(v + \tilde{m} - m) = \begin{cases} 
  i_d & \text{if } v + \tilde{m} - m < m - \tilde{m} \\
  \frac{i_p+i_d}{2} + \frac{m-m}{2(v+m-m)}(i_p-i_d) & \text{if } v + \tilde{m} - m > m - \tilde{m} 
\end{cases}
\] (30)

2. If \( \tilde{m} - m > m - \tilde{m} > 0 \) then the interest rate is

\[
i(v + \tilde{m} - m) = \frac{i_p+i_d}{2} + \frac{m-m}{2(v+m-m)}(i_p-i_d).\]

3. If \( m - \tilde{m} < \tilde{m} - m < 0 \) then the interest rate is given by

\[
i(v + \tilde{m} - m) = \begin{cases} 
  \frac{i_p+i_d}{2} + \frac{m-m}{2(v+m-m)}(i_p-i_d) & \text{if } v + \tilde{m} - m > \tilde{m} - m \\
  i_p & \text{if } v + \tilde{m} - m < \tilde{m} - m
\end{cases}
\] (31)
(4) If $\tilde{m} - m < m - \tilde{m} < 0$ then the interest rate is again
\[
i(v + \tilde{m} - m) = \frac{i_p + i_d}{2} + \frac{\tilde{m} - m}{2(v + m - m)}(i_p - i_d).
\]

As we want to know the marginal value of reserves around $m$, notice that we only have to consider cases (1) and (3), as the other two would yield to a contradiction (as we will set $\tilde{m} = m$). In case (1) where $m - \tilde{m} > m - m$ we have
\[
W(\tilde{m}) = I + \int_{-\tilde{m}}^{2m-\tilde{m}-\tilde{m}} i_d(v + \tilde{m} - m)dF(v) \\
+ \int_{-\tilde{m}}^{-2m-\tilde{m}-\tilde{m}} \left[ \left( \frac{i_p + i_d}{2} \right) (v + \tilde{m} - m) + \frac{\tilde{m} - m}{2}(i_p - i_d) \right] dF(v) \\
+ \int_{-\tilde{m}}^{\tilde{m}} \left[ \left( \frac{i_p + i_d}{2} \right) (v + \tilde{m} - m) + \frac{\tilde{m} - m}{2}(i_p - i_d) \right] dF(v)
\]
with marginal utility
\[
W'(\tilde{m}) = i_d [2F(2m - \tilde{m} - \tilde{m}) - 1] + \left( \frac{i_p + i_d}{2} \right) 2 [1 - F(2m - \tilde{m} - \tilde{m})]
\]

In case (3) where $m - \tilde{m} < \tilde{m} - m < 0$, we have
\[
W(\tilde{m}) = I + \int_{-\tilde{m}}^{\tilde{m}} i_p(v + \tilde{m} - m)dF(v) \\
+ \int_{-\tilde{m}}^{-\tilde{m}} \left[ \left( \frac{i_p + i_d}{2} \right) (v + \tilde{m} - m) + \frac{\tilde{m} - m}{2}(i_p - i_d) \right] dF(v) \\
+ \int_{\tilde{m}}^{\tilde{m}} \left[ \left( \frac{i_p + i_d}{2} \right) (v + \tilde{m} - m) + \frac{\tilde{m} - m}{2}(i_p - i_d) \right] dF(v)
\]
with marginal utility
\[
W'(\tilde{m}) = i_p [2F(\tilde{m} - \tilde{m}) - 1] + (i_p + i_d) [1 - F(\tilde{m} - \tilde{m})]
\]

In equilibrium, $\tilde{m} = m$, so that if $m > \tilde{m}$ then the marginal value of reserves is
\[
W'(m) = i_d [2F(m - \tilde{m}) - 1] + (i_p + i_d) [1 - F(m - \tilde{m})]
\]
while if \( m < \bar{m} \) then the marginal value of reserves is

\[
W'(m) = i_p [2F(\bar{m} - m) - 1] + (i_p + i_d) [1 - F(\bar{m} - m)]
\]

Using the fact that the distribution of shocks is symmetric, we obtain \( F(\bar{m} - m) = 1 - F(m - \bar{m}) \) so that we can simplify the last two expressions to simply

\[
W'(m) = i_d F(m - \bar{m}) + i_p [1 - F(m - \bar{m})].
\]

**Rate volatility with perfect pairing**

\[
\sigma_w^2(i_m) = \int_{-m}^{0} \frac{(\bar{m} - m)^2 (i_p - i_d)^2}{4} v \left[ \frac{1}{v^2} + \frac{[1 - F(|\bar{m} - m|)]^2}{Q(\bar{m} - m)^2} - \frac{2}{v} \frac{1 - F(|\bar{m} - m|)}{Q(\bar{m} - m)} \right] dF(v)
\]

\[
= \frac{(\bar{m} - m)^2 (i_p - i_d)^2}{Q(\bar{m} - m) \frac{4}{4}} \left[ \int_{-m}^{0} \frac{1}{v} dF(v) + \frac{[1 - F(|\bar{m} - m|)]^2}{Q(\bar{m} - m)^2} \int_{-m}^{0} v dF(v) - \frac{2}{Q(\bar{m} - m)} \right]
\]

\[
\leq \frac{(\bar{m} - m)^2 (i_p - i_d)^2}{Q(\bar{m} - m) \frac{4}{4}} \left[ \frac{1 - F(|\bar{m} - m|)}{|\bar{m} - m|} - \frac{[1 - F(|\bar{m} - m|)]^2}{Q(\bar{m} - m)} \right]
\]

\[
= \frac{(i_p - i_d)^2}{4} \frac{|\bar{m} - m| [1 - F(|\bar{m} - m|)]}{Q(\bar{m} - m)} \left[ 1 - \frac{|\bar{m} - m| [1 - F(|\bar{m} - m|)]}{Q(\bar{m} - m)} \right]
\]

Since

\[
Q(\bar{m} - m) \geq |\bar{m} - m| (1 - F(|\bar{m} - m|))
\]

we have

\[
\sigma_w^2(i_m) \leq \frac{(i_p - i_d)^2}{16}
\]

**Proof of Proposition 2.**

We guess and verify that \( \dot{m} = \bar{m} \) (i.e. all banks with enough reserves to satisfy the reserves requirement will become lenders and inversely, while those holding \( \bar{m} \) will be
indifferent). When \( \bar{m} = \bar{m} \) we have

\[
V^b(\bar{m}) = \theta(n) \int_{\overline{m}}^{\overline{m} + \theta} \mathbb{I}_{\{\bar{m} + m_{\ell} > \bar{m}\}} \left[ \bar{m}i_d + (\bar{m} - \bar{m})i_d + \frac{\bar{m} - \bar{m}}{2} (i_p - i_d) \right] dF(m_{\ell}) + \theta(n) \int_{\overline{m}}^{\overline{m} + \theta} \mathbb{I}_{\{\bar{m} + m_{\ell} < \bar{m}\}} \left[ \bar{m}i_d + (\bar{m} - \bar{m})i_p + \frac{\bar{m} - \bar{m}}{2} (i_p - i_d) \right] dF(m_{\ell}) + (1 - \theta(n)) [\bar{m}i_d + (\bar{m} - \bar{m})i_d]
\]

where \( F(m_{\ell}) \) is the distribution of reserves holding of lenders, and where we have used the payoff of the borrower in each possible state. We can now do a change of variable to obtain

\[
V^b(\bar{m}) = \theta(n) \int_{\overline{m} - \bar{m}}^{\overline{m} - \bar{m} - \bar{m}} \left[ \bar{m}i_d + (\bar{m} - \bar{m})i_p + \frac{m_{\ell}(v) - \bar{m}}{2} (i_p - i_d) \right] \frac{dF(v)}{1 - F(\bar{m} - m)} + \theta(n) \int_{\overline{m} - \bar{m} - \bar{m}}^{\overline{m} - \bar{m} - \bar{m} - \bar{m}} \left[ \bar{m}i_d + (\bar{m} - \bar{m})i_p + \frac{m_{\ell}(v) - \bar{m}}{2} (i_p - i_d) \right] \frac{dF(v)}{1 - F(\bar{m} - m)} + (1 - \theta(n)) [\bar{m}i_d + (\bar{m} - \bar{m})i_p]
\]

Notice that

\[
V^b(\bar{m}) = \theta(n) \int_{\overline{m} - \bar{m}}^{\overline{m} - \bar{m} - \bar{m}} \frac{dF(v)}{1 - F(\bar{m} - m)} + (1 - \theta(n))i_d \tag{33}
\]

and we also have

\[
\frac{\partial V^b(\bar{m})}{\partial \bar{m}} = i_p - \theta(n) \frac{(i_p - i_d)}{2} \frac{1 - F(2\bar{m} - m - \bar{m})}{1 - F(\bar{m} - m)} \tag{34}
\]

Turning to the lenders, and using similar steps, we obtain

\[
V^f(\bar{m}) = \frac{n}{1 - n} \theta(n) \int_{\bar{m} - \bar{m} - \bar{m} - \bar{m}}^{\bar{m} - \bar{m} - \bar{m} - \bar{m}} \left[ \bar{m}i_d + \frac{\bar{m} - m_{\ell}(v)}{2} (i_p - i_d) \right] \frac{dF(v)}{F(\bar{m} - m)} \tag{35}
\]

with

\[
V^f(\bar{m}) = \frac{n}{1 - n} \theta(n) \int_{\bar{m} - \bar{m} - \bar{m} - \bar{m}}^{\bar{m} - \bar{m} - \bar{m} - \bar{m}} \frac{dF(v)}{F(\bar{m} - m)} + (1 - \frac{n}{1 - n} \theta(n)) \bar{m}i_d \tag{36}
\]

52
and
\[ \frac{\partial V^t(\tilde{m})}{\partial \tilde{m}} = i_d + \frac{n}{1-n} \theta(n) \frac{(i_p - i_d) F(2\tilde{m} - m - \tilde{m})}{2 F(\tilde{m} - m)}. \] (37)

Now, using (33) and (36), as well as the fact that \( n = \int_{-\infty}^{\tilde{m}} dF(v) = F(\tilde{m} - m) \), we can see that \( V^b(\tilde{m}) = V^t(\tilde{m}) \). Let us now consider the case of a deviating bank. Given all other banks choose the threshold \( \tilde{m} \), a deviating bank with \( \tilde{m} > \bar{m} \) chooses \( B \) instead of \( L \). If this bank is matched with a bank in \( L \) with \( m_b \) then it must be \( m_b + \tilde{m} > 2\bar{m} \), so that from (3) the two banks always trade at the deposit rate \( i_d \). Therefore, the deviator bank’s payoff is
\[ \tilde{V}^b(\tilde{m}) = \theta(n) \int_{\tilde{m}}^{m+0} \mathbb{I}_{(m_b + m > 2\bar{m})} \tilde{m} i_d dF(m_b) + (1 - \theta(n)) \tilde{m} i_d. \]

However, since (35) is greater than \( \tilde{m} i_d \), the deviator bank is worse off. Therefore, given all other banks choose the threshold \( \tilde{m} \), a bank with \( \tilde{m} > \bar{m} \) will choose \( B \). This proves the result.

**Refinement - ex-post consistency**

In this Appendix, we show that imposing the requirement that banks ex-ante choice be consistent with what they do ex-post yields to the equilibrium of Proposition 2 (and the equilibrium is unique). In our context, ex-post consistency means that banks that chose to borrow ex-ante cannot lend ex-post.

By contradiction, suppose there is an equilibrium where all banks with reserves \( m \in [b_0, b_1] \cup [b_2, b_3] \) choose to become borrowers for some numbers \( b_0 < b_1 < b_2 < b_3 \) (where \( b_0 \) and \( b_3 \) are possibly (minus) infinite) while all other banks choose to become lenders. Considering two subsets are without loss of generality as the proof extends when there are more subsets.

First, notice that if \( (b_0, b_1) = (-\infty, \tilde{m}] \) then banks with \( m \in [b_2, b_3] \) with \( b_2 > \tilde{m} \) would prefer to become lenders as they won’t ever benefit from trading if they choose
to become borrowers. So, we will take the case where $b_0 > -\infty$ or $b_1 < \tilde{m}$.

However, those banks with $m < b_0$ would prefer to be borrowers as they would never generate gains from trade if they were to become lenders, as they never meet banks with lower reserves holdings (they could meet banks with higher reserves holdings, but the ex-post consistency refinement imposes that they would not trade in that case). Therefore, only strategies with $b_0 = 1$ can constitute an equilibrium.

Now suppose that $b_1 < b_2 < b_3 < \bar{m}$. We want to show that a bank with reserves $\tilde{m} \in [b_1, b_2]$ would prefer to become a borrower to becoming a lender. If the bank were to choose to become a lender, its payoff would be $V^{l}(\tilde{m}) = \tilde{m}i_d + (\tilde{m} - \tilde{m})i_p$ as this bank would never meet a bank it could either borrow or lend to with positive gains from trade. If the bank were to choose to become a borrower instead, its payoff would be (where $L = \mathbb{R}/(-\infty, b_1] \cup [b_2, b_3]$):

$$V^{b}(\tilde{m}) = \theta(n) \int_{m_{\ell} \in \mathcal{L}} \mathbb{I}_{\{(\tilde{m} + m_{\ell} > 2\tilde{m})\}} \left[ \tilde{m}i_d + (\tilde{m} - \tilde{m})i_d + \frac{\tilde{m} - \tilde{m}}{2}(i_p - i_d) \right] dF_{\ell}(m_{\ell})$$

$$+ \theta(n) \int_{m_{\ell} \in \mathcal{L}} \mathbb{I}_{\{m_{\ell} > \tilde{m}, \tilde{m} + m_{\ell} < 2\tilde{m}\}} \left[ \tilde{m}i_d + (\tilde{m} - \tilde{m})i_p + \frac{m_{\ell} - \tilde{m}}{2}(i_p - i_d) \right] dF_{\ell}(m_{\ell})$$

$$+(1 - \theta(n)) [\tilde{m}i_d + (\tilde{m} - \tilde{m})i_p]$$

Since the sets $\{m_{\ell} : \tilde{m} + m_{\ell} > 2\tilde{m}\} \cap \mathcal{L}$ and $\{m_{\ell} : m_{\ell} > \tilde{m}, \tilde{m} + m_{\ell} < 2\tilde{m}\} \cap \mathcal{L}$ are non-empty and have positive measure, the banks with $\tilde{m} \in [b_1, b_2]$ obtains a higher payoff by being a borrower. Therefore, if $b_3 < \tilde{m}$ then $B = (-\infty, b_3]$ and the argument in Proposition 2 implies that $b_3 = \tilde{m}$.

So, let us now suppose that $b_3 > \tilde{m}$. Then we show that a bank with $\tilde{m} \in [\max\{\tilde{m}, b_2\}, b_3]$ would prefer to become a lender instead of a borrower. If the bank chooses to become a borrower, its payoff would be

$$V^{b}(\tilde{m}) = \tilde{m}i_d$$

as it would either not trade or not generate gains from trade as it already satisfies its
reserve requirement with $\tilde{m} > \bar{m}$. If the bank chooses to become a lender, its payoff is

$$V^\ell(\tilde{m}) = \frac{n}{1 - n} \theta(n) \int_{m_b \in \mathcal{B}} \mathbb{I}_{\{m_b < \tilde{m}, \tilde{m} + m_b > 2\bar{m}\}} \left[ \tilde{m}i_d + \frac{\tilde{m} - m_b}{2}(i_p - i_d) \right] dF_b(v)$$

$$+ \frac{n}{1 - n} \theta(n) \int_{m_b \in \mathcal{B}} \mathbb{I}_{\{\tilde{m} + m_b < 2\bar{m}\}} \left[ \tilde{m}i_d + \frac{\tilde{m} - \bar{m}}{2}(i_p - i_d) \right] dF_b(v)$$

$$+ (1 - \frac{n}{1 - n} \theta(n))\tilde{m}i_d$$

Once again the set of borrowers for which there are gains from trade has a positive measure, so that a bank with $\tilde{m} \in [\max\{\tilde{m}, b_2\}, b_3]$ prefers to become a lender. Hence, $b_3 \leq \tilde{m}$ and we are back in the first case. This shows that with the ex-post consistency refinement, the equilibrium of Proposition 2 is unique.

**Proof of Proposition 3**

Changing variable to integrate over the liquidity shocks $v_\ell$ and $v_b$ instead of reserves, we obtain,

$$\tilde{Q}(\tilde{m} - m) = \int_{-\bar{v}}^{\tilde{m} - m} \int_{\tilde{m} - m}^{\bar{v}} \frac{v_\ell - v_b}{2} \frac{dF(v_\ell)}{1 - F(\tilde{m} - m)} \frac{dF(v_b)}{F(\tilde{m} - m)}$$

and integrating we have

$$\tilde{Q}(\tilde{m} - m) = \frac{E[v] - \int_{-\bar{v}}^{\tilde{m} - m} vdF(v)}{2F(\tilde{m} - m)[1 - F(\tilde{m} - m)]}$$

and assuming that the mean liquidity shock is zero, i.e. $E[v] = 0$, we obtain

$$\tilde{Q}(\tilde{m} - m) = \frac{-\int_{-\bar{v}}^{\tilde{m} - m} vdF(v)}{2F(\tilde{m} - m)[1 - F(\tilde{m} - m)]}.$$

Now we are interested in the derivative of $Q(x) = \min\{F(x), 1 - F(x)\} \tilde{Q}(x)$. So $Q(x)$ is differentiable almost everywhere, as it has a kink at $x = 0$. Assuming that the distribution function $F(v)$ is symmetric, we have $F(x) < 1 - F(x)$ whenever $x < 0$, while $F(x) > 1 - F(x)$ whenever $x > 0$. Therefore, for all $x < 0$ we obtain

$$Q(x) \big|_{x < 0} = \frac{-\int_{-\bar{v}}^{x} vdF(v)}{2[1 - F(x)]},$$

55
with derivative

\[ 2Q'(x) |_{x<0} = \frac{[1 - F(x)] [-xf(x)] + f(x) \left[ -\int_{-\theta}^{x} vdF(v) \right]}{[1 - F(x)]^2}. \]

Hence, \( Q'(x) > 0 \) whenever \( x < 0 \). Also, \( \lim_{x \to -\theta} Q(x) = 0 \).

Now, for all \( x > 0 \) we obtain

\[ Q(x) |_{x>0} = -\frac{\int_{-\theta}^{x} v dF(v)}{2F(x)}, \]

with derivative

\[ 2Q'(x) |_{x>0} = \frac{F(x) [-xf(x)] - f(x) \left[ -\int_{-\theta}^{x} v dF(v) \right]}{F(x)^2} \]

and since \( x > 0 \) we obtain that \( Q'(x) < 0 \) whenever \( x > 0 \). Also, \( \lim_{x \to \theta} Q(x) = 0 \), as the mean shock is zero.

**Average OTC rate**

\[ \mu_i = \int_{\bar{m}}^{\bar{m}+\theta} \int_{\bar{m}}^{m+\theta} \frac{q(m_b, m_\ell)}{Q(\bar{m} - m)} i(m_b, m_\ell) dF_\ell(m_\ell) dF_b(m_b) \]

and using (2) and (13) combined with (3),

\[
\begin{align*}
\mu_i & = \frac{1}{2Q(\bar{m} - m)} \left[ \int_{\bar{m}}^{m} \int_{\bar{m}}^{m+\theta} \left( (m_\ell - m_b) + (m_\ell - \bar{m})i_\ell + (\bar{m} - m_b)i_\ell \right) dF_\ell(m_\ell) dF_b(m_b) \right] \\
& = 1 + \frac{1}{2Q(\bar{m} - m)} \left[ \int_{\bar{m}}^{\bar{m}+\theta} \left( (m + v_\ell - \bar{m})i_\ell + (\bar{m} - m - v_\ell)i_\ell \right) \frac{dF(v_\ell)}{1 - F(\bar{m} - m)} \frac{dF(v_\ell)}{F(\bar{m} - m)} \right] \\
& = 1 + \frac{1}{2Q(\bar{m} - m)} \left[ \int_{\bar{m} - m}^{\bar{m}+\theta} \left( (m + v_\ell - \bar{m})i_\ell \frac{dF(v_\ell)}{1 - F(\bar{m} - m)} + \int_{\bar{m} - m}^{\bar{m}+\theta} (\bar{m} - m - v_\ell)i_\ell \frac{dF(v_\ell)}{F(\bar{m} - m)} \right) \right] \\
& = 1 + \frac{1}{2Q(\bar{m} - m)} \left[ (\bar{m} - m)(i_\ell - i_d) + i_d \int_{\bar{m} - m}^{\bar{m}+\theta} \frac{dF(v)}{1 - F} - i_\ell \int_{\bar{m} - m}^{\bar{m}+\theta} \frac{dF(v)}{F} \right] \\
\end{align*}
\]

where we use \( F = F(\bar{m} - m) \) as a shorthand. Therefore

\[
\mu_i = \frac{1}{2Q(\bar{m} - m)} \left[ (\bar{m} - m)(i_\ell - i_d) + i_d \int_{\bar{m} - m}^{\bar{m}+\theta} \frac{dF(v)}{1 - F} [E[v] - \int_{-\theta}^{\bar{m}+\theta} v \frac{dF(v)}{F}] \left( \frac{i_d}{1 - F} + i_\ell \frac{dF(v)}{F} \right) \right]
\]
Hence, using (13) we have

$$\mu_i = (1 - F)i_t + F_i_d + \frac{1}{2Q(\bar{m} - m)} \left[ (\bar{m} - m)(i_t - i_d) + \frac{i_d}{1 - F}E[v] \right]$$

and if $E[v] = 0$ we obtain

$$\mu_i = (1 - F(\bar{m} - m))i_t + F(\bar{m} - m)i_d + \frac{(\bar{m} - m)}{Q(\bar{m} - m)} \frac{(i_t - i_d)}{2}$$

$$= F(\bar{m} - m)i_t + [1 - F(\bar{m} - m)]i_d + \left[ 1 - 2F(\bar{m} - m) + \frac{(\bar{m} - m)}{2Q(\bar{m} - m)} \right] (i_t - i_d)$$

Hence, the average OTC rate is equal to the rate in Poole (1968) whenever $\bar{m} = m$. The relative position of the average OTC rate with respect to the Poole rate then depends on the expression in brackets. Using $x = \bar{m} - m$ we have

$$\mu'_i(x) = f(x)(i_d - i_t) + \frac{(i_t - i_d)\tilde{Q}(x) - x\tilde{Q}'(x)}{Q(x)^2}$$

and

$$\tilde{Q}'(x) = \frac{f(x)}{2} \frac{F(x)[1 - F(x)] [-x] - [1 - 2F(x)] \left[ - \int_{-\infty}^{x} v dF(v) \right]}{F(x)^2[1 - F(x)]^2}$$

Therefore $x > 0$ implies $\tilde{Q}'(x) < 0$ and $x < 0$ implies $\tilde{Q}'(x) > 0$, so that $x\tilde{Q}'(x) < 0$.

**Volatility**

For computational purpose, we can rearrange the expression for the volatility as follows:
where the last inequality follows from the fact that is equal to 
\[ i(v_\ell, v_b) - \mu_i \]

\[ (i_\ell - i_d)^2 \frac{v_\ell - v_b}{2\bar{Q}} \left[ \frac{\bar{m} - m - v_\ell}{v_\ell - v_b} + c \right]^2 
= (i_\ell - i_d)^2 \frac{v_\ell - v_b}{2\bar{Q}} \left[ \frac{\bar{m} - m - v_\ell}{v_\ell - v_b} + c \right]^2 
= (i_\ell - i_d)^2 \frac{v_\ell - v_b}{2\bar{Q}} \left[ \frac{\bar{m} - m - v_\ell}{v_\ell - v_b} \right]^2 + \frac{v_\ell - v_b}{2\bar{Q}} c^2 + \frac{1}{\bar{Q}} (\bar{m} - m - v_\ell) c \]

where
\[ c = F(\bar{m} - m) - \frac{(\bar{m} - m)}{2\bar{Q}(\bar{m} - m)} \]

Notice that, as \( \bar{Q} \geq 0 \), we obtain \( c \geq 0 \) whenever \( m \geq \bar{m} \), while \( c \leq F(\bar{m} - m) \) otherwise. In the sequel, we will assume that \( c \geq 0 \), as we are mostly interested in the case where \( m \geq \bar{m} \). So

\[ \sigma_w^2(\mu_i) = \int_{-\bar{v}}^{\bar{m} - m} \int_{m - m}^{\bar{m} - m} \frac{q(v_\ell, v_b)}{Q(\bar{m}, m)} [i(v_\ell, v_b) - \mu_i]^2 \frac{dF(v_\ell)}{1 - F(\bar{m} - m)} \frac{dF(v_b)}{F(\bar{m} - m)} \]

is equal to

\[ \sigma_w^2(\mu_i) = (i_\ell - i_d)^2 \left[ \int_{-\bar{v}}^{\bar{m} - m} \int_{m - m}^{\bar{m} - m} \left\{ \frac{1}{2\bar{Q}} \frac{(\bar{m} - m - v_\ell)^2}{v_\ell - v_b} + \frac{1}{\bar{Q}} (\bar{m} - m - v_\ell) c \right\} \frac{dF(v_\ell)}{1 - F(\bar{m} - m)} \frac{dF(v_b)}{F(\bar{m} - m)} \right] \]

\[ \leq (i_\ell - i_d)^2 \left[ \int_{-\bar{v}}^{\bar{m} - m} \int_{m - m}^{\bar{m} - m} \left\{ \frac{1}{2\bar{Q}} \frac{(\bar{m} - m - v_\ell)^2}{v_\ell - v_b} \frac{dF(v_\ell)}{1 - F(\bar{m} - m)} \frac{dF(v_b)}{F(\bar{m} - m)} + c^2 \right\} \right] \]

where the last inequality follows from the fact that \( v_b \leq \bar{m} - m \) over the space of integration. Therefore
\[
\sigma_w^2(\mu_i) \leq (i_\ell - i_d)^2 \left[ \int_{-\bar{v}}^{\bar{v}} \int_{-\bar{m}}^{\bar{m}} \frac{1}{2Q} \left( v_\ell - (\bar{m} - m) \right)^2 \frac{1}{1 - F(\bar{m} - m)} + c^2 \right]
\]

\[
= (i_\ell - i_d)^2 \left[ \int_{-\bar{v}}^{\bar{v}} \int_{-\bar{m}}^{\bar{m}} \frac{1}{2Q} \left( v_\ell - (\bar{m} - m) \right)^2 \frac{1}{1 - F(\bar{m} - m)} - \frac{(\bar{m} - m)}{2Q} + c^2 \right]
\]

\[
= (i_\ell - i_d)^2 \left[ \frac{1}{2Q} \left( E[v] - \int_{-\bar{m}}^{\bar{m}} v_\ell dF(v_\ell) \right) - \frac{(\bar{m} - m)}{2Q} + c^2 \right]
\]

\[
= (i_\ell - i_d)^2 \left[ \frac{1}{2Q} \left( 2\bar{Q}F(\bar{m} - m) - \frac{(\bar{m} - m)}{2Q} + c^2 \right) \right]
\]

\[
= (i_\ell - i_d)^2 \left[ \frac{2}{2\bar{Q}} \left( F(\bar{m} - m) - \frac{(\bar{m} - m)}{2Q} + c^2 \right) \right]
\]

\[
= (i_\ell - i_d)^2 (c + c^2)
\]

We now describe a lower bound for the OTC rate volatility:

\[
\sigma_w^2(\mu_i) = (i_\ell - i_d)^2 \left[ \int_{-\bar{v}}^{\bar{v}} \int_{-\bar{m}}^{\bar{m}} \frac{1}{2Q} \left( \frac{v_\ell - v_b}{v_\ell - v_b} \right)^2 + \frac{1}{\bar{Q}} (\bar{m} - m - v_\ell) c \right] \frac{dF(v_\ell)}{1 - F(\bar{m} - m)}
\]

\[
\geq (i_\ell - i_d)^2 \left[ \int_{-\bar{v}}^{\bar{v}} \int_{-\bar{m}}^{\bar{m}} \frac{1}{2Q} \left( \frac{v_\ell - v_b}{v_\ell - v_b} \right)^2 + \frac{1}{\bar{Q}} (v_b - v_\ell) c \right] \frac{dF(v_\ell)}{1 - F(\bar{m} - m)} \frac{F(v_\ell)}{F(v_\ell)}
\]

\[
= (i_\ell - i_d)^2 \left[ \int_{-\bar{v}}^{\bar{v}} \int_{-\bar{m}}^{\bar{m}} \frac{1}{2Q} \left( \frac{v_\ell - \bar{m} - m}{v_\ell - v_b} \right)^2 + \frac{1}{\bar{Q}} (v_\ell - v_b) \right] \frac{dF(v_\ell)}{1 - F(\bar{m} - m)} + \frac{dF(v_\ell)}{F(\bar{m} - m)} - 2c + c^2
\]

Counterparty risk

To find the risk premium, we again have to consider four types of trades: (1) meetings where \( m_b + m_\ell \geq 2\bar{m}, m_b < \bar{m} \), and \( m_\ell > \bar{m} \), (2) meetings where \( m_b + m_\ell \geq 2\bar{m}, m_b > \bar{m} \), and \( m_\ell > \bar{m} \), (3) meetings where \( m_b + m_\ell < 2\bar{m}, m_b < \bar{m} \), and \( m_\ell > \bar{m} \), and (4) meetings where \( m_b + m_\ell < 2\bar{m}, m_b < \bar{m} \), and \( m_\ell < \bar{m} \). We assume that if the borrower defaults, then he gets nothing.

We now solve the rate in meeting (1). Since the rates are given by equalizing surplus from trade, rates in meetings (1) will have to satisfy

\[
(1 - \delta) \left\{ (m_b + q - \bar{m})i_d + \bar{m}i_d - q^2 \right\} = (1 - \delta) \left\{ (m_\ell - q - \bar{m})i_d + \bar{m}i_d + (1 - \delta)q^2 \right\} - \left\{ (m_\ell - \bar{m})i_d + \bar{m}i_d \right\}
\]

59
notice that the lender only gets paid when it does not disappear and the borrower does not disappear either. We assume that the borrower always has to pay, as this is akin to the fact that all liabilities are still due, even in bankruptcy case. We can simplify the expression above to find

\[ i^\delta = \frac{m_\ell - \bar{m}}{m_\ell - m_b} i_d + \frac{\bar{m} - m_b}{m_\ell - m_b} i_\ell + \frac{\delta}{2} \frac{i^\delta}{2} \]

\[ i^\delta = \frac{2}{2 - \delta} i \]

We now solve for the rate in meeting (2). Since the rates are given by equalizing surplus from trade, rates in meetings (2) will have to satisfy

\[
(1 - \delta) \left\{ (m_b + q - \bar{m})i_d + \bar{m}i_d - q^\delta - [(m_b - \bar{m})i_d + \bar{m}i_d] \right\} = \\
(1 - \delta) \left\{ (m_\ell - q - \bar{m})i_d + \bar{m}i_d + (1 - \delta)qi^\delta - [(m_\ell - \bar{m})i_d + \bar{m}i_d] \right\}
\]

Simplifying, we obtain

\[ i^\delta = \frac{2}{2 - \delta} i_d \]

\[ i^\delta = \frac{2}{2 - \delta} i \]

We now solve for the rate in meeting (3). In this case, both the borrower and the lender have to borrow at the lending facility after they exit the interbank market. Rates in meetings (3) will have to satisfy

\[
(1 - \delta) \left\{ -(\bar{m} - (m_b + q))i_p + \bar{m}i_d - qi^\delta - [(\bar{m} - m_b)i_p + \bar{m}i_d] \right\} = \\
(1 - \delta) \left\{ -(m_\ell - (m_\ell - q))i_p + \bar{m}i_d + (1 - \delta)qi^\delta - [(m_\ell - \bar{m})i_d + \bar{m}i_d] \right\}
\]

Simplifying, we obtain

\[ i^\delta = \frac{2}{2 - \delta} \left[ \frac{(m_\ell - \bar{m})}{(m_\ell - m_b)} i_d + \frac{(\bar{m} - m_b)}{(m_\ell - m_b)} i_\ell \right] \]

\[ i^\delta = \frac{2}{2 - \delta} i \]

Finally, we now solve for the rate in meeting (4). In this case, both the borrower and the lender have to borrow at the lending facility after they exit the interbank market,
whether or not they traded on the interbank market. Rates in meetings (4) will have to satisfy

\[
(1 - \delta) \left\{ -(\bar{m} - (m_b + q))i_p + \bar{m}i_d - qi^\delta - [(\bar{m} - m_b)i_p + \bar{m}i_d] \right\} = \\
(1 - \delta) \left\{ -(\bar{m} - (m_\ell - q))i_p + \bar{m}i_d + (1 - \delta)qi^\delta - [(\bar{m} - m_\ell)i_p + \bar{m}i_d] \right\}
\]

Simplifying, we obtain

\[
i^\delta = \frac{2}{2 - \delta} i_\ell \]
\[
= \frac{2}{2 - \delta} i
\]
Payoff with counterparty risk  We now compute the payoffs, under each cases. In case (1) for the borrower:

\[ i_d(m_b + \frac{m_\ell - m_b}{2} - \bar{m}) + i_d\bar{m} - \frac{m_\ell - m_b}{2}i\delta = \]

\[ i_d \left( m_b + \frac{m_\ell - m_b}{2} \right) - \frac{m_\ell - m_b}{2}i\delta = \]

\[ i_d m_b + \frac{m_\ell - m_b}{2} \left[ i_d - \left( \frac{2}{2 - \delta} \right) \right] \]

\[ i_d m_b + \frac{1}{2} \left[ i_d (m_\ell - m_b) - \frac{2}{2 - \delta} \right] \left( (m_\ell - \bar{m})i_d + (\bar{m} - m_b)i_p \right) = \]

\[ i_d m_b + \frac{1}{2} \left[ \bar{m} - m_b \right] \left[ (m_\ell - \bar{m})i_d \right] + (\bar{m} - m_b) \left[ \frac{2}{2 - \delta} \right] \]
In case (1) for the lender:

\[
\begin{align*}
    i_d \left( m_{\ell} - \frac{m_{\ell} - m_b}{2} - \bar{m} \right) + i_d \bar{m} + \frac{m_{\ell} - m_b}{2} i_d \delta &= \\
    i_d \left( m_{\ell} - \frac{m_{\ell} - m_b}{2} \right) + \frac{m_{\ell} - m_b}{2} i_d \delta &= \\
    i_d m_{\ell} - \frac{m_{\ell} - m_b}{2} \left[ i_d - \left( \frac{2}{2 - \delta} \right) \right] &= \\
    i_d m_{\ell} - \frac{1}{2} \left[ i_d (m_{\ell} - m_b) - \left( \frac{2}{2 - \delta} \right) (m_{\ell} - \bar{m}) i_d + (\bar{m} - m_b) i_p \right] &= \\
    i_d m_{\ell} - \frac{1}{2} \left[ i_d (m_{\ell} - \bar{m} + \bar{m} - m_b) - \left( \frac{2}{2 - \delta} \right) (m_{\ell} - \bar{m}) i_d + (\bar{m} - m_b) i_p \right] &= \\
    i_d m_{\ell} - \frac{1}{2} \left[ (m_{\ell} - \bar{m}) i_d \left( 1 - \frac{2}{2 - \delta} \right) + (\bar{m} - m_b) \left[ i_d - \left( \frac{2}{2 - \delta} \right) \right] \right] &= \\
    i_d m_{\ell} - \frac{1}{2} \left[ -\frac{\delta}{2 - \delta} (m_{\ell} - \bar{m}) i_d + (\bar{m} - m_b) \frac{2(i_d - i_p) - \delta i_d}{2 - \delta} \right] &= \\
    i_d m_{\ell} + \frac{\delta}{2 - \delta} \left( \frac{m_{\ell} - \bar{m}}{2} \right) i_d + \frac{(\bar{m} - m_b)}{2} \frac{2(i_p - i_d) + \delta i_d}{2 - \delta} &= \\
    i_d m_{\ell} + \frac{(\bar{m} - m_b)}{2} (i_p - i_d) + \frac{\delta}{2 - \delta} \left[ \frac{(\bar{m} - m_b)}{2} (i_p - i_d) + \frac{(m_{\ell} - m_b)}{2} i_d \right] &= \\
    P^\ell(m_{\ell}, m_b) + \frac{\delta}{2 - \delta} \left[ \frac{(m_{\ell} - m_b)}{2} i_p + \frac{(m_{\ell} - \bar{m})}{2} i_d \right] &= \\
    P^\ell(m_{\ell}, m_b) + \frac{\delta}{2 - \delta} \left[ \frac{(\bar{m} - m_b)}{2} i_p + \frac{(m_{\ell} - m_b)}{2} i_d \right] &=
\end{align*}
\]
In case (3) for the borrower:

\[ i_p(m_b + \frac{m_\ell - m_b - \bar{m}}{2} + i_d \bar{m} - \frac{m_\ell - m_b - \bar{m}}{2} i^\delta) = \]

\[ i_p(m_b - \bar{m}) + i_d \bar{m} + \frac{m_\ell - m_b}{2} [i_p - i^\delta] = \]

\[ i_p(m_b - \bar{m}) + i_d \bar{m} + \frac{m_\ell - m_b}{2} [i_p - \left(\frac{2}{2 - \delta}\right) i] = \]

\[ i_p(m_b - \bar{m}) + i_d \bar{m} + \frac{m_\ell - m_b}{2} \left[ i_p - \left(\frac{2}{2 - \delta}\right) \left( \frac{m_\ell - \bar{m}}{m_\ell - m_b} i_d + \frac{\bar{m} - m_b}{m_\ell - m_b} i_p \right) \right] = \]

\[ i_p(m_b - \bar{m}) + i_d \bar{m} + \frac{m_\ell - m_b - \bar{m} + \bar{m} - m_b}{2} - \left(\frac{1}{2 - \delta}\right) ((m_\ell - \bar{m}) i_d + (\bar{m} - m_b) i_p) = \]

\[ i_p(m_b - \bar{m}) + i_d \bar{m} + \frac{(m_\ell - \bar{m})}{2} \left[ i_p - \frac{2}{2 - \delta} i_d \right] - i_p \frac{\bar{m} - m_b}{2} \frac{\delta}{2 - \delta} = \]

\[ i_p(m_b - \bar{m}) + i_d \bar{m} + \frac{(m_\ell - \bar{m})}{2} (i_p - i_d) - \frac{(m_\ell - m_b) \delta}{2} \frac{i_p}{2 - \delta} = \]

\[ P^b(m_\ell, m_b) + \frac{\delta}{2 - \delta} \left[ \frac{(m_\ell - \bar{m})}{2} (i_p - i_d) - \frac{(m_\ell - m_b)}{2} i_p \right] = \]

\[ P^b(m_\ell, m_b) + \frac{\delta}{2 - \delta} \left[ \frac{(m_\ell - \bar{m})}{2} (1 + i_p - i_d) - \frac{(m_\ell - \bar{m} + \bar{m} - m_b)}{2} i_p \right] = \]

\[ P^b(m_\ell, m_b) - \frac{\delta}{2 - \delta} \left[ \frac{(m_\ell - \bar{m})}{2} i_d + \frac{(\bar{m} - m_b)}{2} i_p \right] = \]
In case (3) for the lender:

\[ i_p(m_\ell - \frac{m_\ell - m_b}{2} - \bar{m}) + i_d\bar{m} + \frac{m_\ell - m_b}{2} i_p \delta = \]

\[ i_p(m_\ell - \bar{m}) + i_d\bar{m} - \frac{m_\ell - m_b}{2} [i_p - i_p^2] = \]

\[ i_p(m_\ell - \bar{m}) + i_d\bar{m} - \frac{m_\ell - m_b}{2} \left[ i_p - \left( \frac{2}{2 - \delta} \right) i_d \right] = \]

\[ i_p(m_\ell - \bar{m}) + i_d\bar{m} - \frac{m_\ell - m_b}{2} \left[ i_p - \left( \frac{2}{2 - \delta} \right) \left( \frac{m_\ell - \bar{m}}{m_\ell - m_b} i_d + \frac{\bar{m} - m_b}{m_\ell - m_b} i_p \right) \right] = \]

\[ i_p(m_\ell - \bar{m}) + i_d\bar{m} - \frac{m_\ell - \bar{m} + \bar{m} - m_b}{2} + \left( \frac{1}{2 - \delta} \right) \left( (m_\ell - \bar{m})i_d + (\bar{m} - m_b)i_p \right) = \]

\[ i_p(m_\ell - \bar{m}) + i_d\bar{m} - \frac{(m_\ell - \bar{m})}{2} \left[ i_p - \frac{2}{2 - \delta} i_d \right] + \frac{i_p (\bar{m} - m_b)}{2} \delta \]

\[ i_p(m_\ell - \bar{m}) + i_d\bar{m} - \frac{(m_\ell - \bar{m})}{2} + \frac{2}{2 - \delta} i_d + \frac{i_p (\bar{m} - m_b)}{2} \delta = \]

\[ i_p(m_\ell - \bar{m}) + i_d\bar{m} + \frac{(m_\ell - \bar{m})}{2} i_d + \frac{i_p (\bar{m} - m_b)}{2} \delta \]

\[ i_d m_\ell + \frac{(m_\ell - \bar{m})}{2} (i_p - i_d) + \frac{(m_\ell - \bar{m})}{2} \delta i_d + \frac{\delta}{2 - \delta} i_p (\bar{m} - m_b) = \]

\[ i_d m_\ell + \frac{(m_\ell - \bar{m})}{2} (i_p - i_d) + \frac{(m_\ell - \bar{m})}{2} \delta i_d + \frac{\delta}{2 - \delta} i_p (\bar{m} - m_b) = \]

\[ P^p(m_\ell, m_b) + \frac{\delta}{2 - \delta} \left[ \frac{(m_\ell - \bar{m})}{2} i_d + \frac{(\bar{m} - m_b)}{2} i_p \right] \]

\[ \frac{V^b(\tilde{m}; \delta)}{1 - \delta} = V^b(\bar{m}; 0) - \theta(n) \int_{\tilde{m} + \frac{\bar{m} - m}{\Delta - 1}}^{\bar{m} + \frac{m - \bar{m}}{\Delta - 1}} \frac{\delta}{2 - \delta} i(p, \tilde{m}) q^b(m_\ell, \tilde{m}) dF_\ell(m_\ell) \]

\[ -\theta(n) \int_{\tilde{m}}^{m + \frac{m - \bar{m}}{\Delta - 1}} \left\{ P^b(m_\ell, \bar{m}; 0) - [(\bar{m} - m)b + i_d\bar{m}] \right\} dF_\ell(m_\ell) \]
Notice that $q^\delta(\bar{m} - \bar{m}, \bar{m}) = 0$ and also $P^b(\bar{m} - \bar{m}, \bar{m}; 0) = (\bar{m} - \bar{m})i_p + i_d\bar{m}$. Hence the derivative of $V^b(\bar{m}; \delta)$ is

$$
\frac{1}{1 - \delta} \frac{\partial V^b(\bar{m}; \delta)}{\partial \bar{m}} = \frac{\partial V^b(\bar{m})}{\partial \bar{m}} - \theta(n) \int_{\bar{m}}^{\bar{m} + \frac{\bar{m} - \bar{m}}{\Delta - 1}} \left[ \frac{\partial P^b(\bar{m}, \bar{m}; 0)}{\partial \bar{m}} - i_p \right] dF(\bar{m})
$$

$$
+ \theta(n) \int_{\bar{m} + \frac{\bar{m} - \bar{m}}{\Delta - 1}}^{\bar{m} + \theta} \frac{\delta}{2 - \delta} \frac{i_p}{2} dF(\bar{m})
$$

Notice that $\frac{\partial P^b(\bar{m}, \bar{m}; 0)}{\partial \bar{m}} - i_p \leq 0$, so that all the extra terms (multiplied by $\theta(n)$) are positive. Hence

$$
\frac{\partial V^b(\bar{m}; \delta)}{\partial \bar{m}} > (1 - \delta) \frac{\partial V^b(\bar{m})}{\partial \bar{m}}
$$

We can now turn to lenders. Since $\hat{m} = \bar{m}$, we can rewrite the value of becoming a lender as

$$
\frac{V^\ell(\bar{m}; \delta)}{1 - \delta} = V^\ell(\bar{m}; 0) + \frac{n}{1 - n} \theta(n) \int_{\bar{m}}^{\bar{m}} \frac{\delta}{2 - \delta} i(\bar{m}, m_b) q^\delta(\bar{m}, m_b) dF_b(m_b)
$$

$$
- \frac{n}{1 - n} \theta(n) \int_{\bar{m} - \frac{\bar{m} - \bar{m}}{\Delta - 1}}^{\bar{m} - \frac{\bar{m} - \bar{m}}{\Delta - 1}} \left[ P^\ell(\bar{m}, m_b; 0) - i_d\bar{m} \right] dF_b(m_b).
$$

and as $q^\delta(\bar{m}, \bar{m} - (\Delta - 1)(\bar{m} - \bar{m})) = 0$, and $P^\ell(\bar{m}, \bar{m} - m - (\Delta - 1)(\bar{m} - \bar{m}); 0) = i_d\bar{m}$, the derivative is

$$
\frac{1}{1 - \delta} \frac{\partial V^\ell(\bar{m}; \delta)}{\partial \bar{m}} = \frac{\partial V^\ell(\bar{m}; 0)}{\partial \bar{m}} + \frac{n}{1 - n} \theta(n) \int_{\bar{m} - m - \frac{\bar{m} - \bar{m}}{\Delta - 1}}^{\bar{m} - \frac{\bar{m} - \bar{m}}{\Delta - 1}} \frac{\delta}{2 - \delta} \frac{i_d}{2} F_b(m_b)
$$

$$
- \frac{n}{1 - n} \theta(n) \int_{\bar{m} - \frac{\bar{m} - \bar{m}}{\Delta - 1}}^{\bar{m} - \frac{\bar{m} - \bar{m}}{\Delta - 1}} \left[ \frac{\partial P^\ell(\bar{m}, m_b; 0)}{\partial \bar{m}} - i_d \right] dF_b(m_b),
$$

since $\frac{\partial P^\ell(\bar{m}, m_b; 0)}{\partial \bar{m}} \geq i_d$ in the region of interest, we cannot say if an additional unit of reserves is more valuable with or without counterparty risk.

**Auction rate with counterparty risk**

To find each bank’s willingness to pay, we need to define the value of exiting this centralized stage with reserves $m$. This is

$$
\frac{W(m)}{1 - \delta} = \int_{\bar{m} - \bar{m}}^{\bar{m} - m} V^b(m + v) dF(v) + \int_{\bar{m} - \bar{m}}^{\bar{m} + \theta} V^\ell(m + v) dF(v).
$$
Therefore,

\[ W'(m) = \int_{\tilde{\theta}}^{\tilde{\theta}} \frac{\partial V^b(m + v; \delta)}{\partial m} dF(v) + \int_{\tilde{\theta}}^{\tilde{\theta}} \frac{\partial V^l(m + v; \delta)}{\partial m} dF(v) \]

**References**

To be added