# The Irrelevance of Market Incompleteness for the Price of Aggregate Risk*

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**Abstract**

In a model with a large number of agents who have CRRA-type preferences, the lack of insurance markets for idiosyncratic labor income risk only affects the equilibrium risk-free rate, but it has no effect on the premium for aggregate risk in equilibrium. Despite missing markets, the representative agent prices the excess returns on stocks correctly. As a result, there is no link between the extent of self-insurance against idiosyncratic income risk and risk premia. This result holds even when households face binding borrowing constraints.

## 1 Introduction

This paper shows that there is no link between the extent to which idiosyncratic risk is traded away by households and the size of the risk premium that stocks command over

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risk-free bonds in financial markets. In a standard incomplete markets model populated by a continuum of agents who have CRRA-preferences, the part of idiosyncratic risk that is orthogonal to the aggregate shocks only affects the equilibrium risk-free rate, but it has no effect on the premium for aggregate risk in equilibrium. Thus the Consumption-CAPM derived by Breeden (1979) and Lucas (1978) prices the excess returns on the stock. The extent to which households manage to insure against idiosyncratic income risk is therefore irrelevant for risk premia.

Our results deepen the equity premium puzzle, because we show that Mehra and Prescott's (1985) statement of the puzzle applies to a much larger class of incomplete market models.[1] Our results also imply that macroeconomists can obtain reliable estimates of key parameters, such as the intertemporal elasticity of substitution, from aggregate consumption and security markets data even when markets are incomplete and households face binding constraints. In contrast to the micro-estimates, these are the right estimates when households have heterogeneous preferences.[2]

This result holds regardless of the persistence of the idiosyncratic shocks, as long as these shocks are not permanent, and it is robust to the introduction of various borrowing and solvency constraints, regardless of the tightness of these constraints, as long as these constraints allow for the existence of a stationary equilibrium. Our result also survives when agents have non-time-additive preferences.

In addition, we show that adding markets does not always lead to more risk sharing. In particular, if the logarithm of aggregate consumption follows a random walk, trading claims on payoffs that are contingent on *aggregate shocks* does not help agents to smooth their consumption. Introducing these assets leaves interest rates and asset prices unaltered. However, if there is predictability in aggregate consumption growth, agents want to hedge their portfolio against interest rate shocks, creating a role for trade in a richer menu of assets. The risk premium irrelevance result, however, still applies. Finally, we also show that idiosyncratic uninsurable income risk does not contribute any variation in the conditional market price of risk, beyond what is built into aggregate consumption

[1] Weil's (1989) statement of the risk-free rate puzzle, on the contrary, does not.

[2] Browning, Hansen and Heckman (2000) provide a comprehensive survey of the evidence for preference heterogeneity.

growth.

As an extension, we allow for idiosyncratic shocks that interact with the aggregate shocks. This restores a role for risk sharing in determining risk premia, but it is limited. We show analytically that the model precludes large effects on the risk premium through this channel: if there were any large effects, the increase in risk triggers a precautionary effect and this delivers complete risk sharing in spite of the market incompleteness. So, as we turn on Mankiw's mechanism by feeding in more variation in the cross-sectional dispersion of income shocks, this is offset by an increase in the amount of equilibrium risk sharing. This is true even if the idiosyncratic shocks are very persistent, as long as they are not permanent. Our result is in sharp contrast with Constantinides and Duffie (1996)'s existence result in an environment with permanent idiosyncratic shocks. They show the cross-sectional distribution of idiosyncratic shocks can be manipulated to deliver any distribution of risk premia in the case of permanent shocks.

In the quest towards the resolution of the equity premium puzzle identified by Hansen and Singleton (1983) and Mehra and Prescott (1985) uninsurable idiosyncratic income risk has been introduced into standard dynamic general equilibrium models.[3] Incomplete insurance of household consumption against idiosyncratic income risk was believed to increase the aggregate risk premium. Heaton and Lucas (1996), in their seminal paper, motivate their analysis as follows:

> The motivation for considering the interaction between trading frictions and asset prices in this environment is best understood by reviewing the findings of a number of recent papers. Telmer (1993) and Lucas (1994) examine a similar model with transitory idiosyncratic shocks and without trading costs. Surprisingly, they find that even though agents cannot insure against idiosyncratic shocks, predicted asset prices are similar to those with complete markets.' This occurs because when idiosyncratic shocks are transitory, consumption can be effectively smoothed by accumulating financial assets after good shocks and selling assets after bad shocks.

[3] For examples, see the work of Ayiagari and Gertler (1991), Telmer (1993), Lucas (1994), Heaton and Lucas (1996), Krusell and Smith (1998), and Marcet and Singleton (1999).

Heaton and Lucas (1996) attribute the failure of incomplete market models (or their partial success) in matching moments of asset prices to the household's success in smoothing consumption in this class of models. Lucas (1994) concludes that:

> agents effectively self-insure by trading to offset the idiosyncratic shocks when they have limited access to either a stock market or a bond market. Thus, even limited access to capital markets implies that asset prices will be similar to those predicted by the representative agent model, a finding that deepens the equity premium puzzle.

Our paper shows *analytically* that there is essentially no link between the extent to which idiosyncratic risk can be traded away by households and the size of the risk premium. The main contribution of our paper is to argue that Telmer (1993), Lucas (1994), Heaton and Lucas (1996), Marcet and Singleton (1999) and others in this literature have reached the right conclusions -namely that adding uninsurable idiosyncratic income risk to standard models does not alter the asset pricing implications of the model - but not for the right reasons. These authors have argued that households manage to self-insure quite well by trading a single bond, and as result, the risk premium is not affected by aggregate risk.[4]

We show that the extent of self-insurance does not matter. We can make our solvency constraints arbitrarily tight or make the income process highly persistent, and our theoretical result still goes through. Mankiw (1986)'s analysis of the standard incomplete markets Euler equation already established that time variation in the cross-sectional dispersion of equilibrium household consumption growth is one recipe for increasing risk premia in the case of convex marginal utility. Our work shows that solvency constraints and transaction costs in incomplete market models cannot produce this time variation as an equilibrium feature unless it is explicitly and exogenously built into the labor income process.

[4]In his survey of the equity premium literature, Kocherlakota (1996) concludes that:

> ..given this ability to self-insure, the behavior of the risk-free rate and the equity premium remain largely unaffected by the absence of markets as long as idiosyncratic shocks are not highly persistent.

Allocations and prices in our model with idiosyncratic and aggregate risk as well as incomplete markets can be backed out from the allocations and interest rates in a stationary equilibrium of a model with only idiosyncratic risk (as in Bewley (1986), Huggett (1993) or Aiyagari (1994)). This results implies an algorithm for computing equilibria in this model which is much simpler than the auctioneer algorithm devised by Lucas (1994) and its extension to economies with a continuum of agents. Under a transformed, aggregate-risk-neutral probability measure, there is a stationary recursive equilibrium for our economy characterized by an invariant measure over wealth and endowments, despite the presence of aggregate shocks. Hence, there is no need for computing a law of motion for this measure, or approximating it by a finite number of moments, as in Krusell and Smith (1997, 1998). This result goes through even when aggregate consumption growth is not distributed i.i.d., provided that a complete menu of claims on aggregate shocks is traded. In addition, we extend this result to allow for interaction between the aggregate and idiosyncratic shocks, in the case of i.i.d. aggregate shocks.

The paper is structured as follows. In section 2, we lay out the physical environment of our model. This section also demonstrates how we can transform the stochastically growing economy into an economy with a constant endowment. In section 3 we study this stationary model, called the Bewley model. The next section introduces the Arrow model, the model with aggregate uncertainty and a full set of Arrow securities that pay contingent on the realization of this aggregate uncertainty. We show that a stationary equilibrium of the Bewley model can be mapped into an equilibrium of the Arrow economy just by scaling up allocations by the aggregate endowment. In section 5 we derive the same result for the Bond model where only a one-period risk-free bond can be traded. After briefly discussing the representative agent model, section 6 shows that risk premia in the representative agent model and the Arrow model (and by implication, in the Bond model) are identical. Section 7 investigates the robustness of our results with respect to the assumptions about the underlying stochastic income process, and shows in particular that our results are robust to specific forms of dependence of idiosyncratic on aggregate shocks, as well as on aggregate shocks being correlated over time. Finally, section 8 concludes; all proofs are contained in the appendix.

# 2 Environment

Our exchange economy is populated by a continuum of individuals of measure 1. There is a single nonstorable consumption good. The aggregate endowment of this good is stochastic. Each individual's endowment depends on the realization of an idiosyncratic and an aggregate shock. This economy is identical to the one described by Lucas (1994), except that ours is populated by a continuum of agents (as in Bewley (1986), Aiyagari and Gertler (1991), Huggett (1993) and Aiyagari (1994)), instead of just two agents.

## 2.1 Representation of Uncertainty

We denote the current aggregate shock by $z_t \in Z$ and the current idiosyncratic shock by $y_t \in Y$ . For simplicity, both $Z$ and $Y$ are assumed to be finite. Furthermore, let $z^t = (z_0, \ldots, z_t)$ and $y^t = (y_0, \ldots, y_t)$ denote the history of aggregate and idiosyncratic shocks. As shorthand notation, we use $s_t = (y_t, z_t)$ and $s^t = (y^t, z^t)$. We let the economy start at initial node $z_0$. Conditional on idiosyncratic shock $y_0$ and thus $s_0 = (y_0, z_0)$, the probability of a history $s^t$ is given by $\pi_t(s^t|s_0)$. We assume that these shocks follow a first order Markov process with transition probabilities given by $\pi(s'|s)$.

## 2.2 Preferences and Endowments

Consumers rank stochastic consumption streams $\{c_t(s^t)\}$ according to the following homothetic utility function:

$$U(c)(s_0) = \sum_{t=0}^{\infty} \sum_{s^t \geq s^0} \beta^t \pi(s^t|s_0) \frac{c_t(s^t)}{1-\gamma}^{1-\gamma} \tag{1}$$

where $\gamma$ is the coefficient of relative risk aversion and $\beta \in (0,1)$ is the constant time discount factor. We define $U(c)(s^t)$ to be the continuation expected lifetime utility from consumption allocation $c = \{c_t(s^t)\}$ in node $s^t$. This utility can be constructed recursively as follows:

$$U(c)(s^t) = u(c_t(s^t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) U(c)(s^t, s_{t+1})$$

where we made use of the Markov assumption for the underlying stochastic process.

The economy's aggregate endowment process $\{e_t\}$ depends only on the aggregate event history; we let $e_t(z^t)$ denote the aggregate endowment at node $z^t$. Each agent draws a 'labor income' share $\eta(y_t, z_t)$ as a fraction of the aggregate endowment in each period. Her labor income share only depends on the current individual and aggregate event. We denote the resulting individual labor income process by $\{\eta_t\}$, with

$$\eta_t(s^t) = \eta(y_t, z_t)e_t(z^t) \tag{2}$$

where $s^t = (s^{t-1}, y_t, z_t)$. We assume $\eta(y_t, z_t) > 0$ in all states of the world.

Furthermore, we assume that the law of large numbers holds, so that $\pi(s^t|s_0)$ is not only a household's individual probability of having income $\eta_t(s^t)$, but also the deterministic fraction of the population having that income. We do *not* assume that the idiosyncratic shocks $y$ are uncorrelated over time.

In addition, there is a Lucas tree that yields a constant share $\alpha$ of the total aggregate endowment as capital income, so that the total dividends of the tree are given by $\alpha e_t(z^t)$ in each period. The remaining fraction of the total endowment accrues to individuals as labor income, so that $1 - \alpha$ denotes the labor income share. Therefore, by construction, the labor share of the aggregate endowment equals the sum over all individual labor income shares:

$$\sum_{y_t \in Y} \Pi_{z_t}(y_t)\eta(y_t, z_t) = (1 - \alpha), \tag{3}$$

for all $z_t$, where $\Pi_{z_t}(y_t)$ represents the cross-sectional distribution of idiosyncratic shocks $y_t$, conditional on the aggregate shock $z_t$. By the law of large numbers, the fraction of agents who draw $y$ in state $z$ only depends on $z$. An increase in the capital share $\alpha$ translates into proportionally lower individual labor income shares $\eta(y, z)$ for all $(y, z)$.[5]

At time 0, the agents are endowed with initial wealth $\theta_0$. This wealth represents the value of an agent's share of the Lucas tree producing the dividend flow in units of time 0 consumption, as well as the value of her labor endowment at date 0. We use $\Theta_0$ to denote

[5]Our setup nests the baseline calibration of Heaton and Lucas (1996), except for the fact that they allow for some minor variation in the capital share $\alpha$ depending on $z$.

the initial joint distribution of wealth and idiosyncratic shocks $(\theta_0, y_0)$.

This concludes our description of the physical environment. Many of our results are derived using a de-trended version of our economy, with a constant aggregate endowment and a transition probability matrix that is adjusted for stochastic growth as well. The agents in this de-trended economy, discussed now, have stochastic time discount factors.

## 2.3 Transformation of Growth Economy into a Stationary Economy

The stochastic growth rate of the endowment of the economy is denoted $\lambda(z^{t+1}) = e_{t+1}(z^{t+1})/e_t(z^t)$. We assume that aggregate endowment growth only depends on the current aggregate state:

**Condition 2.1.** *Aggregate endowment growth is a function of the current aggregate shock only:*

$$\lambda(z^{t+1}) = \lambda(z_{t+1})$$

Using this assumption, we can transform our growing economy into a stationary economy with a stochastic time discount rate and a growth-adjusted probability matrix, following Alvarez and Jermann (2001), who use this transformation in a complete markets economy. First, we define growth deflated consumption allocations (or consumption shares) as

$$\hat{c}_t(s^t) = \frac{c_t(s^t)}{e_t(z^t)} \text{ for all } s^t. \tag{4}$$

Next, we define the *growth-adjusted* probabilities and the growth-adjusted discount factor as:

$$\hat{\pi}(s_{t+1}|s_t) = \frac{\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\gamma}}{\sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\gamma}} \text{ and } \hat{\beta}(s_t) = \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\gamma}.$$

Note that $\hat{\pi}$ is a well-defined Markov matrix in that $\sum_{s_{t+1}} \hat{\pi}(s_{t+1}|s_t) = 1$ for all $s_t$ and that $\hat{\beta}(s_t)$ is stochastic as long as the original Markov process is not *iid* over time. Finally, let $\hat{U}(\hat{c})(s^t)$ denote the lifetime expected continuation utility in node $s^t$, under the new transition probabilities and discount factor, defined over consumption shares $\{\hat{c}_t(s^t)\}$ :

$$\hat{U}(\hat{c})(s^t) = u(\hat{c}_t(s^t)) + \hat{\beta}(s_t) \sum_{s_{t+1}} \hat{\pi}(s_{t+1}|s_t)\hat{U}(\hat{c})(s^t, s_{t+1}) \tag{5}$$

In the appendix we prove that this transformation does not alter the agents' ranking of different consumption streams.

**Proposition 2.1.** *Households rank consumption share allocations in the de-trended economy in exactly the same way as they rank the corresponding consumption allocations in the original growing economy: for any $s^t$ and any two consumption allocations $c, c'$*

$$U(c)(s^t) \geq U(c')(s^t) \Longleftrightarrow \hat{U}(\hat{c})(s^t) \geq \hat{U}(\hat{c}')(s^t)$$

*where the transformation of consumption into consumption shares is given by* (4).

This result is crucial for demonstrating that equilibrium allocations $c$ for the stochastically growing economy can be found by solving for equilibrium allocations $\hat{c}$ in the transformed economy.

## 2.4 Independence of Idiosyncratic Shocks from Aggregate Conditions

Next, we assume that idiosyncratic shocks are independent of the aggregate shocks. This assumption is crucial for most of the results in this paper.

**Condition 2.2.** *Individual endowment shares $\eta(y_t, z_t)$ are functions of the current idiosyncratic state $y_t$ only, that is $\eta(y_t, z_t) = \eta(y_t)$. Also, transition probabilities of the shocks can be decomposed as*

$$\pi(z_{t+1}, y_{t+1}|z_t, y_t) = \varphi(y_{t+1}|y_t)\phi(z_{t+1}|z_t).$$

That is, individual endowment *shares* and the transition probabilities of the idiosyncratic shocks are independent of the aggregate state of the economy $z$. In this case, the growth-adjusted probability matrix $\hat{\pi}$ and the re-scaled discount factor is obtained by adjusting only the transition probabilities for the *aggregate* shock, $\phi$, but not the transition probabilities for the idiosyncratic shocks:

$$\hat{\pi}(s_{t+1}|s_t) = \varphi(y_{t+1}|y_t)\hat{\phi}(z_{t+1}|z_t), \text{ and } \hat{\phi}(z_{t+1}|z_t) = \frac{\phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma}}.$$

Furthermore, the growth-adjusted discount factor only depends on the aggregate state $z_t$:

$$\hat{\beta}(z_t) = \beta \sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma} \tag{6}$$

The first part of our analysis assumes that the aggregate shocks are independent over time:

**Condition 2.3.** *Aggregate endowment growth is i.i.d.:*

$$\phi(z_{t+1}|z_t) = \phi(z_{t+1}).$$

In this case the growth rate of aggregate endowment is uncorrelated over time, so that the logarithm of the aggregate endowment follows a random walk with drift.[6] As a result, the growth-adjusted discount factor is a constant: $\hat{\beta}(z_t) = \hat{\beta}$, since $\hat{\phi}(z_{t+1}|z_t) = \hat{\phi}(z_{t+1})$.

There are two competing effects on the growth-adjusted discount rate: consumption growth itself makes agents more impatient, while the consumption risk makes them more patient.[7]

## 2.5 A Quartet of Economies

In order to derive our results, we study four models, whose main characteristics are summarized in table 1. The first three economies are endowment economies with aggregate

[6] In section 7 we show that our results survive the introduction of persistence in the growth rates if a complete set of contingent claims on aggregate shocks is traded.

[7] This growth-adjusted measure is obviously connected to the risk-neutral measure commonly used in asset pricing (see e.g. Harrison and Kreps, 1979). Under our hatted measure, agents can evaluate utils from consumption streams while abstracting from aggregate risk; under a risk-neutral measure, agents can price payoffs by simply discounting at the risk-free rate.

shocks. The economies differ along two dimensions, namely whether agents can trade a full set of Arrow securities against aggregate shocks, and whether agents face idiosyncratic risk, in addition to aggregate risk. Idiosyncratic risk, if there is any, is never directly insurable.

| **Table 1: Summary of Four Economies** | | | |
|---|---|---|---|
| *Model* | *Aggregate Shocks* | *Idiosyncr. Shocks* | *Arrow Securities* |
| Bond | Yes | Yes | No |
| Arrow | Yes | Yes | Yes |
| Rep. Agent | Yes | No | Yes |
| Bewley | No | Yes | N/A |

We want to understand asset prices in the first model in the table. This model has idiosyncratic and aggregate risk, as well as incomplete markets. We call this the Bond model, since besides using stocks agents can only insure against idiosyncratic and aggregate shocks by trading a single bond. The standard representative agent complete markets model (model III in the table) lies on the other end of the spectrum. This is the model studied by Lucas (1978) and Breeden (1979). There is no idiosyncratic risk and a full menu of Arrow securities to for the representative agent to hedge against aggregate uncertainty. Though our analysis we will demonstrate that in the Bond model the standard representative agent Euler equation is satisfied:

$$E_t\left[\beta\left(\lambda_{t+1}\right)^{-\gamma}\left(R^s_{t+1}-R_t\right)\right]=0 \tag{7}$$

where $R^s_{t+1}$ is the return on the stock, $R_t$ is the return on the bond and $\lambda_{t+1}$ is the growth rate of the aggregate endowment.

Hence, the aggregate risk premium is identical to that of this representative agent model. Constantinides (1982) had already shown that, in the case of complete markets, even if agents are heterogeneous in wealth, there exists a representative agent who satisfies the Euler equation for excess returns (7) and also the Euler equation for bonds:

$$E_t\left[\beta\left(\lambda_{t+1}\right)^{-\gamma}R_t\right]=1. \tag{8}$$

The key to Constantinides' result is that markets are complete. We show that the first Euler equation in (7) survives market incompleteness and binding solvency constraints. The second one does not. To show this result, we employ a third model, the Arrow model (model II in the table). In the latter, households trade a full set of Arrow securities against aggregate risk, but not against idiosyncratic risk. The basis of our asset pricing findings is a result that shows that equilibria in the Bond and the Arrow model can be found by first determining equilibria in a model with *only* idiosyncratic risk (the Bewley model, model IV in the table) and then by simply scaling consumption allocations in that model by the stochastic aggregate endowment.

**Outline** We therefore start in section 3 by characterizing equilibria for the Bewley economy, a de-trended economy with a constant aggregate endowment in which agents trade a single discount bond and a stock.[8] This economy merely serves as a computational device. Next, we turn to the stochastically growing economy (with different market structures), the one whose asset pricing implications we are interested in.

# 3 The Bewley Model

In this economy the aggregate endowment is constant. The households face idiosyncratic shocks $y$ that follow a Markov process with transition probabilities $\varphi(y'|y)$. The household's preferences over consumption shares $\{\hat{c}(y^t)\}$ are as defined in equation (5), with the time discount factor $\hat{\beta}$ as defined in equation (28). The adjusted discount factor is $\hat{\beta}$ constant, because the aggregate shocks are i.i.d. (see Condition (2.3)).

[8]One of the two assets will be redundant for the households, so that this model is nothing else but a standard Bewley model studied in Bewley (1986), Huggett (1993) or Aiyagari (1994). The presence of both assets will make it easier to demonstrate our equivalence results with respect to the Bond and Arrow model later on.

## 3.1 Market Structure

Agents trade only a riskless discount bond and shares in a Lucas tree that yields safe dividends of $\alpha$ in every period. The price of the Lucas tree at time $t$ is denoted by $v_t$.[9] The riskless bond is in zero net supply. Each household is indexed by an initial condition $(\theta_0, y_0)$, where $\theta_0$ denotes its wealth (including period 0 labor income) at time 0.

The household chooses consumption $\{\hat{c}_t(\theta_0, y^t)\}$, bond positions $\{\hat{a}_t(\theta_0, y^t)\}$ and share holdings $\{\hat{\sigma}_t(\theta_0, y^t)\}$ to maximize its normalized expected utility $\hat{U}(\hat{c})(s^0)$, subject to a standard budget constraint:[10]

$$\hat{c}_t(y^t) + \frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t)\hat{v}_t = \eta(y_t) + \hat{a}_{t-1}(y^{t-1}) + \hat{\sigma}_{t-1}(y^{t-1})(\hat{v}_t + \alpha)$$

Finally, each household faces one of two types of borrowing constraints. The first one restricts household wealth at the end of the current period. The second one restricts household wealth at the beginning of the next period:[11]

$$\begin{aligned}
\frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t)\hat{v}_t \quad &\geq \quad \hat{K}_t(y^t) \text{ for all } y^t \\
\hat{a}_t(y^t) + \hat{\sigma}_t(y^t)(\hat{v}_{t+1} + \alpha) \quad &\geq \quad \hat{M}_t(y^t) \text{ for all } y^t
\end{aligned}$$

## 3.2 Equilibrium in the Bewley Model

The definition of equilibrium in this model is standard.

**Definition 3.1.** *For an initial distribution $\Theta_0$ over $(\theta_0, y_0)$, a competitive equilibrium for the Bewley model consists of trading strategies $\{\hat{a}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$, consumption allocations $\{\hat{c}_t(\theta_0, y^t)\}$, interest rates and share prices $\{\hat{R}_t, \hat{v}_t\}$ such that*

1. *Given prices, allocations solve the household maximization problem*

[9] The price of the tree is nonstochastic due to the absence of aggregate risk.

[10] We suppress dependence on $\theta_0$ for simplicity whenever there is no room for confusion.

[11] For the Bewley model this distinction is redundant, but it will become meaningful in our models with stochastically growing endowment.

*2. The goods markets and asset markets clear in all periods t*

$$\begin{aligned}\int \sum_{y^t} \varphi(y^t|y_0)\hat{c}_t(\theta_0, y^t)d\Theta_0 &= 1 \\ \int \sum_{y^t} \varphi(y^t|y_0)\hat{a}_t(\theta_0, y^t)d\Theta_0 &= 0 \\ \int \sum_{y^t} \varphi(y^t|y_0)\hat{\sigma}_t(\theta_0, y^t)d\Theta_0 &= 1\end{aligned}$$

Without aggregate risk, the risk-free one period bond and the risk-free stock are perfect substitutes for households, so that the absence of arbitrage implies the following restriction on equilibrium stock prices and interest rates:

$$\hat{R}_t = \frac{\hat{v}_{t+1} + \alpha}{\hat{v}_t}$$

In addition, household portfolios are indeterminate, and without loss of generality we focus on the case in which households simply trade the stock, not the bond: $\hat{a}_t(\theta_0, y^t) \equiv 0$.[12]

A stationary Bewley equilibrium is a constant interest rate $\hat{R}$, a share price $\hat{v}$, optimal household allocations and a time-invariant measure $\Phi$ over income shocks and financial wealth. We define a stationary recursive competitive equilibrium in section (A.1) of the appendix, and also outline a (straightforward) algorithm for computing it there.

# 4 Arrow Economy

We now turn to our main object of interest, the stochastically growing economy. We start off with the Arrow market structure in which households can trade the stock and a complete menu of contingent claims on aggregate shocks. Idiosyncratic shocks are still uninsurable. We demonstrate in this section that the allocations we compute for the stationary Bewley economy can be mapped into equilibrium allocations and prices in the

[12] Alternatively, we could have let agents simply trade in the bond and adjust the net supply of bonds to account for the positive capital income $\alpha$ in the aggregate. We only introduce both assets into the Bewley economy to make the map into allocations in our growing Arrow and Bond models simpler.

stochastically growing Arrow economy.

## 4.1 Market Structure and Equilibrium

Let $a_t(s^t, z_{t+1})$ denote the quantity purchased of a security that pays off one unit of the consumption good if aggregate shock in the next period is $z_{t+1}$, irrespective of the idiosyncratic shock $y_{t+1}$. Its price today is given by $q_t(z^t, z_{t+1})$. In addition, households trade shares in the Lucas tree. We use $\sigma_t(s^t)$ to denote the number of shares a household with history $s^t = (y^t, z^t)$ purchases today and we let $v_t(z^t)$ denote the price of one share.

An agent starting period $t$ with initial wealth $\theta_t(s^t)$ buys consumption commodities in the spot market and trades securities subject to the usual budget constraint:

$$c_t(s^t) + \sum_{z_{t+1}} a_t(s^t, z_{t+1}) q_t(z^t, z_{t+1}) + \sigma_t(s^t) v_t(z^t) \leq \theta_t(s^t). \tag{9}$$

If next period's state is $s^{t+1} = (s^t, y_{t+1}, z_{t+1})$, her wealth is given by her labor income, the payoff from the contingent claim purchased in the previous period as well as the value of her position on the stock, including dividends:

$$\begin{aligned} \theta_{t+1}(s^{t+1}) &= \underbrace{\eta(y_{t+1}, z_{t+1}) e_{t+1}(z_{t+1})}_{\textit{labor income}} + \underbrace{a_t(s^t, z_{t+1})}_{\textit{contingent payoff}} \\ &\quad + \underbrace{\sigma_t(s^t) \left[ v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z_{t+1}) \right]}_{\textit{value of shares in Lucas tree}} \end{aligned} \tag{10}$$

In addition to the budget constraints, the households' trading strategies are subject to solvency constraints of one of two types. The first type of constraint imposes a lower bound on the value of the asset portfolio at the end of the period today,

$$\sum_{z_{t+1}} a_t(s^t, z_{t+1}) q_t(z^t, z_{t+1}) + \sigma_t(s^t) v_t(z^t) \geq K_t(s^t), \tag{11}$$

while the second type imposes state-by-state lower bounds on net wealth tomorrow,

$$a_t(s^t, z_{t+1}) + \sigma_t(s^t)\left[v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z_{t+1})\right] \geq M_t(s^t, z_{t+1}) \text{ for all } z_{t+1}. \quad (12)$$

The definition of an equilibrium in this economy is standard. Each household is assigned a label that consists of its initial financial wealth $\theta_0$ and its initial state $s_0 = (y_0, z_0)$. A household of type $(\theta_0, s_0)$ chooses consumption allocations $\{c_t(\theta_0, s^t)\}$, trading strategies for Arrow securities $\{a_t(\theta_0, s^t, z_{t+1})\}$ and shares $\{\sigma_t(\theta_0, s^t)\}$ to maximize her expected utility (1), subject to the budget constraints (9) and subject to solvency constraints (11) or (12). We are ready to define an Arrow equilibrium.[13]

**Definition 4.1.** *For initial aggregate state $z_0$ and distribution $\Theta_0$ over $(\theta_0, y_0)$, a competitive equilibrium for the Arrow economy consists of household allocations $\{a_t(\theta_0, s^t, z_{t+1})\}$, $\{\sigma_t(\theta_0, s^t)\}$, $\{c_t(\theta_0, s^t)\}$ and prices $\{q_t(z^t, z_{t+1})\}$, $\{v_t(z^t)\}$ such that*

1. *Given prices, household allocations solve the household maximization problem*
2. *The goods market clears for all $z^t$,*

$$\int \sum_{y^t} \frac{\pi(y^t, z^t | y_0, z_0)}{\pi(z^t | z_0)} c_t(\theta_0, s^t) d\Theta_0 = e_t(z^t)$$

3. *The asset markets clear for all $z^t$*

$$\begin{aligned} \int \sum_{y^t} \frac{\pi(y^t, z^t | y_0, z_0)}{\pi(z^t | z_0)} \sigma_t(\theta_0, s^t) d\Theta_0 &= 1 \\ \int \sum_{y^t} \frac{\pi(y^t, z^t | y_0, z_0)}{\pi(z^t | z_0)} a_t(\theta_0, s^t, z_{t+1}) d\Theta_0 &= 0 \textit{ for all } z_{t+1} \in Z \end{aligned}$$

We now impose the restrictions on the solvency constraints that make them proportional to the aggregate endowment in the economy:

[13]Several elements are worth commenting on. First, since there exists a full set of Arrow securities spanning the aggregate uncertainty, the stock is a redundant asset and we could have formulated the household problem and thus the equilibrium definition without explicitly mentioning the stock. Incorporating the stock explicitly, though, will make our asset pricing and equivalence results to follow clearer. Second, by Walras' law the goods market clearing condition (or one of the asset market clearing conditions) is redundant.

**Condition 4.1.** *We assume the borrowing constraints only depend on the aggregate history through the level of the aggregate endowment. That is, we assume*

$$K_t(y^t, z^t) = \hat{K}_t(y^t)e_t(z^t)$$

*and*

$$M_t(y^t, z^t, z_{t+1}) = \hat{M}_t(y^t)e_{t+1}(z^{t+1}).$$

If the constraints did not have this feature in a stochastically growing economy, the constraints would become more or less binding as the economy grows, clearly not a desirable feature[14].

Next, we deflate allocations by the aggregate endowment and we demonstrate that the resulting equilibrium allocations are the same as the allocations in a Bewley equilibrium.

## 4.2 Equilibrium in the De-trended Arrow Model

Households rank consumption shares $\{\hat{c}_t\}$ in exactly the same way as original consumption streams $\{c_t\}$, so we can solve for an equilibrium in the detrended economy. Dividing the budget constraint (9) by $e_t(z^t)$, and using equation (10) yields the deflated budget constraint:

$$\begin{aligned} & \hat{c}_t(s^t) + \sum_{z_{t+1}} \hat{a}_t(s^t, z_{t+1})\hat{q}_t(z^t, z_{t+1}) + \sigma_t(s^t)\hat{v}_t(z^t) \\ \leq \quad & \eta(y_t) + \hat{a}_{t-1}(s^{t-1}, z_t) + \sigma_{t-1}(s^{t-1})\left[\hat{v}_t(z^t) + \alpha\right], \end{aligned} \tag{13}$$

where we have defined the deflated Arrow positions $\hat{a}_t(s^t, z_{t+1}) = \frac{a_t(s^t, z_{t+1})}{e_{t+1}(z^{t+1})}$ and prices $\hat{q}_t(z^t, z_{t+1}) = q_t(z^t, z_{t+1})\lambda(z_{t+1})$ as well as deflated stock prices $\hat{v}_t(z^t) = \frac{v_t(z^t)}{e_t(z^t)}$. Similarly,

[14]It is easy to show that solvency constraints that are not too tight in the sense of Alvarez and Jermann (2000) satisfy this conditionIn the incomplete markets literature, the borrowing constraints usually have this feature (see e.g. Heaton and Lucas (1996)). If not, one cannot compute a stationary equilibrium. It is more common in this literature to impose growing short-sale constraints on stocks and bonds separately instead of total financial wealth, but this is done mostly for computational reasons, to bound the state space. In fact, the bounds should be directly on total financial wealth instead, if the solvency constraints are to prevent default (see e.g. Zhang (1996) and Alvarez and Jermann (2000)). .

deflating the solvency constraints (11) and (12) using condition (4.1 ) yields:

$$\begin{aligned}\sum_{z_{t+1}} \hat{a}_t(s^t, z_{t+1})\hat{q}_t(z^t, z_{t+1}) + \sigma_t(s^t)\hat{v}_t(z^t) &\geq \hat{K}_t(y^t) &(14)\\ \hat{a}_t(s^t, z_{t+1}) + \sigma_t(s^t)\left[\hat{v}_{t+1}(z^{t+1}) + \alpha\right] &\geq \hat{M}_t(y^t) \text{ for all } z_{t+1}. &(15)\end{aligned}$$

Finally, the goods market clearing condition is given by[15]:

$$\int \sum_{y^t} \pi(y^t|y_0)\hat{c}_t(\theta_0, s^t)d\Theta_0 = 1. \tag{16}$$

The asset market clearing conditions are exactly the same as before. In the stationary economy, the household maximizes $\hat{U}(\hat{c})(s_0)$ by choosing consumption, Arrow securities and shares of the Lucas tree, subject to the budget constraint (13) and the solvency constraint (14) or (15) in each node $s^t$. The definition of a competitive equilibrium in the de-trended Arrow economy is straightforward.

**Definition 4.2.** *For initial aggregate state $z_0$ and distribution $\Theta_0$ over $(\theta_0, y_0)$, a competitive equilibrium for the de-trended Arrow economy consists of trading strategies $\{\hat{a}_t(\theta_0, s^t, z_{t+1})\}$, $\{\hat{\sigma}_t(\theta_0, s^t)\}$, $\{\hat{c}_t(\theta_0, s^t)\}$ and prices $\{\hat{q}_t(z^t, z_{t+1})\}$, $\{\hat{v}_t(z^t)\}$ such that*

1. *Given prices, allocations solve the household maximization problem*
2. *The goods market clears, that is, equation* (16) *holds for all $z^t$.*
3. *The asset markets clear*

$$\begin{aligned}\int \sum_{y^t} \varphi(y^t|y_0)\hat{\sigma}_t(\theta_0, s^t)d\Theta_0 &= 1\\ \int \sum_{y^t} \varphi(y^t|y_0)\hat{a}_t(\theta_0, s^t, z_{t+1})d\Theta_0 &= 0 \text{ for all } z_{t+1} \in Z\end{aligned}$$

The first order conditions and complementary slackness conditions, together with the appropriate transversality condition, are listed in the appendix in section (A.2). These are necessary and sufficient conditions for optimality on the household side. Now, we

[15]The conditional probabilities simplify due to condition (2.2).

are ready to establish the equivalence between equilibria in the Bewley model and in the Arrow model.

## 4.3 Equivalence Results

Our first result states that if the consumption shares in the de-trended economy do not depend on the aggregate history $z^t$, then it follows that the interest rates in this economy are deterministic.

**Proposition 4.1.** *In the de-trended Arrow economy, if there exists a competitive equilibrium with equilibrium consumption allocations* $\{\hat{c}_t(\theta_0, y^t)\}$ *, then there is a deterministic interest rate process* $\left\{\hat{R}_t^A\right\}$ *and equilibrium prices* $\{\hat{q}_t(z^t, z_{t+1})\}$ *, that satisfy:*

$$\hat{q}_t(z^t, z_{t+1}) = \frac{\hat{\phi}(z_{t+1})}{\hat{R}_t^A} \tag{17}$$

The interest rate in the deflated economy can depend on time, but not on the aggregate state of the economy, simply because nothing else in the Euler equation does:

$$1 \;=\; \hat{\beta}\hat{R}_t^A \underbrace{\sum_{z_{t+1}} \hat{\phi}(z_{t+1})}_{drops\ out} \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \tag{18, 19}$$

The aggregate shocks $z$ do not affect the transformed discount factor $\hat{\beta}$, because of the i.i.d assumption, and the transition probabilities for the aggregate shocks disappear from the Euler equation altogether, because they are independent from the $y$ shocks.[16] By summing over aggregate states tomorrow on both sides of equation (17), we can compute the interest rate:

$$\hat{R}_t^A = \frac{1}{\sum_{z_{t+1}} \hat{q}_t(z^t, z_{t+1})}.$$

This interest rate in the Arrow model is also the equilibrium interest rate in the Bewley

[16]Finally, the dependence of $\hat{R}_t^A$ on time $t$ is not surprising since, for an arbitrary initial distribution of assets $\Theta_0$, we cannot expect the equilibrium to be stationary. In the same way we expect that $\hat{v}_t(z^t)$ is only a function of $t$ as well, but not of $z^t$.

model of section (3). Once we have found interest rates for the de-trended economy, $\hat{R}_t^A$, we can back out the implied interest rate for the original growing Arrow economy.

**Corollary 4.1.** *Risk-free interest rates in the original Arrow model are given by*

$$R_t^A = \hat{R}_t^A * \frac{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{-\gamma}} \tag{20}$$

In the absence of aggregate risk (constant $\lambda$), the risk-free rates in the original and deflated economy are related by $R_t^A = \hat{R}_t^A \lambda_{t+1}$where $\lambda_{t+1}$ is the gross growth rate of endowment between period $t$ and $t+1$.

**Trading Strategies** The next question is which trading strategies for Arrow securities would support consumption allocations in the de-trended Arrow model that do not depend on aggregate shocks as well, that is $\hat{c}_t(\theta_0, s^t) = \hat{c}_t(\theta_0, y^t)$. The next corollary provides the answer.

**Corollary 4.2.** *This equilibrium is supported by aggregate history-invariant Arrow security trades, that is $\hat{a}_t(\theta_0, s^t, z_{t+1}) = \hat{a}_t(\theta_0, y^t)$ for all $z^{t+1}$.*

This is not surprising, because the consumption shares do not depend on the aggregate history. This brings us to the main result, the mapping between the stochastically growing Arrow economy and the Bewley economy.

**Theorem 4.1.** *An equilibrium of the Bewley model $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$ and $\{\hat{R}_t, \hat{v}_t\}$ can be made into an equilibrium for the Arrow economy with growth, $\{a_t(\theta_0, s^t, z_{t+1})\}$, $\{\sigma_t(\theta_0, s^t)\}$, $\{c_t(\theta_0, s^t)\}$ and $\{q_t(z^t, z_{t+1})\}$, $\{v_t(z^t)\}$, with*

$$\begin{aligned}
c_t(\theta_0, s^t) &= \hat{c}_t(\theta_0, y^t) e_t(z^t) \\
\sigma_t(\theta_0, s^t) &= \hat{\sigma}_t(\theta_0, y^t) \\
a_t(\theta_0, s^t, z_{t+1}) &= \hat{a}_t(\theta_0, y^t) e_{t+1}(z^{t+1}) \\
v_t(z^t) &= \hat{v}_t e_t(z^t) \\
q_t(z^t, z_{t+1}) &= \frac{1}{\hat{R}_t} * \frac{\phi(z_{t+1})\lambda(z_{t+1})^{-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1})\lambda(z_{t+1})^{1-\gamma}}
\end{aligned}$$

We can solve for an equilibrium in the Bewley economy (section 3), including the risk free interest rate $\hat{R}_t$, and we can deduce the equilibrium allocations and prices for the Arrow economy from those in the Bewley economy. The key to this result is that households in the Bewley model face exactly the same Euler equations as the households in the de-trended version of the Arrow economy (as in equation (18)).

This result has several important implications. First, the existence and uniqueness proofs in the literature for equilibria in the Bewley model directly carry over to the stochastically growing economy[17]. In addition, the moments of the wealth distribution vary over time but proportionally to the aggregate endowment: e.g. the ratio of the mean to the standard deviation of the wealth distribution is constant if we start the corresponding Bewley economy off from its invariant wealth distribution.

Second, without loss of generality, we can focus on equilibria in the Arrow model in which Arrow securities are not traded, because, in the Bewley model, the bonds are a redundant asset. Therefore it follows from our equivalence result that we can support the equilibrium allocation in the Arrow model with the following Arrow security trades

$$\hat{a}_t(\theta_0, s^t, z_{t+1}) = \hat{a}_t(\theta_0, y^t) = 0 \text{ for all } z^{t+1}.$$

We will use this equivalence result with the Bewley model to show that asset prices in the Arrow economy are identical to those in the representative agent economy, except for the lower interest rate (and a higher price/dividend ratio for stocks).

Finally, we conclude this section by showing that the previous equivalence result does not depend on the absence of solvency constraints.

**Corollary 4.3.** *If $\hat{a}_t(\theta_0, y^t)$ and $\hat{\sigma}_t(\theta_0, y^t)$ satisfy the constraints in the de-trended Arrow model (equivalently, in the Bewley model), then $\sigma_t(\theta_0, s^t) = \hat{\sigma}_t(\theta_0, y^t)$ and $a_t(\theta_0, s^t, z_{t+1}) = \hat{a}_t(\theta_0, y^t)e_{t+1}(z^{t+1})$ also satisfy the solvency constraints in the stochastically growing Arrow model.*

Note that this is true regardless of the tightness of the solvency constraints. Even if the borrowing constraints bind, nothing in the Euler equation of the detrended Arrow

[17] To prove existence, Aiyagari (1994) assumes i.i.d. idiosyncratic shocks but Huggett (1993) does not.

economy depends on $z$:

$$1 = \hat{\beta}\hat{R}_t^A \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} + \mu_t(y^t) + \hat{R}_t^A \kappa_t(y^t) \ \forall z_{t+1},$$

where $\mu_t(y^t)$ and $\kappa_t(y^t)$ are the multipliers on the solvency constraints. These multipliers are obtained from the computation of the Bewley equilibrium, and hence do not depend on the aggregate history either.

**Continuum** Importantly, our results do not go through in economies with a finite number of agents. In that case, asset prices depend on the cross-product of individual histories $y^t$ in addition to the history of aggregate shocks $z^t$.

# 5 *IC* Economy

We now turn our attention to the model whose asset pricing implications we are really interested in, namely the model with a stock and a single uncontingent bond. This section establishes the equivalence of equilibria in the *IC* model and the Bewley model by showing that optimality conditions in the detrended Arrow and IC economy are identical. In addition, we show that agents do not even trade bonds in the benchmark case with i.i.d. aggregate endowment growth shocks.

## 5.1 Market Structure and Equilibrium

In the bond economy, agents only trade a one-period discount bond and a stock. An agent who starts period $t$ with initial wealth $\theta_t(s^t)$ buys consumption commodities in the spot market and trades a one-period bond and the stock, subject to budget constraint:

$$c_t(s^t) + \frac{b_t(s^t)}{R_t(z^t)} + \sigma_t(s^t) v_t(z^t) \leq \theta_t(s^t). \tag{21}$$

$b_t(s^t)$ denotes the amount of bonds purchased and $R_t(z^t)$ is the gross interest rate from period $t$ to $t+1$. Wealth tomorrow in state $s^{t+1} = (s^t, y_{t+1}, z_{t+1})$ is given by

$$\begin{aligned}\theta_{t+1}(s^{t+1}) &= \underbrace{\eta(y_{t+1})e_{t+1}(z_{t+1})}_{\textit{labor income}} + \underbrace{b_t(s^t)}_{\textit{bond payoff}} \\ &\quad + \underbrace{\sigma_t(s^t)\left[v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z_{t+1})\right]}_{\textit{value of shares in Lucas tree}}. \end{aligned} \tag{22}$$

As was the case in the Arrow model, short-sales of the bond and the stock are constrained by a lower bound on the value of the portfolio today,

$$\frac{b_t(s^t)}{R_t(z^t)} + \sigma_t(s^t)v_t(z^t) \geq K_t(s^t), \tag{23}$$

or a state-by-state constraint on the value of the portfolio tomorrow,

$$b_t(s^t) + \sigma_t(s^t)\left[v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z_{t+1})\right] \geq M_t(s^t, z_{t+1}) \text{ for all } z_{t+1}. \tag{24}$$

Since $b_t(s^t)$ and $\sigma_t(s^t)$ are chosen before $z_{t+1}$ is realized, at most one of the constraints (24) will be binding at a given time. The definition of an equilibrium for the IC model follows directly:

**Definition 5.1.** *For initial aggregate state $z_0$ and distribution $\Theta_0$ over $(\theta_0, y_0)$, a competitive equilibrium for the* IC *economy consists of trading strategies $\{b_t(\theta_0, s^t)\}$, $\{c_t(\theta_0, s^t)\}$, $\{\sigma_t(\theta_0, s^t)\}$ and interest rates $\{R_t(z^t)\}$ and share prices $\{v_t(z^t)\}$ such that*

1. *Given prices, allocations solve the household maximization problem*

2. *The goods market clears*

$$\int \sum_{y^t} \frac{\pi(y^t, z^t | y_0, z_0)}{\pi(z^t | z_0)} c_t(\theta_0, s^t) d\Theta_0 = e_t(z^t)$$

*3. The asset markets clear for all $z^t$*

$$\int \sum_{y^t} \frac{\pi(y^t, z^t | y_0, z_0)}{\pi(z^t | z_0)} \sigma_t(\theta_0, s^t) d\Theta_0 = 1$$

$$\int \sum_{y^t} \frac{\pi(y^t, z^t | y_0, z_0)}{\pi(z^t | z_0)} b_t(\theta_0, s^t) d\Theta_0 = 0.$$

We now show that the equilibria in the Arrow and the *IC* model coincide. As a corollary, it follows that the asset pricing implications of both models are identical. In order to do so we first transform the growing economy into a stationary, de-trended economy.

## 5.2 Equilibrium in the De-trended *IC* Model

Dividing the budget constraint (21) by $e_t(z^t)$ we obtain, using (22),

$$\hat{c}_t(s^t) + \frac{\hat{b}_t(s^t)}{R_t(z^t)} + \sigma_t(s^t)\hat{v}_t(z^t) \leq \eta(y_t) + \frac{\hat{b}_{t-1}(s^{t-1})}{\lambda(z_t)} + \sigma_{t-1}(s^{t-1}) \left[\hat{v}_t(z^t) + \alpha\right],$$

where we define the deflated bond position $\hat{b}_t(s^t) = \frac{b_t(s^t)}{e_t(z^t)}$. Using condition (4.1), the solvency constraints in the de-trended economy are simply:

$$\frac{\hat{b}_t(s^t)}{R_t(z^t)} + \sigma_t(s^t)\hat{v}_t(z^t) \geq \hat{K}_t(y^t), \text{ or}$$

$$\frac{\hat{b}_t(s^t)}{\lambda(z_{t+1})} + \sigma_t(s^t) \left[\hat{v}_{t+1}(z^{t+1}) + \alpha\right] \geq \hat{M}_t(y^t) \text{ for all } z_{t+1}.$$

The definition of equilibrium in the de-trended IC model is straightforward and hence omitted[18]. We now show that equilibrium consumption allocations in the de-trended *IC* model coincide with those of the Arrow model. For simplicity we first abstract from binding borrowing constraints and then extend our results to that case later on.

[18] We list the first order conditions for household optimality and the transversality conditions in section (A.3) of the appendix.

### 5.3 Equivalence Results

The Euler equation for bonds in the detrended *IC* model for an unconstrained household is given by:

$$1 = \hat{\beta}\hat{R}_t^{IC}\sum_{y_{t+1}}\varphi(y_{t+1}|y_t)\frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))},$$

It is identical to the detrended Arrow economy' Euler equation for bonds, as long as the equilibrium risk-free rate $\hat{R}_t^{IC}$ in the bond economy is the same as that in the detrended Arrow economy. This explains the following proposition.

**Proposition 5.1.** *If borrowing constraints in both models never bind, the Euler equations in the de-trended* IC *model are identical to the Euler equations in the de-trended Arrow model. The equilibrium risk-free rates in both models coincide* $\left\{R_t^{IC}(z^t) = R_t^A(z^t)\right\}$.

**Trading Strategies** We now investigate the trading strategies that support the equilibrium consumption allocation in the IC model. Net wealth in the de-trended version of the IC model can only depend on the idiosyncratic shock history, as does de-trended consumption $\hat{c}_t(y^t)$

**Lemma 5.1.** *In the de-trended bond economy, net wealth only depends on* $y^t$.

This implies that in the growth economy with aggregate uncertainty, wealth at the beginning of the period,

$$b_{t-1}(s^{t-1}) + \sigma_{t-1}(s^{t-1})\left[v_t(z^t) + \alpha e_t(z^t)\right] \tag{25}$$

has to be proportional to $e_t(z^t)$. From inspecting (25) it is obvious that this is only possible if $v_t(z^t)$ is proportional to aggregate endowment $e_t(z^t)$ and if $b_{t-1}(s^{t-1}) = 0$! In the appendix we therefore show the following proposition.

**Proposition 5.2.** *In the* IC *model, the bond market is inoperative in that* $b_{t-1}(s^{t-1}) = \hat{b}_{t-1}(s^{t-1}) = 0$ *for all* $s^{t-1}$.

All the risk sharing is done by trading stocks, because agents want to keep their net wealth proportional to the level of the aggregate endowment. The stock position of households in the IC model is simply given by the net wealth position in the Arrow economy $\hat{\theta}_t(y^{t-1})$ deflated by the cum dividend price-dividend ratio,

$$\sigma^{IC}_{t-1}(y^{t-1}) = \frac{\hat{\theta}_{t-1}(y^{t-1})}{[\hat{v}_t + \alpha]} = \frac{\hat{a}_{t-1}(y^{t-1})}{[\hat{v}_t + \alpha]} + \sigma^{A}_{t-1}(y^{t-1}),$$

where $\hat{\theta}_{t-1}(y^{t-1})$, as before, denotes the total net wealth position in the deflated Arrow economy at the start of $t$:[19]

$$\hat{a}_{t-1}(y^{t-1}) + \sigma^{A}_{t-1}(y^{t-1})\,[\hat{v}_t + \alpha] = \hat{\theta}_{t-1}(y^{t-1}).$$

Also, it is important to note that the stock positions in the *IC* model are not necessarily equal to the stock positions in the Arrow economy. A household that is long in bonds in the Arrow economy simply has a higher stock position in the *IC* economy, while a household that is short in the bond market in the Arrow economy holds less stocks in the *IC* economy. The net borrowing and lending in the IC model is done in the stock market, not in the bond market[20]. Consequently, we have the following result:[21].

**Theorem 5.1.** *An equilibrium of a stationary Bewley economy* $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$ *and* $\{\hat{R}_t, \hat{v}_t\}$ *can be made into an equilibrium for the* IC *economy with growth,* $\{b_t(\theta_0, s^t)\}$ , $\{c_t(\theta_0, s^t)\}$ , $\{\sigma_t(\theta_0, s^t)\}$ *and* $\{R_t(z^t)\}$ *and* $\{v_t(z^t)\}$ *where*

$$\begin{aligned}
c_t(\theta_0, s^t) &= \hat{c}_t(\theta_0, y^t)e_t(z^t) \\
\sigma^{IC}_t(\theta_0, s^t) &= \hat{\sigma}^{IC}_t(\theta_0, y^t) = \frac{\hat{a}_t(\theta_0, y^t)}{[\hat{v}_{t+1} + \alpha]} + \sigma^{A}_t(\theta_0, y^t) \\
v_t(z^t) &= \hat{v}_t e_t(z^t) \\
R_t(z^t) &= \hat{R}_t(z^t) * \frac{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{-\gamma}}
\end{aligned}$$

*and bond holdings given by* $b_t(\theta_0, s^t) = 0$.

[19]Remember that in the deflated Arrow model households find it optimal to only trade state-uncontingent assets, see Corollary 4.2

[20]This explains why net wealth at the start of the period is proportional to $e_t(z^t)$ : all the households are long or short in the stock.

[21]Even in the Arrow model, trade in bonds is not needed to implement the equilibrium consumption allocation, and thus we could consider only equilibria in the Arrow model with $\hat{a}_t(y^t) = 0$. .

Absent any binding borrowing constraints, we can solve for equilibria in a standard Bewley model economy and then map this equilibrium into one for both the Arrow economy and the *IC* economy *with* aggregate uncertainty. The risk-free interest rate and the price of the Lucas tree coincide in the stochastic Arrow and *IC* economies. Finally, without loss of generality, we can restrict attention to equilibria in which bonds are not traded; consequently transaction costs in the bond market would not change our results. Transaction costs in the stock market of course would (see section (7)).

In the *IC* economy, households do not have a motive for trading bonds, unless there are short-sale constraints on stocks. We do not deal with this case. In addition, the no-trade result depends critically on the i.i.d assumption for aggregate shocks, as we will show in section (7). If the aggregate shocks are not i.i.d, agents want to hedge against the interest rate shocks -these show up as aggregate taste shocks in the de-trended economy.

**Borrowing constraints** The next corollary establishes that our previous results survive the introduction of borrowing constraints.

**Corollary 5.1.** *The asset allocation for the IC model from the previous theorem satisfies the borrowing constraints* (23) *or* (24) *if and only if the associated asset allocation* $a_t(\theta_0, s^t, z_{t+1})$ *and* $\sigma_t^A(\theta_0, s^t)$ *satisfies the borrowing constraints in the stochastically growing Arrow economy*

This result follows from the fact that the new wealth at the end of the period and the beginning of the next period coincides in equilibria of both models. As a consequence, we have established that even with borrowing constraints, the equilibria in the Arrow and *IC* model coincide. We now compare these asset prices to those emerging from the standard representative agent model.

Finally, we note that these results extend to the case of recursive utility as well.

# 6 Asset Pricing Implications

This section shows that the multiplicative risk premium on a claim to aggregate consumption in the *IC* economy -and the Arrow economy- equals the risk premium in the representative agent economy. Idiosyncratic income risk only lowers the risk-free rate.

## 6.1 Consumption-CAPM

Our benchmark is the representative agent model. The representative agent owns a claim to the aggregate 'labor' income stream $\{(1-\alpha)e_t(z^t)\}$ and she can trade a stock (a claim to the dividends $\alpha e_t(z^t)$ of the Lucas tree), a bond and a complete set of Arrow securities[22].

First, we show that the Breeden-Lucas Consumption-CAPM also prices excess returns on the stock in the IC model and the Arrow model. $R^s$ denotes the return on a claim to aggregate consumption.

**Lemma 6.1.** *The Consumption-CAPM prices excess returns in the Arrow economy and the* IC *Economy:*

$$E_t\left[\left(R^s_{t+1}-R_t\right)\beta\left(\lambda_{t+1}\right)^{-\gamma}\right]=0$$

This follows directly from the Euler equation in (18). In spite of the market incompleteness, one can estimate the coefficient of risk aversion directly from aggregate consumption data and the excess return on stocks, as in Hansen and Singleton (1982)[23].

## 6.2 Risk Premia

Not surprisingly, the equilibrium risk premium is identical to the one in the representative agent economy[24]. Since the risk-free rate is higher than in the Arrow and *IC* model, the price of the stock is correspondingly lower. However, the multiplicative risk premium is the same in all three models.

[22] see section (A.4) in the Appendix for a complete description.
[23] In the bond economy, we can only use return data for a claim to aggregate consumption.
[24] This does not immediately follow from Lemma 6.1.

The stochastic discount factors that prices stochastic payoffs in the representative agent economy and the Arrow economy only differ by a non-random multiplicative term, equal to the ratio of (growth-deflated) risk-free interest rates in the two models. We use the superscript $RE$ to denote the Representative Agent economy.

**Proposition 6.1.** *In the Arrow economy, there is a unique SDF given by*

$$m_{t+1}^{A} = m_{t+1}^{RE}\kappa_t$$

*with a non-random multiplicative term given by:*

$$\kappa_t = \frac{\hat{R}_t^{RE}}{\hat{R}_t^{A}} \geq 1$$

Note that the term $\kappa_t$ is straightforward to compute since $\hat{R}_t^A$ equals the interest rate in the Bewley model discussed in section (3). What about the $IC$ economy? We have shown, using the equivalence result between the Arrow and the $IC$ model, that the Arrow economy's stochastic discount factor $m_{t+1}^A$ is also a valid stochastic discount factor in the $IC$ economy. So, our results below carry over to the $IC$ economy as well.

The proof that risk premia are unchanged between the representative agent model and the Arrow model now follows directly from the previous decomposition of the SDF[25].

Let $R_{t,j}\left[\{d_{t+k}\}\right]$ denote the $j$-period holding return on claim to $\{d_{t+k}\}$. Consequently $R_{t,1}[1]$ is the gross risk-free rate and $R_{t,1}[\alpha e_{t+k}]$ is the one-period holding return on a $k$-period strip of the aggregate endowment (a claim to $\alpha$ times the aggregate endowment $k$ periods from now). With this notation in place, we can state our main result.

**Theorem 6.1.** *The multiplicative risk premium in the Arrow economy equals that in the representative agent economy*

$$1 + \nu_t^A = 1 + \nu_t^{RE} = \frac{E_t R_{t,1}\left[\{e_{t+k}\}\right]}{R_{t,1}[1]}$$

Thus, the extent to which households smooth idiosyncratic income shocks (in the

[25]The proof strategy follows Alvarez and Jermann (2001) who derive a similar result in the context of a *complete markets* model populated by two agents that face endogenous solvency constraints.

Arrow model or in the IC model) amount has absolutely no effect on the size of the risk premia; it merely lowers the risk-free rate. Exactly the same result applies to the IC model as well[26]

# 7 Robustness and Extensions of the Main Results

In this section, we investigate how robust our results are to two important assumptions we have maintained so far. First, we demonstrate that our assumption that the aggregate shocks are *i.i.d* over time, which means the growth rates of the aggregate endowment are *i.i.d* over time, is not crucial for our results. Second, we show that our main results go through even if idiosyncratic and aggregate shocks are not orthogonal to each other. More specifically, we can allow the Markov transition matrix for idiosyncratic shocks to depend on the aggregate state, as long as the variance of the idiosyncratic shocks themselves are not affected by the aggregate state of the world.

## 7.1 Non-iid Aggregate Shocks

When the aggregate shocks $z$ follow an arbitrary, finite state, Markov chain, the growth-adjusted time discount factor $\hat{\beta}(z)$ depends on the current aggregate state, and, as a result, the aggregate endowment shock acts as an aggregate taste shock in the deflated economy. This shock renders all households more or less impatient. Of course, households are not able to insure against this shock at all, since it affects all households in the same way. As a consequence, non-iid aggregate endowment shocks, acting as taste shocks in the deflated economy, only affect the price/dividend ratio and the interest rate, but not the risk premium. However, in this case, there is trade in the bond market.

**Stationary Bewley economy** In order to establish these results, we define an *ergodic* economy, as a computational device. Agents in this ergodic economy discount future

[26] Cochrane and Hansen (1992) had already established a similar aggregation result for the case in which households face market wealth constraints, but in a complete markets environment. We show this result survives even if households trade only a stock and a bond.

utility flows using the average time discount factor as

$$\widetilde{\beta} = \sum_{z'} \hat{\Pi}(z')\hat{\beta}(z')$$

where $\hat{\Pi}(z')$ is the invariant distribution over the aggregate shock. In order to construct equilibrium allocations, we first compute the equilibrium allocations and interest rates in the *ergodic* Bewley economy, exactly as described in section (3). In a second step, we make these allocations and interest rates into an equilibrium of the *actual* Arrow economy with time-varying discount factors by adjusting the risk-free interest rate in proportion to the taste shock $\hat{\beta}(z)$. We do not change the allocations, and we back out the implied state-contingent bond positions.

**Arrow Model** From section (4.3), we know that the Bewley allocations in the ergodic economy can be implemented without any trade in Arrow securities. We can now back out the Arrow prices for the deflated economy from the prices in the deflated Bewley economy.

**Proposition 7.1.** *In the detrended Arrow economy with non-iid aggregate shocks the equilibrium risk-free interest rates* $\left\{\hat{R}_t^A\right\}$ *are given by*

$$\hat{R}_t^A(z_t) = \widetilde{R}_t * \frac{\widetilde{\beta}}{\hat{\beta}(z_t)}$$

*where* $\widetilde{R}_t$ *is the interest rate in the ergodic Bewley model with constant time discount factor* $\widetilde{\beta}$*. Equilibrium Arrow prices* $\{\hat{q}_t(z^t, z_{t+1})\}$ *satisfy:*

$$\hat{q}_t(z^t, z_{t+1}) = \frac{\hat{\phi}(z_{t+1}|z_t)}{\hat{R}_t^A(z_t)}. \tag{26}$$

The interest rate in the deflated economy depends on the current aggregate state (and possibly on time) , but not on the entire history of aggregate shocks $z^t$. Thus, share prices

$\hat{v}_t(z_t)$ are only functions of $t$ as well as the current shock $z_t$, but not of the entire shock history $z^t$.[27]

**Trading Strategies** In this case, the trading strategies are not quite as simple anymore; the households do take contingent bond positions. Let $\{\hat{c}_t(\theta_0, y^t), \widetilde{a}_t(\theta_0, y^t) = 0, \hat{\sigma}_t(\theta_0, y^t)\}$ denote the equilibrium allocations and trading strategies from the constant discount factor economy, and let the state prices be defined recursively, as follows:

$$\hat{Q}_t(z^\tau | z_t) = \hat{q}_t(z_{\tau-1}, z_\tau)\hat{q}_t(z_{\tau-2}, z_{\tau-1}) \ldots \hat{q}_t(z_{t+1}, z_t)$$

**Proposition 7.2.** *In the economy with persistent aggregate shocks, the households do take contingent bond positions, for each $y^t, z_t$:*

$$\begin{aligned}\frac{\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)}{\lambda(z_t)} &= \hat{c}_t(y^t, \theta_0) - \eta(y_t) \\ &+ \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \hat{Q}_\tau(z^\tau | z_t) \sum_{\eta^\tau} (\hat{c}_\tau(y^\tau, \theta_0) - \eta(y_\tau)) \\ &- \sigma_{t-1}(y^{t-1}) \left[\hat{v}_t(z_t) + \alpha\right].\end{aligned}$$

These contingent bond positions (i) satisfy the budget constraint by construction and (ii) they are shown to clear the bond market, in the appendix. Note that these contingent bond positions allow the households to hedge against the interest rate shock. There is another way of writing these bond positions that emphasizes the interest rate hedging aspect more clearly.

[27] If we start the ergodic economy off from the right initial wealth distribution (see section (A.1) in the appendix), a stationary equilibrium obtains in the ergodic economy, in which the risk-free rate $\widetilde{R}_t = \widetilde{R}$ is constant over time. As a result, the interest rate in the actual deflated economy can be stated as $\hat{R}^A(z)$, simply because $\hat{\beta}(z)\hat{R}^A(z) = \widetilde{\beta}\widetilde{R}$ is constant as well. The equilibrium risk free rate simply increases when everybody in the economy becomes more impatient, and it decreases when everyone becomes more patient.

**Lemma 7.1.** *The bond positions hedge against the interest rate shocks:*

$$\begin{aligned}\frac{\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)}{\lambda(z_t)} &= \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \left(\hat{Q}_\tau(z^\tau|z_t) - \widetilde{Q}_\tau\right) \sum_{\eta^\tau} \left(\hat{c}_\tau(y^\tau, \theta_0) - \eta(y_\tau)\right) \\ &\quad -\sigma_{t-1}(y^{t-1})\alpha \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \left(\hat{Q}_\tau(z^\tau|z_t) - \widetilde{Q}_\tau\right)\end{aligned}$$

All else equal, the larger the shocks to the interest rates, the larger the bond positions the households take. In the appendix, we show that on average, these bond positions are still zero, as in the case of i.i.d. shocks. Our main result, theorem (4.1), still goes through.

**Theorem 7.1.** *An equilibrium of a stationary Bewley economy, populated by households with constant discount factor* $\widetilde{\beta}$, $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t) = 0, \hat{\sigma}_t(\theta_0, y^t)\}$ *and* $\{\hat{R}_t, \hat{v}_t\}$ *can be made into an equilibrium for the Arrow economy with growth,* $\{a_t(\theta_0, s^t, z_{t+1})\}$, $\{\sigma_t(\theta_0, s^t)\}$, $\{c_t(\theta_0, s^t)\}$ *and* $\{q_t(z^t, z_{t+1})\}$, $\{v_t(z^t)\}$, *with*

$$\begin{aligned}c_t(\theta_0, s^t) &= \hat{c}_t(\theta_0, y^t) \\ \sigma_t(\theta_0, s^t) &= \hat{\sigma}_t(\theta_0, y^t) \\ a_t(\theta_0, s^t, z_{t+1}) &= \hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)e_t(z^t) \text{ defined above} \\ v_t(z_t) &= \sum_{z_{t+1}} \frac{\hat{\phi}(z_{t+1}|z_t)}{\lambda(z_{t+1})} \left[\frac{v_{t+1}(z_{t+1}) + \alpha e_{t+1}(z_{t+1})}{\hat{R}_t^A(z_t)}\right] \\ \hat{R}_t^A(z_t) &= \frac{\widetilde{R}_t \widetilde{\beta}}{\hat{\beta}(z_t)} \\ q_t(z^t, z_{t+1}) &= \frac{1}{\hat{R}_t^A(z_t)} * \frac{\phi(z_{t+1}|z_t)\lambda(z_{t+1})^{-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma}}\end{aligned}$$

The proof of this result proceeds exactly as before; in addition, however, we have to check whether at the Bewley asset trading strategies and the state-contingent interest rates (and thus stock prices) the budget constraint is still satisfied, as stated in the proposition. However, the net wealth positions in the deflated Arrow economy will depend on the current aggregate state $z_t$.

**Risk Premia** Of course, this implies that our baseline irrelevance result for risk premia survives the introduction of non-i.i.d. aggregate shocks, provided that a complete menu of aggregate-state-contingent securities is traded. These aggregate taste shocks only affect interest rates and price/dividend ratios, not risk premia.

**Solvency Constraints** Sofar we have abstracted from solvency constraints. Remember that we originally defined the solvency constraints as $K_t(s^t) = \hat{K}_t(y^t)e_t(z^t)$ and $M_{t+1}(s^{t+1}) = \hat{M}_t(y^t)e_t(z^{t+1})$. The allocations computed in the ergodic economy using $\hat{K}_t(y^t)$ and $\hat{M}_t(y^t)$ as solvency constraints, satisfy a modified version of the solvency constraints $K_t(s^t)$ and $M_{t+1}(s^{t+1})$.

**Lemma 7.2.** *The allocations from the ergodic economy satisfy the modified solvency constraints:*

$$\begin{aligned} K_t^*(s^t) &= K_t(s^t) + \sum_{z_{t+1}} q_t(z^t, z_{t+1}) \begin{pmatrix} \widetilde{a}_t(s^t, z_{t+1}) \\ -a_t(s^t, z_{t+1}) \end{pmatrix} \\ M_{t+1}^*(s^{t+1}) &= M_{t+1}(s^{t+1}) + \widetilde{a}_t(s^t, z_{t+1}) - a_t(s^t, z_{t+1}) \end{aligned}$$

This means that our result for non-i.i.d. aggregate shocks is not quite robust to the introduction of binding solvency constraints: if the allocations satisfy the constraints in the ergodic Bewley economy, they satisfy these modified solvency constraints in the actual Arrow economy, but not the ones we originally specified, because of the aggregate-state-contingent bond positions. However, as is easy to verify using the result in Corollary (B.1) in the appendix, these modified solvency constraints coincide with the actual ones on average (averaged across $z$ shocks), and the violations are likely to be small for plausible calibrations of the aggregate endowment growth process.

***IC* Economy** For the *IC* model, the same equivalence result obviously no longer holds, because the market for state contingent Arrow securities is now operative in the Arrow economy. When there is predictability in aggregate consumption growth, households actually find it optimal to trade the state-contingent Arrow securities, but the market structure in the bond economy prevents them from doing so.

## 7.2 Interaction between Aggregate and Idiosyncratic Risk

Next, we relax the independence assumption. In the case of i.i.d. aggregate shocks, a version of our irrelevance result still goes through, even when there is interaction between the aggregate and the idiosyncratic risk. That is the focus of this section. To compute the equilibria, we use a version of the Bewley economy with a twisted transition probability matrix.

**Condition 7.1.** *Individual endowment shares $\eta(y_t, z_t)$ are functions of the current idiosyncratic state $y_t$ only, that is $\eta(y_t, z_t) = \eta(y_t)$. Also, transition probabilities of the shocks can be decomposed as*

$$\pi(z_{t+1}, y_{t+1}|z_t, y_t) = \varphi(y_{t+1}, z_{t+1}|y_t)\phi(z_{t+1}).$$

That is, individual endowment *shares* of the idiosyncratic shocks are independent of the aggregate state of the economy $z$, but the transition probabilities are not. We maintain our assumption that the aggregate shocks are independent over time:

$$\phi(z_{t+1}|z_t) = \phi(z_{t+1}).$$

For example, a larger fraction of households could draw low/high income shocks in the bad state of the world (see e.g. Krusell and Smith (1998)).

The growth-adjusted probability matrix $\widehat{\pi}$ is obtained, as before, by adjusting only the transition probabilities for the aggregate shock, $\phi$, but not the transition probabilities for the idiosyncratic shocks:

$$\begin{aligned} \widehat{\pi}(s_{t+1}|s_t) &= \varphi(y_{t+1}, z_{t+1}|y_t)\widehat{\phi}(z_{t+1}), \text{ where} \\ \widehat{\phi}(z_{t+1}) &= \frac{\phi(z_{t+1})\lambda(z_{t+1})^{1-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1})\lambda(z_{t+1})^{1-\gamma}}. \end{aligned} \tag{27}$$

Furthermore, the growth-adjusted discount factor only depends on the aggregate state $z_t$:

$$\widehat{\beta} = \beta \sum_{z_{t+1}} \phi(z_{t+1})\lambda(z_{t+1})^{1-\gamma} \tag{28}$$

### 7.2.1 Arrow Economy

We start in the Arrow economy. Let $\hat{R}_t^{Be}$ denote the equilibrium interest rate for the *Bewley economy with the unconditional twisted transition matrix:*

$$\widehat{\varphi}(y'|y) = \sum_{z'} \hat{\phi}(z')\varphi(y', z'|y)$$

and let $\{\hat{c}_{t+1}(y^t)\}$ denote the equilibrium allocations. We proceed in the same way as before, i.e. by conjecturing that the consumption shares in the detrended Arrow economy do not depend on the aggregate histories. Given our conjecture, we obtain a simple expression for the state prices. And, then we check that the Euler equations of the households are satisfied.

As before, the state prices in the Bewley economy with interaction can be backed out from the allocations as follows:

$$\hat{q}_t(z_{t+1}) = \frac{\hat{\phi}(z_{t+1})}{\hat{R}_t^A} \sum_{y_{t+1}} \varphi(y_{t+1}|y_t, z_{t+1}) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \tag{29}$$

To prove this claim, we start by assuming the borrowing constraints do not bind, as before. The Euler equations for the household problem, for the contingent claim, are given by:

$$1 = \frac{\widehat{\beta}\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t, z_{t+1})} \sum_{y_{t+1}} \varphi(y_{t+1}|y_t, z_{t+1}) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \ \forall z_{t+1}$$

We assume that the state prices can be written only as a function of the current aggregate shock $z_{t+1}$ :

$$\hat{q}_t(z_{t+1}) = \hat{\phi}(z_{t+1})\widehat{\beta} \sum_{y_{t+1}} \varphi(y_{t+1}|y_t, z_{t+1}) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \tag{30}$$

Then, in a second step, we check that these state prices together with the allocations satisfy the Euler equation. We know the interest rate $\hat{R}_t^{Be}$ in the Bewley economy is deterministic (i.e. it does not depend on the aggregate shocks $z$):

$$1 = \widehat{\beta}\hat{R}_t^{Be} \sum_{z_{t+1}} \widehat{\varphi}(y_{t+1}|y_t) \sum_{y_{t+1}} \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))}$$

Now, we need to check that these conjectured allocations and the conjectured prices satisfy the Euler equations. Let us start with the Euler equation for the contingent claim, by summing across aggregate states $z_{t+1}$:

$$\sum_{z_{t+1}} \hat{q}_t(z_{t+1}) = \sum_{z_{t+1}} \hat{\phi}(z_{t+1})\widehat{\beta} \sum_{y_{t+1}} \varphi(y_{t+1}|y_t, z_{t+1}) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))}$$
$$\widehat{\beta} \sum_{y_{t+1}} \left[ \sum_{z_{t+1}} \hat{\phi}(z_{t+1})\varphi(y_{t+1}|y_t, z_{t+1}) \right] \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))},$$

and we know from the definition of the hatted transition probabilities that:

$$\sum_{z_{t+1}} \hat{\phi}(z_{t+1})\varphi(y_{t+1}|y_t, z_{t+1}) = \widehat{\varphi}(y_{t+1}|y),$$

which in turn implies that

$$\sum_{z_{t+1}} \hat{q}_t(z_{t+1}) = \widehat{\beta} \sum_{y_{t+1}} \widehat{\varphi}(y_{t+1}|y) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} = \frac{1}{\hat{R}_t^{Be}}.$$

The expression on the right hand side is the Euler equation in the Bewley economy. So, the Euler equation in the detrended Arrow economy is satisfied if the Bewley economy's Euler equation is satisfied.

**Market Clearing** But there is a loose end: market clearing. Market clearing in the transformed Bewley economy implies that on average, across aggregate histories, the market for consumption goods clears in the Arrow economy:

$$\int \sum_{y^t} \widehat{\varphi}(y^t|y_0)\hat{c}_t(\theta_0, y^t)d\Theta_0 = 1 = \int \sum_{z^t, y^t} \hat{\phi}(z^t)\widehat{\varphi}(y^t|y_0, z^t)\hat{c}_t(\theta_0, y^t)d\Theta_0$$

where $\hat{\phi}(z^t)$ denotes the probability of observing history $z^t$ and $\widehat{\varphi}$ is defined as:

$$\widehat{\varphi}(y^t|y_0, z^t) = \hat{\phi}(z_t)\varphi(y_t|z_t, y_{t-1})\widehat{\varphi}(y^{t-1}|y_0, z^{t-1}),$$

but the market does not necessarily clear in each aggregate state separately. Before, the probability of drawing $y^t$ did not depend on $z^t$, which meant market clearing in the Arrow economy was satisfied trivially. But, we will show this economy does not generate aggregate history dependence either, and as a consequence, a version of our baseline result continues to hold.

**No History Dependence** So, let $\mu(z^t)$ denote

$$\int \sum_{y^t} \widehat{\varphi}(y^t|y_0, z^t)\hat{c}_t(\theta_0, y^t)d\Theta_0.$$

If $\mu > 1$, demand for aggregate consumption exceeds the aggregate endowment; if $\mu < 1$, demand falls short. On average however the market clears since $\sum_{z^t} \hat{\phi}(z^t)\mu(z^t) = 1$. From the i.i.d. property of aggregate endowment growth, we also know that :

$$\sum_{z^t} \hat{\phi}(z^t)\mu(z^t) = 1 = \sum_{z^t} {}_{z_\tau \leq z^t}\hat{\phi}(z_\tau)\mu(z^t),$$

for all $z^t$, but this suggests that $\mu(z_t)$ is Markovian and that is satisifies

$$\sum_{z^t} \hat{\phi}(z_t)\mu(z_t) = 1$$

Instead of simply $\hat{c}_t(\theta_0, y^t)$, our conjectured allocation for the detrended Arrow economy is $\frac{\hat{c}_t(\theta_0, y^t)}{\mu_t(z_t)}$. These shares sum to one by construction in each aggregate state $z_t$.

Because of the (conjectured) Markovian nature of $\mu(\cdot)$, this turns out to require only a minor modification of our procedure. We will end up computing equilibria for a tweaked version of the Bewley economy.

**Tweaked version of Bewley economy** So, given the Markov property of $\mu(z)$, all we have to do is redefine the transition probabilities:

$$\widehat{\varphi}(y'|y) = \sum_{z'} \hat{\phi}(z')\frac{\mu(z')^\gamma}{\sum_{z'} \hat{\phi}(z')\mu(z')^\gamma}\varphi(y'|y, z'),$$

and we also define the adjusted discount factor accordingly:

$$\widetilde{\beta} = \widehat{\beta} \sum_{z'} \hat{\phi}(z') \mu(z')^{\gamma}.$$

So, now we solve for an equilibrium in the Bewley economy with transition matrix $\widehat{\varphi}(y'|y)$ and discount factor $\widetilde{\beta}$. From these equilibrium allocations, we can back out the equilibrium state prices for the detrended Arrow economy.

**Proposition 7.3.** *The state prices for the detrended Arrow economy satisfy:*

$$\hat{q}_t(z_t, z_{t+1}) = \hat{\phi}(z_{t+1}) \widehat{\beta} \sum_{y_{t+1}} \varphi(y_{t+1}|y_t, z_{t+1}) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \frac{\mu(z_{t+1})^{\gamma}}{\mu(z_t)^{\gamma}}$$

Once, we have solved this problem, we can back out the interest rates in the detrended Arrow economy as follows:

$$\hat{R}^A_t(z_t) = \hat{R}^{Be}_t \mu(z_t)^{\gamma}.$$

**Fixed Point** Of course, we do not know $\mu(\cdot)$ to begin with. This has to be computed itself as well. Define the following operator that takes a $\mu(\cdot)$ function and produces a new one:

$$T(\mu_i) = \mu_{i+1}$$

We are solving for a fixed point of this operator by solving a sequence of Bewley economies populated by agents with time discount factor $\widetilde{\beta}_i$ and transition matrix $\widehat{\varphi}_i(y'|y)$, where

$$\widehat{\varphi}_i(y'|y) = \sum_{z'} \hat{\phi}(z') \frac{\mu_i(z')^{\gamma}}{\sum_{z'} \hat{\phi}(z') \mu_i(z')^{\gamma}} \varphi(y'|y, z'),$$

and we also define the adjusted discount factor accordingly:

$$\widetilde{\beta}_i = \widehat{\beta} \sum_{z'} \hat{\phi}(z') \mu_i(z')^{\gamma}.$$

**Trading Strategies** The households do take contingent bond positions. Let $\{\hat{c}_t(\theta_0, y^t), \widetilde{a}_t(\theta_0, y^t) = 0,\ \hat{\sigma}_t(\theta_0, y^t)\}$ denote the equilibrium allocations and trading strategies from the constant discount factor economy, and let the state prices be defined recursively, as follows:

$$\hat{Q}_t(z^\tau|z_t) = \hat{q}_t(z_{\tau-1}, z_\tau)\hat{q}_t(z_{\tau-2}, z_{\tau-1})\ldots\hat{q}_t(z_{t+1}, z_t)$$

**Proposition 7.4.** *In the economy with persistent aggregate shocks, the households do take contingent bond positions, for each $y^t, z_t$:*

$$\begin{aligned}\frac{\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)}{\lambda(z_t)} &= \frac{\hat{c}_t(y^t, \theta_0)}{\mu(z_t)} - \eta(y_t) \\ &+ \sum_{\tau=t+1}^{\infty}\sum_{z^\tau}\hat{Q}_\tau(z^\tau|z_t)\sum_{\eta^\tau}\left(\frac{\hat{c}_\tau(y^\tau, \theta_0)}{\mu(z_\tau)} - \eta(y_\tau)\right) \\ &-\sigma_{t-1}(y^{t-1})\left[\hat{v}_t(z_t) + \alpha\right].\end{aligned}$$

These results naturally lead to the following theorem.

**Theorem 7.2.** *An equilibrium of the Bewley model with twisted transition probabilities $\widehat{\varphi}(y\prime|y)$ and adjusted discount factor $\widetilde{\beta}$ given by allocations $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$ and prices $\{\hat{R}_t^{Be}, \hat{v}_t\}$ can be made into an equilibrium for the Arrow economy with growth, $\{a_t(\theta_0, s^t, z_{t+1})\}$, $\{\sigma_t(\theta_0, s^t)\}$, $\{c_t(\theta_0, s^t)\}$ and $\{q_t(z^t, z_{t+1})\}$, $\{v_t(z^t)\}$, with*

$$\begin{aligned}c_t(\theta_0, s^t) &= \frac{\hat{c}_t(\theta_0, y^t)}{\mu(z_t)}e_t(z^t) \\ \sigma_t(\theta_0, s^t) &= \hat{\sigma}_t(\theta_0, y^t) \\ a_t(\theta_0, s^t, z_{t+1}) &= \hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)e_{t+1}(z^{t+1}) \\ v_t(z^t) &= \hat{v}_t e_t(z^t) \\ q_t(z^t, z_{t+1}) &= \frac{1}{\hat{R}_t^{Be}} * \frac{\phi(z_{t+1})\mu(z_{t+1})^\gamma\lambda(z_{t+1})^{-\gamma}}{\sum_{z_{t+1}}\phi(z_{t+1})\lambda(z_{t+1})^{1-\gamma}} \\ &\frac{\sum_{y_{t+1}}\varphi(y_{t+1}|y_t, z_{t+1})\frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))}}{\sum_{y_{t+1}}\widehat{\varphi}(y_{t+1}|y_t)\frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))}}\end{aligned}$$

Proof of Theorem 7.2:

*Proof.* identical to proof of Theorem 4.1. ■

The prices and allocation can be deduced from those in the Bewley economy with the twisted transition probability matrix. In this Bewley economy, there exists a stationary RCE under standard conditions. As before, this implies that the wealth distribution under this transformed measure does not change, in spite of the interaction between the aggregate and the idiosyncratic shocks. As a result, we can easily characterize the effect of the interaction between aggregate and idiosyncratic shocks on prices.

Finally, this result continues to hold, even when households face binding borrowing constraints.

**Corollary 7.1.** *If $\hat{a}_t(\theta_0, y^t)$ and $\hat{\sigma}_t(\theta_0, y^t)$ satisfy the constraints in the de-trended Arrow model (equivalently, in the Bewley model), then $\sigma_t(\theta_0, s^t) = \hat{\sigma}_t(\theta_0, y^t)$ and $a_t(\theta_0, s^t, z_{t+1}) = \hat{a}_t(\theta_0, y^t)e_{t+1}(z^{t+1})$ also satisfy the solvency constraints in the stochastically growing Arrow model.*

### 7.2.2 Asset Pricing Implications

The interaction contributes a third multiplicative component to the SDF.

**Proposition 7.5.** *In the Arrow economy, there is a unique SDF given by*

$$m_{t+1}^{A} = \frac{\hat{R}_t^{RE}}{\hat{R}_t^{A}} m_{t+1}^{RE} \xi_{t+1},$$

*with:*

$$\xi_{t+1} = \frac{\mu(z_{t+1})^{\gamma}}{\mu(z_t)^{\gamma}} \frac{E\left[\left(\frac{\hat{c}_{t+1}(y^t, y_{t+1})}{\hat{c}_t(y^t))}\right)^{-\gamma} | z_{t+1}, y^t\right]}{E\left[\left(\frac{\hat{c}_{t+1}(y^t, y_{t+1})}{\hat{c}_t(y^t)}\right)^{-\gamma} | y^t\right]}.$$

$\xi_{t+1}$ is the multiplicative addition to the SDF contributed by the interaction between aggregate and idiosyncratic shocks. It is the ratio of the expected marginal utility growth conditional on the aggregate state tomorrow, to expected marginal utility growth, easy to compute from the Bewley equilibrium allocations, and it only depends on the current aggregate shock $z_{t+1}$. In the standard case without interaction between $z$ and $y$ shocks, the ratio

$$\frac{E\left[\left(\frac{\hat{c}_{t+1}(y^t,y_{t+1})}{\hat{c}_t(y^t))}\right)^{-\gamma}|z_{t+1},y^t\right]}{E\left[\left(\frac{\hat{c}_{t+1}(y^t,y_{t+1})}{\hat{c}_t(y^t)}\right)^{-\gamma}|y^t\right]}$$

is one (see Proposition 6.1) and $\frac{\mu(z_{t+1})^{\gamma}}{\mu(z_t)^{\gamma}}$ is one as well, for all $z$. However, here it depends on $z_{t+1}$, but only in a limited way, through its effect on the transition probabilities. There is no direct effect on the consumption allocations except through $\mu$. Furthermore, the aggregate history plays no role.

As a result, the price/dividend ratio of stocks is Markov in $z$, because it reflects the changes in interest rate, and the interest rate is also Markov in $z$. The heterogeneity does not contribute any dynamics to the conditional standard deviation and mean of the SDF, simply because it does not directly impact the household trading strategies. The wealth distribution is essentially constant, and as a result, the aggregate history plays no role.

**Risk Sharing** When it comes to the risk premium, $\xi_{t+1}$ restores a role for the extent of risk sharing in determining the risk premium, but in a more subtle way: the cross-sectional dispersion cannot be manipulated to deliver an arbitrary risk premium, unlike what happens in the case of permanent shocks.

The reason is that these incomplete market models *endogenously* generate more risk sharing when the cross-sectional dispersion mechanism is turned on. The common aggregate shocks ($\mu's$) are the only direct effect of the interaction between aggregate and idiosyncratic shocks on allocations. The larger the increase in the cross-sectional dispersion, the larger one would expect the volatility of $\mu(\cdot)$ to be across aggregate states $z$, and hence the larger the effect on the risk premium. The next example explains why this intuition cannot be entirely correct.

**Example 1.** *Suppose all agents are ex ante identical. Consider the case with two aggregate states: in the high $z_{hi}$ state, all agents get the same $y = 1$; in the low $z_{lo}$ state, there is dispersion in the $y's$. In this case, we naturally expect $\mu(z_{lo}) > 1$ and $\mu(z_{hi}) < 1$, simply* *because the consumption shares $\frac{\widehat{c}(y^t)}{\mu(z_t)}$ integrate to one on average,*

$$\frac{1}{2}\mu(z_{lo}) + \frac{1}{2}\mu(z_{hi}) = 1,$$

*Now, if the $\mu'$s vary enough, then, for large enough $\gamma$,*

$$\widetilde{\beta} = \widehat{\beta} \sum_{z'} \hat{\phi}(z')\mu(z')^{\gamma} \rightarrow 1$$

*But, if this is the case, we get perfect risk sharing in the Bewley economy: $\widehat{c}(y^t) = 1$ for all $y^t$ (Levine and Zame, 2000). This in turn implies that $\mu = 1$ for all $z'$. Note that this limiting result holds true regardless of the persistence of the income process.*

As we turn on the cross-sectional dispersion mechanism, we push the time discount factor of agents in the transformed Bewley economy to one:

$$\widetilde{\beta} = \widehat{\beta} \sum_{z'} \hat{\phi}(z')\mu(z')^{\gamma} \rightarrow 1.$$

This reflects the *precautionary* effect: agents want to self-insure better against the $y$ shocks and this manifests itself in a decrease in the risk-free rate or an increase in the time discount factor in the transformed Bewley economy. If the precautionary effect is strong enough, which depends on $\gamma$ as well, then perfect risk sharing will obtain, and this is more likely the stronger the Mankiw-CD-STY effect. So in fact, the $\mu$ shocks cannot be 'large', because if they were, agents would share all the idiosyncratic risk, in which case the $\mu$ shocks disappear. This precludes large effects on the risk premium: if they were large, $\widetilde{\beta}$ would be close to one, but then that generates a contradiction.

# 8 Related Literature and Conclusion

Our paper makes contact with the literature on aggregation. Constantinides (1982), building on work by Negishi (1960) and Wilson (1968), derives an aggregation result for heterogenous agents in complete market models, implying that assets can be priced off the intertemporal marginal rate of substitution of an agent who consumes the aggregate endowment. We extend his result to incomplete market models.

Most of the work on incomplete markets and risk premia documents the moments of model-generated data for particular calibrations, but there are few analytical results. Levine and Zame (2002) show that in economies populated by agents with infinite horizons, the equilibrium allocations in the limit, as their discount factors go to one, converge to the complete markets allocations. Consequently the pricing implications of the incomplete markets model converge to that of the representative agent model as households become perfectly patient. We provide a qualitatively similar equivalence result that applies only to the risk premium, but the result does not, however, depend on the time discount factor of households. For households with CARA utility, closed form solutions of the individual decision problem in incomplete markets models with idiosyncratic risk are sometimes available, as Willen (1999) shows.[28] We use CRRA preferences for our results, and we obtain unambiguous (and negative) implications of uninsurable income risk for the equity premium. Finally, Lettau (2006) shows that if household consumption (in logs) consists of an aggregate and an idiosyncratic part, the latter does not affect risk premia. We show that this actually what happens in equilibrium in a large class of incomplete market models.

Most related to our study is the work by Constantinides and Duffie (1996), who consider an environment in which agents face permanent, idiosyncratic income shocks and can trade stocks and bonds. Their equilibrium is characterized by no trade in financial markets. By choosing the right stochastic income process, CD deliver equilibrium asset prices with all the desired properties. Krebs (2005) extends this result to a production economy. In his world, as in ours, the wealth distribution is not required as a state variable to characterize equilibria, in spite of the presence of aggregate shocks, but, as in CD (1996), the equilibrium is autarkic, so that households cannot diversify any of their risk. In our models households will be able to do so, and the equilibrium features trade in assets, but we can still fully characterize equilibrium asset prices.

Finally, our paper establishes the existence of a recursive competitive equilibrium with only asset holdings in the state space, albeit under a transformed measure. Kubler and

[28] Eliminating wealth effects simplifies the analysis.

Schmedders (2002) establish the existence of such an equilibrium, but only under very strong conditions. Miao (2004) relaxes these conditions, but he includes continuation utilities in the state space.

We already know from Mankiw (1986)'s analysis of the household's Euler equation that time variation in the cross-sectional dispersion of equilibrium household consumption growth can increase the size of risk premia. Our work shows that incomplete market models cannot produce this time variation as an equilibrium feature unless it is explicitly built into the primitives of the model.

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# A Additional Definitions

## A.1 Recursive Competitive Equilibrium

As stated above, without loss of generality we can abstract from trade in bonds. Let $A \subset \mathbf{R}$ denote the set of possible share holdings $\mathcal{B}(A)$ the Borel $\sigma-$algebra of $A$. $Y$ is the set of possible labor income draws and $\mathcal{P}(Y)$ is the power-set of $Y$. Define the state space as $Z = A \times Y$ and $\mathcal{B}(Z) = \mathcal{P}(Y) \times \mathcal{B}(A)$. We define the households' recursive problem for the case of a stationary recursive equilibrium in which the cross-sectional distribution $\Phi$ and prices are constant:

$$
\begin{aligned}
v(\sigma, y) &= \max_{c \geq 0, \sigma'} \left\{ u(c) + \hat{\beta} \sum_{y'} \varphi(y'|y) v(\sigma', y') \right\} \\
& \quad \text{s.t.} \\
c + \sigma' \hat{v} &= y + \sigma \left[ \hat{v} + \alpha \right] \\
\sigma' \hat{v} &\geq \hat{K} \text{ or } \sigma' \left[ \hat{v} + \alpha \right] \geq \hat{M}
\end{aligned}
$$

**Definition A.1.** *A stationary recursive competitive equilibrium (RCE) for the Bewley economy is a value function $v : Z \to R$, policy functions for the household $\sigma' : Z \to R$ and $c : Z \to R$, a stock price $\hat{v}$ and a measure $\Phi$ such that*

1. *$v, \sigma, c'$ are measurable with respect to $\mathcal{B}(Z)$, $v$ satisfies the household's Bellman equation and $\sigma', c$ are the associated policy functions, given $\hat{v}$*

2. *Market clearing*

$$\begin{aligned} \int c d\Phi &= 1 \\ \int \sigma' d\Phi &= 1 \end{aligned}$$

3. *The distribution is stationary*

$$\Phi(\mathcal{A}, \mathcal{Y}) = \int Q((a, y), (\mathcal{A}, \mathcal{Y}))\Phi(da \times dy)$$

*where $Q$ is the Markov transition function induced by the exogenous Markov chain $\hat{\phi}$ and the decision rule $\sigma'$.*

Existence of a stationary RCE in the Bewley economy can now be guaranteed under the standard conditions (see Huggett (1993) and Aiyagari (1994) (Aiyagari 1994)). More importantly, the standard algorithm to compute stationary equilibria in Bewley models can be applied.

**Algorithm A.1.** 1. *Guess a price $\hat{v}$.*

2. *Solve the recursive household problem.*

3. *Find the stationary measure $\Phi$ associated with the household decision rule $\sigma$ and the exogenous Markov chain $\varphi$.*

4. *Compute the excess demand in the stock market,*

$$d(\hat{v}) = \int \sigma' d\Phi - 1$$

*5. If $d(\hat{v}) = 0$, we are done, if not, adjust the guess for $\hat{v}$ and repeat 2.-5.*

**Remark 1.** *Once the equilibrium stock price $\hat{v}$ is determined, we can easily derive the equilibrium interest rate as*

$$\hat{R} = \frac{\hat{v} + \alpha}{\hat{v}}$$

## A.2 Optimality Conditions for Arrow Economy

Finally we list the optimality conditions. Attaching Lagrange multiplier $\mu_t(s^t) \geq .0$ to the constraint (14) and $\kappa_t(s^t, z_{t+1}) \geq 0$ to the constraint (15), the Euler equations of the detrended Arrow economy are given by:

$$1 = \frac{\hat{\beta}(s_t)}{\hat{q}_t(z^t, z_{t+1})} \sum_{s_{t+1}} \hat{\pi}(s_{t+1}|s_t) \frac{u'(\hat{c}_{t+1}(s^t, s_{t+1}))}{u'(\hat{c}_t(s^t))} \tag{31}$$

$$+\mu_t(s^t) + \frac{\kappa_t(s^t, z_{t+1})}{\hat{q}_t(z^t, z_{t+1})} \; \forall z_{t+1}. \tag{32}$$

$$1 = \hat{\beta}(s_t) \sum_{s_{t+1}} \hat{\pi}(s_{t+1}|s_t) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)}\right] \frac{u'(\hat{c}_{t+1}(s^t, s_{t+1}))}{u'(\hat{c}_t(s^t))}$$

$$+\mu_t(s^t) + \sum_{z_{t+1}} \kappa_t(s^t, z_{t+1}) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)}\right]. \tag{33}$$

Only one of the two Lagrange multiplier is relevant, depending on which version of the shortsale constraint we are considering. Finally, the complementary slackness conditions for the Lagrange multipliers are given by

$$\mu_t(s^t) \left[\sum_{z_{t+1}} \hat{a}_t(s^t, z_{t+1}) \hat{q}_t(z^t, z_{t+1}) + \sigma_t(s^t) \hat{v}_t(z^t) - \hat{K}_t(y^t)\right] = 0$$

$$\kappa_t(s^t, z_{t+1}) \left[\hat{a}_t(s^t, z_{t+1}) + \sigma_t(s^t) \left[\hat{v}_{t+1}(z^{t+1}) + \alpha\right] - \hat{M}_t(y^t)\right] = 0$$

The first order conditions and complementary slackness conditions, together with the appropriate transversality condition, are necessary and sufficient conditions for optimality on the household side.

In addition, we list the appropriate transversality conditions:

$$\lim_{t\to\infty}\sum_{s^t}\hat{\beta}^t\hat{\pi}(s^t|s_0)u'(c_{t+1}(y^{t+1},z^{t+1}))[\hat{a}_t(s^t,z_{t+1})-\hat{M}_t(y^t)]=0,$$

and

$$\lim_{t\to\infty}\sum_{s^t}\hat{\beta}^t\hat{\pi}(s^t|s_0)u'(c_t(y^t,z^t))[\hat{a}_t(s^t)-\hat{K}_t(y^t)]=0.$$

## A.3 Optimality Conditions for *IC* Economy

In the detrended *IC* economy the Euler equations read as

$$\begin{aligned}
1 &= \hat{\beta}(s_t)\sum_{s_{t+1}}\hat{\pi}(s_{t+1}|s_t)\left[\frac{R_t(z^t)}{\lambda(z_{t+1})}\right]\frac{u'(\hat{c}_{t+1}(s^t,s_{t+1}))}{u'(\hat{c}_t(s^t))} \\
&\quad +\mu_t(s^t)+\sum_{z_{t+1}}\kappa_t(s^t,z_{t+1})\left[\frac{R_t(z^t)}{\lambda(z_{t+1})}\right] \qquad (34)\\
&= \hat{\beta}(s_t)\sum_{s_{t+1}}\hat{\pi}(s_{t+1}|s_t)\left[\frac{\hat{v}_{t+1}(z^{t+1})+\alpha}{\hat{v}_t(z^t)}\right]\frac{u'(\hat{c}_{t+1}(s^t,s_{t+1}))}{u'(\hat{c}_t(s^t))} \\
&\quad +\mu_t(s^t)+\sum_{z_{t+1}}\kappa_t(s^t,z_{t+1})\left[\frac{\hat{v}_{t+1}(z^{t+1})+\alpha}{\hat{v}_t(z^t)}\right], \qquad (35)
\end{aligned}$$

with complementary slackness conditions given by:

$$\begin{aligned}
\mu_t(s^t)\left[\frac{\hat{b}_t(s^t)}{R_t(z^t)}+\sigma_t(s^t)\hat{v}_t(z^t)-\hat{K}_t(s^t)\right] &= 0\\
\kappa_t(s^t,z_{t+1})\left[\frac{\hat{b}_t(s^t)}{\lambda(z_{t+1})}+\sigma_t(s^t)\left[\hat{v}_{t+1}(z^{t+1})+\alpha\right]-\hat{M}_t(s^t,z_{t+1})\right] &= 0
\end{aligned}$$

In addition, we list the appropriate transversality conditions:

$$\lim_{t\to\infty}\sum_{s^t}\hat{\beta}^t\hat{\pi}(s^t|s_0)u'(\hat{c}_t(y^t,z^t))\left[\frac{\hat{b}_t(s^t)}{R_t(z^t)}\right]=0.$$

$$\lim_{t\to\infty}\sum_{s^t}\hat{\beta}^t\hat{\pi}(s^t|s_0)u'(\hat{c}_t(y^t,z^t))\sigma_{t-1}(s^{t-1})\left[\hat{v}_t(z^t)+\alpha\right]=0.$$

## A.4 Representative Agent Model

The budget constraint reads as

$$\begin{aligned} & c_t(z^t) + \sum_{z_{t+1}} a_t(z^t, z_{t+1}) q_t(z^t, z_{t+1}) + \sigma_t(z^t) v_t(z^t) \\ \leq \; & e_t(z^t) + a_{t-1}(z^{t-1}, z_t) + \sigma_{t-1}(z^{t-1}) \left[ v_t(z^t) + \alpha e_t(z_t) \right] \end{aligned}$$

After deflating by the aggregate endowment $e_t(z^t)$, the budget constraint reads as

$$\begin{aligned} & \hat{c}_t(z^t) + \sum_{z_{t+1}} \hat{a}_t(z^t, z_{t+1}) \hat{q}_t(z^t, z_{t+1}) + \sigma_t(z^t) \hat{v}_t(z^t) \\ \leq \; & 1 + \hat{a}_{t-1}(z^{t-1}, z_t) + \sigma_{t-1}(z^{t-1}) \left[ \hat{v}_t(z^t) + \alpha \right], \end{aligned}$$

where $\hat{a}_t(z^t, z_{t+1}) = \frac{a_t(z^t, z_{t+1})}{e_{t+1}(z^{t+1})}$ and $\hat{q}_t(z^t, z_{t+1}) = q_t(z^t, z_{t+1}) \lambda(z_{t+1})$ as well as $\hat{v}_t(z^t) = \frac{v_t(z^t)}{e_t(z^t)}$, as before in the Arrow model. Obviously, in an equilibrium of this model the representative agent consumes the aggregate endowment. Asset pricing in this economy is very simple.

**Lemma A.1.** *Equilibrium asset prices are given by*

$$\begin{aligned} \hat{q}_t(z^t, z_{t+1}) &= \hat{\beta} \hat{\phi}(z_{t+1}) \text{ for all } z_{t+1} \\ \hat{v}_t(z^t) &= \hat{\beta} \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left[ \hat{v}_{t+1}(z^{t+1}) + \alpha \right] \end{aligned} \tag{36}$$

## A.5 Recursive Utility

We consider the class of preferences due to Epstein and Zin (1989) . Let $V(c^i)$ denote the utility derived from consuming $c^i$ :

$$V(c^i) = \left[ (1-\beta) c_t^{1-\rho} + \beta (\mathcal{R}_t V_1)^{1-\rho} \right]^{\frac{1}{1-\rho}}, \tag{37}$$

where the risk-adjusted expectation operator is defined as:

$$\mathcal{R}_t V_{t+1} = \left( E_t V_{t+1}^{1-\alpha} \right)^{1/1-\alpha}.$$

$\alpha$ governs risk aversion and $\rho$ governs the willingness to substitute consumption intertemporally. These preferences impute a concern for the timing of the resolution of uncertainty to agents. In the special case where $\rho = \frac{1}{\alpha}$, these preferences collapse to standard power utility preferences with CRRA coefficient $\alpha$. As before, we can define *growth-adjusted* probabilities and the growth-adjusted discount factor as:

$$\hat{\pi}(s_{t+1}|s_t) = \frac{\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}}{\sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}}$$

$$\text{and } \hat{\beta}(s_t) = \beta \left( \sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha} \right)^{\frac{1-\rho}{1-\alpha}}$$

As before, $\hat{\beta}(s_t)$ is stochastic as long as the original Markov process is not *iid* over time. Note that the adjustment of the discount rate is affected by both $\rho$ and $\alpha$. If $\rho = \frac{1}{\alpha}$, this transformation reduces to the case we discussed in section (2).

Finally, let $\hat{V}_t(\hat{c})(s^t)$ denote the lifetime expected continuation utility in node $s^t$, under the new transition probabilities and discount factor, defined over consumption shares $\{\hat{c}_t(s^t)\}$ :

$$\hat{V}_t(\hat{c})(s^t) = \left[(1-\beta)\hat{c}_t^{1-\rho} + \hat{\beta}(s_t)(\hat{\mathcal{R}}_t \hat{V}_{t+1}(s^{t+1}))^{1-\rho}\right]^{\frac{1}{1-\rho}},$$

where $\mathcal{R}$ denotes the following operator:

$$\hat{\mathcal{R}}_t V_{t+1} = \left(\hat{E}_t \hat{V}_{t+1}^{1-\alpha}\right)^{1/1-\alpha}.$$

and $\hat{E}$ denotes the expectation operator under the hatted measure $\hat{\pi}$.

**Proposition A.1.** *Households rank consumption share allocations in the de-trended economy in exactly the same way as they rank the corresponding consumption allocations in the original growing economy: for any $s^t$ and any two consumption allocations $c, c'$*

$$V(c)(s^t) \geq V(c')(s^t) \Longleftrightarrow \hat{V}(\hat{c})(s^t) \geq \hat{V}(\hat{c}')(s^t)$$

*where the transformation of consumption into consumption shares is given by* (4).

**Detrended Arrow Economy** We proceed as before, by conjecturing that the equilibrium consumption shares only depend on $y^t$. Our first result states that if the consumption shares in the de-trended economy do not depend on the aggregate history $z^t$, then it follows that the interest rates in this economy are deterministic.

**Proposition A.2.** *In the de-trended Arrow economy, if there exists a competitive equilibrium with equilibrium consumption allocations* $\left\{\hat{c}_t(\theta_0, y^t)\right\}$, *then there is a deterministic interest rate process* $\left\{\hat{R}_t^A\right\}$ *and equilibrium prices* $\left\{\hat{q}_t(z^t, z_{t+1})\right\}$, *that satisfy:*

$$\hat{q}_t(z^t, z_{t+1}) = \frac{\hat{\phi}(z_{t+1})}{\hat{R}_t^A} \tag{38}$$

All the results basically go through. We can map an equilibrium of the Bewley economy into an equilibrium of the detrended Arrow economy.

**Theorem A.1.** *An equilibrium of the Bewley model* $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$ *and* $\{\hat{R}_t, \hat{v}_t\}$ *can be made into an equilibrium for the Arrow economy with growth,* $\left\{a_t(\theta_0, s^t, z_{t+1})\right\}$, $\{\sigma_t(\theta_0, s^t)\}$, $\left\{c_t(\theta_0, s^t)\right\}$ *and* $\left\{q_t(z^t, z_{t+1})\right\}$, $\{v_t(z^t)\}$, *with*

$$\begin{aligned}
c_t(\theta_0, s^t) &= \hat{c}_t(\theta_0, y^t) e_t(z^t) \\
\sigma_t(\theta_0, s^t) &= \hat{\sigma}_t(\theta_0, y^t) \\
a_t(\theta_0, s^t, z_{t+1}) &= \hat{a}_t(\theta_0, y^t) e_{t+1}(z^{t+1}) \\
v_t(z^t) &= \hat{v}_t e_t(z^t) \\
q_t(z^t, z_{t+1}) &= \frac{1}{\hat{R}_t} * \frac{\phi(z_{t+1})\lambda(z_{t+1})^{-\alpha}}{\sum_{z_{t+1}} \phi(z_{t+1})\lambda(z_{t+1})^{1-\alpha}}
\end{aligned}$$

As a result, even for an economy with agents who have these Epstein-Zin preferences, the risk premium is not affected[29].

# B Proofs

Proof of Proposition 2.1:

*Proof.* First, we show that households rank consumption streams $\{c_t(s^t)\}$ in the original econ-

[29]However, the extension to non-i.i.d. aggregate consumption growth is non-trivial, because the taste shocks affect continuation utilities, and simply adjusting the interest rate may not be sufficient.

omy in exactly the same way as they rank growth-deflated consumption streams

$$\hat{c}_t(s^t) = \frac{c_t(s^t)}{e_t(z^t)}.$$

where $s^t = (z^t, y^t)$. Denote $U(c)(s^t)$ as continuation utility of an agent from consumption stream $c$, starting at history $s^t$. This continuation utility follows the simple recursion

$$U(c)(s^t) = u(c_t(s^t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) U(c)(s^t, s_{t+1}),$$

where it is understood that $(s^t, s_{t+1}) = (z^t, z_{t+1}, y^t, y_{t+1})$. Divide both sides by $e_t(s^t)^{1-\gamma}$ to obtain

$$\frac{U(c)(s^t)}{e_t(z^t)^{1-\gamma}} = u(\hat{c}_t(s^t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) \frac{e_{t+1}(z^{t+1})^{1-\gamma}}{e_t(z^t)^{1-\gamma}} \frac{U(c)(s^t, s_{t+1})}{e_{t+1}(z^{t+1})^{1-\gamma}}.$$

Define a new utility index $\hat{U}(\cdot)$ as follows:

$$\hat{U}(\hat{c})(s^t) = \frac{U(c)(s^t)}{e_t(z^t)^{1-\gamma}},$$

it follows that

$$\begin{aligned}
\hat{U}(\hat{c})(s^t) &= u(\hat{c}_t(s^t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) \lambda(z_{t+1})^{1-\gamma} \hat{U}(\hat{c})(s^t, s_{t+1}) \\
&= u(\hat{c}_t(s^t)) + \hat{\beta}(s_t) \sum_{s_{t+1}} \hat{\pi}(s_{t+1}|s_t) \hat{U}(\hat{c})(s^t, s_{t+1})
\end{aligned}$$

Thus it follows, for two consumption streams $c$ and $c'$, that

$$U(c)(s^t) \geq U(c')(s^t) \text{ if and only if } \hat{U}(\hat{c})(s^t) \geq \hat{U}(\hat{c}')(s^t)$$

i.e. the household orders original and growth-deflated consumption streams in exactly the same way. ■

Proof of Proposition 4.1:

*Proof.* First, we suppose the borrowing constraints are not binding, which is the easiest case. Assume the equilibrium allocations only depend on $y^t$, not on $z^t$. Then conditions 2.2 and 2.3 imply that the Euler equations of the Arrow economy, for the contingent claim and the stock respectively, read as follows:

$$1 = \frac{\hat{\beta}\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t, z_{t+1})} \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \; \forall z_{t+1} \tag{39}$$

$$1 = \hat{\beta} \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left[ \frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)} \right] * \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))}. \tag{40}$$

In the first Euler equation, the only part that depends on $z_{t+1}$ is $\frac{\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t, z_{t+1})}$ which therefore implies that $\frac{\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t, z_{t+1})}$ cannot depend on $z_{t+1}$ : $\hat{q}_t(z^t, z_{t+1})$ is proportional to $\hat{\phi}(z_{t+1})$. Thus define $\hat{R}^A_t(z^t)$ by

$$\hat{q}_t(z^t, z_{t+1}) = \frac{\hat{\phi}(z_{t+1})}{\hat{R}^A_t(z^t)} \tag{41}$$

as the risk-free interest rate in the stationary Arrow economy. Using this condition, the Euler equation in (39) simplifies to the following expression:

$$1 = \hat{\beta}\hat{R}^A_t(z^t) \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \tag{42}$$

First, notice that apart from $\hat{R}^A_t(z^t)$ noting in this condition depends on $z^t$, so we can choose $\hat{R}^A_t(z^t) = \hat{R}^A_t$. ■

Proof of Corollary 4.1 :

*Proof.* The prices for Arrow securities in the original economy are given by

$$q_t(z^t, z_{t+1}) = \frac{\hat{q}_t(z^t, z_{t+1})}{\lambda(z_{t+1})} = \frac{\hat{\phi}(z_{t+1})}{\hat{R}^A_t \lambda(z_{t+1})}$$

and using the definition of the growth-adjusted transition probability $\hat{\phi}(z_{t+1})$ :

$$q_t(z^t, z_{t+1}) = \frac{1}{\hat{R}^A_t} * \frac{\phi(z_{t+1}|z_t)\lambda(z_{t+1})^{-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma}} \tag{43}$$

Note that $q_t(z^t, z_{t+1})$ is proportional to $\phi(z_{t+1}|z_t)\lambda(z_{t+1})^{-\gamma}$, but not to $\phi(z_{t+1}|z_t)$.

If we denote by $R^A_t = \frac{1}{\sum_{z_{t+1}} q_t(z^t, z_{t+1})}$ the implied risk-free interest rate in the original economy, we can easily back it out from the risk-free rate in the detrended Arrow economy:

$$R_t^A(z^t) = \hat{R}_t^A(z^t) * \frac{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{-\gamma}} \tag{44}$$

And, obviously, $R_t^A(z^t) = R_t^A$, the risk-free rate in the Arrow economy does not depend on time either. ■

Proof of Corollary 4.2:

*Proof.* Then (also using the conjecture above and the form of $\hat{q}_t(z^t, z_{t+1})$) consider the budget constraint in the detrended Arrow economy:

$$\begin{aligned}
& \hat{c}_t(y^t) + \sum_{z_{t+1}} \hat{a}_t(y^t, z^t, z_{t+1})\hat{q}_t(z^t, z_{t+1}) + \sigma_t(y^t)\hat{v}_t \\
= \; & \eta(y_t) + \hat{a}_{t-1}(y^{t-1}, z^t) + \sigma_{t-1}(y^{t-1})\left[\hat{v}_t + \alpha\right] \\
& \hat{c}_t(y^t) + \frac{\sum_{z_{t+1}} \hat{a}_t(y^t, z^t, z_{t+1})\hat{\phi}(z_{t+1})}{\hat{R}_t^A} + \sigma_t(y^t)\hat{v}_t \\
= \; & \eta(y_t) + \hat{a}_{t-1}(y^{t-1}, z^t) + \sigma_{t-1}(y^{t-1})\left[\hat{v}_t + \alpha\right]
\end{aligned}$$

Note that only the contingent claims quantities themselves depend on aggregate histories. So, it is obvious we can simply choose $\hat{a}_t(s^t, z_{t+1}) = \frac{a_t(s^t, z_{t+1})}{e_{t+1}(z^{t+1})} = \hat{a}_t(y^t)$,that is, holdings of Arrow securities in the transformed economy also only depend on the idiosyncratic shock history. ■

Proof of Theorem 4.1:

*Proof.* The crucial observation is that at the prices $\{q_t(z^t, z_{t+1})\}$, $\{v_t(z^t)\}$ the consumption allocation $\{c_t(\theta_0, s^t)\}$ satisfies the Euler equations of the original problem, because

$$\begin{aligned}
& \frac{\hat{\beta}(s_t)}{\hat{q}_t(z^t, z_{t+1})} \sum_{y_{t+1}} \hat{\pi}(s_{t+1}|s_t) \frac{u'(\hat{c}_{t+1}(s^t, s_{t+1}))}{u'(\hat{c}_t(s^t))} \\
= \; & \frac{\beta \sum_{y_{t+1}} \pi(s_{t+1}|s_t) \left(\frac{e_{t+1}(z^{t+1})}{e_t(z^t)}\right)^{1-\gamma}}{q_t(z^t, z_{t+1})\frac{e_{t+1}(z^{t+1})}{e_t(z^t)}} \left(\frac{c_{t+1}(s^t, s_{t+1})}{c_t(s^t)}\right)^{-\gamma} \left(\frac{e_{t+1}(z^{t+1})}{e_t(z^t)}\right)^{\gamma} \\
= \; & \frac{\beta}{q_t(z^t, z_{t+1})} \sum_{y_{t+1}} \pi(s_{t+1}|s_t) \left(\frac{c_{t+1}(s^t, s_{t+1})}{c_t(s^t)}\right)^{-\gamma}
\end{aligned}$$

and

$$
\begin{aligned}
& \hat{\beta}(s_t) \sum_{s_{t+1}} \hat{\pi}(s_{t+1}|s_t) \left[ \frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)} \right] \frac{u'(\hat{c}_{t+1}(s^t, s_{t+1}))}{u'(\hat{c}_t(s^t))} \\
= \; & \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) \left( \frac{e_{t+1}(z^{t+1})}{e_t(z^t)} \right)^{1-\gamma} \left[ \frac{v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z^{t+1})}{v_t(z^t)} \right] \left( \frac{e_{t+1}(z^{t+1})}{e_t(z^t)} \right)^{-1} \\
& * \left( \frac{c_{t+1}(s^t, s_{t+1})}{c_t(s^t)} \right)^{-\gamma} \left( \frac{e_{t+1}(z^{t+1})}{e_t(z^t)} \right)^{\gamma} \\
= \; & \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) \left[ \frac{v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z^{t+1})}{v_t(z^t)} \right] \left( \frac{c_{t+1}(s^t, s_{t+1})}{c_t(s^t)} \right)^{-\gamma},
\end{aligned}
$$

which shows that the non-detrended variables satisfy the Euler equations of the non-detrended economy without borrowing constraints. Especially it shows that it was suitable to define $\hat{q}_t(z^t, z_{t+1})$ and $\hat{v}_t(z^t)$ in the way we have above. ■

Proof of Corollary 4.3:

*Proof.* Suppose the stationary Bewley allocation $\{\hat{a}_t(y^t), \hat{\sigma}_t(y^t)\}$ satisfies the constraint

$$
\frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t)\hat{v}_t \geq \hat{K}_t(y^t)
$$

which seems the natural constraint to impose on the stationary Bewley economy. We want to show that the allocation for the stochastic Arrow economy satisfies the borrowing constraint if the allocation for the stationary Bewley economy does. Multiply both sides by $e_t(z^t)\hat{\phi}(z_{t+1})$ to obtain

$$
\frac{a_t(s^t, z_{t+1})\hat{\phi}(z_{t+1})}{\hat{R}_t \lambda(z_{t+1})} + \sigma_t(s^t) v_t(z^t) \hat{\phi}(z_{t+1}) \geq K_t(s^t)\hat{\phi}(z_{t+1})
$$

Using the fact that

$$
q_t(z^t, z_{t+1}) = \frac{\hat{\phi}(z_{t+1})}{\hat{R}_t \lambda(z_{t+1})}
$$

and summing over all $z_{t+1}$ yields

$$
\sum_{z_{t+1}} q_t(z^t, z_{t+1}) a_t(s^t, z_{t+1}) + \sigma_t(s^t) v_t(z^t) \geq K_t(s^t),
$$

exactly the constraint of the stochastic Arrow economy. So for this borrowing constraint and the Arrow economy the above Theorem goes through unchanged. Next consider the state-by state wealth constraint. In the stationary Bewley economy we may impose

$$\hat{a}_t(y^t) + \hat{\sigma}_t(y^t)\left[\hat{v}_{t+1} + \alpha\right] \geq \hat{M}_t(y^t)$$

Multiplying by $e_{t+1}(z^{t+1})$ yields

$$a_t(s^t, z_{t+1}) + \sigma_t(s^t)\left[v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z^{t+1})\right] \geq M_t(s^t, z_{t+1}) \text{ for all } z_{t+1}$$

and thus also for this constraint the Theorem above goes through, because, if the borrowing constraints are binding, we now know that the detrended version of the Arrow economy and the Bewley economy have the same Euler equations, even if the constraints bind: the Lagrangian multipliers $\mu_t(y^t)$ and $\kappa_t(y^t)$ from the Bewley economy, together with the allocations $\left\{\hat{c}_t(\theta_0, y^t)\right\}$, are also the right multipliers for the detrended Arrow economy. Proposition (4.1) is still valid in the case of binding constraints. ■

Proof of Proposition 5.1:

*Proof.* Again suppose the borrowing constraints are not binding. Conditions 2.2 and 2.3, and our conjecture that the allocations only depend on $y^t$ imply that the Euler equations of the *IC* economy read as

$$1 = \hat{\beta} R_t(z^t) \sum_{z_{t+1}} \frac{\hat{\phi}(z_{t+1})}{\lambda(z_{t+1})} * \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \quad (45)$$

$$1 = \hat{\beta} \sum_{s_{t+1}} \hat{\phi}(z_{t+1}) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)}\right] * \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \quad (46)$$

Define

$$\hat{R}_t^{IC}(z^t) = R_t(z^t) * \frac{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma}} \quad (47)$$

as the risk-free interest rate in the transformed *IC* economy. It is immediate that equation (45) becomes

$$1 = \hat{\beta} \hat{R}_t^{IC}(z^t) \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \quad (48)$$

which is identical to the Euler equation (42) in the Arrow economy. This suggests (since the

Euler equations are identical in both economies) that we have

$$\hat{R}_t^{IC}(z^t) = \hat{R}_t^A(z^t) \tag{49}$$

Notice that since

$$R_t(z^t) \equiv R_t^{IC}(z^t) = \hat{R}_t^{IC}(z^t) * \frac{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{-\gamma}} \tag{50}$$

it then follows that

$$R_t^{IC}(z^t) = R_t^A(z^t) \tag{51}$$

that is, the implied risk-free rates in the original Arrow and $IC$ economies are identical. ■

Proof of Lemma 5.1

*Proof.* Recall that the budget constraint for the transformed $IC$ economy is

$$\hat{c}_t(s^t) + \frac{\hat{b}_t(s^t)}{R_t(z^t)} + \sigma_t(s^t)\hat{v}_t(z^t) \leq \eta(y_t) + \frac{\hat{b}_{t-1}(s^{t-1})}{\lambda(z_t)} + \sigma_{t-1}(s^{t-1})\left[\hat{v}_t(z^t) + \alpha\right]$$

Using the results from above

$$\hat{c}_t(y^t) + \frac{\hat{b}_t(s^t)}{R_t} + \sigma_t(y^t)\hat{v}_t \leq \eta(y_t) + \frac{\hat{b}_{t-1}(s^{t-1})}{\lambda(z_t)} + \sigma_{t-1}(y^{t-1})\left[\hat{v}_t + \alpha\right]$$

The budget constraint reads as

$$\hat{c}_t(y^t) - \eta(y_t) + \frac{\hat{b}_t(s^t)}{R_t} + \sigma_t(s^t)\hat{v}_t = \frac{\hat{b}_{t-1}(s^{t-1})}{\lambda(z_t)} + \sigma_{t-1}(s^{t-1})\left[\hat{v}_t + \alpha\right]$$

and from the absence of arbitrage

$$\frac{R_t}{\lambda_{t+1}} = \frac{\hat{v}_{t+1} + \alpha}{\hat{v}_t}$$

Next period's budget constraint reads as

$$\hat{c}_{t+1}(y^{t+1}) - \eta(y_{t+1}) + \frac{\hat{b}_{t+1}(s^{t+1})}{R_{t+1}} + \sigma_{t+1}(s^{t+1})\hat{v}_{t+1} = \frac{\hat{b}_t(s^t)}{\lambda(z_{t+1})} + \sigma_t(s^t)\left[\hat{v}_{t+1} + \alpha\right]$$

Multiplying both sides by $\hat{\phi}(z_{t+1})$ and then summing with respect to $z_{t+1}$ yields

$$
\begin{aligned}
& \hat{c}_{t+1}(y^{t+1}) - \eta(y_{t+1}) + \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left( \frac{\hat{b}_{t+1}(s^{t+1})}{R_{t+1}} + \sigma_{t+1}(s^{t+1}) \hat{v}_{t+1} \right) \\
= \; & \hat{b}_t(s^t) \sum_{z_{t+1}} \frac{\hat{\phi}(z_{t+1})}{\lambda(z_{t+1})} + \sigma_t(s^t) \left[ \hat{v}_{t+1} + \alpha \right] \\
= \; & \hat{R}_t \left[ \frac{\hat{b}_t(s^t)}{R_t} + \sigma_t(s^t) \hat{v}_t \right]
\end{aligned}
$$

(the last step requires the use of the definition of $\hat{R}_t$ and a little algebra). Thus

$$
\frac{\hat{b}_t(s^t)}{R_t} + \sigma_t(s^t)\hat{v}_t = \frac{\hat{c}_{t+1}(y^{t+1}) - \eta(y_{t+1})}{\hat{R}_t} + \frac{1}{\hat{R}_t} \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left( \frac{\hat{b}_{t+1}(s^{t+1})}{R_{t+1}} + \sigma_{t+1}(s^{t+1}) \hat{v}_{t+1} \right)
$$

Repeating this and noting that by assumption future $\hat{c}$'s do not depend on $z_{t+\tau}$'s suggests

$$
\hat{c}_t(y^t) - \eta(y_t) + \sum_{\tau=t+1}^{\infty} \frac{\hat{c}_\tau(y^\tau) - \eta(y_\tau)}{{}_{j=t}^{\tau-1} \hat{R}_j} = \frac{\hat{b}_{t-1}(s^{t-1})}{\lambda(z_t)} + \sigma_{t-1}(s^{t-1}) \left[ \hat{v}_t + \alpha \right]
$$

which simply says that if consumption in the detrended bond economy only depends on idiosyncratic shocks, then the wealth households come into the period with can only depend on this as well, that is,

$$
\frac{\hat{b}_{t-1}(s^{t-1})}{\lambda(z_t)} + \sigma_{t-1}(s^{t-1}) \left[ \hat{v}_t + \alpha \right]
$$

cannot depend on $z_t$. ■

Proof of Proposition 5.2:

*Proof.* Let us start with the case of 2 aggregate states. We consider the agent's wealth at the start of period $t$ in the *IC* economy, for the case with two aggregate states:

$$
b_{t-1}(s^{t-1}) + \sigma_{t-1}(s^{t-1}) \left[ v_t(z^t) + \alpha e_t(z^t) \right] = \theta_{t-1}(s^{t-1}, hi),
$$

and, similarly, in the low state:

$$
b_{t-1}(s^{t-1}) + \sigma_{t-1}(s^{t-1}) \left[ v_t(z^t) + \alpha e_t(z^t) \right] e_{t-1}(z^{t-1}) = \theta_{t-1}(s^{t-1}, lo),
$$

where the $\theta_{t-1}(s^{t-1}, z_t)$ are the contingent claim positions (total, including stocks) in the Arrow economy *at the start of t*. We deflate the wealth positions of our agent in each aggregate state, to obtain:

$$\hat{b}_{t-1}(s^{t-1}) + \sigma_{t-1}^{IC}(s^{t-1}) \left[\hat{v}_t(z^t) + \alpha\right] \lambda(hi) = \hat{\theta}_{t-1}(s^{t-1}, hi)\lambda(hi)$$

and, similarly, in the low state:

$$\hat{b}_{t-1}(s^{t-1}) + \sigma_{t-1}^{IC}(s^{t-1}) \left[\hat{v}_t(z^t) + \alpha\right] \lambda(lo) = \hat{\theta}_{t-1}(s^{t-1}, lo)\lambda(lo)$$

This implies that the stock position of the agent is given by:

$$\sigma_{t-1}^{IC}(s^{t-1}) \left[\hat{v}_t(z^t) + \alpha\right] = \frac{\left[\hat{\theta}_{t-1}(s^{t-1}, hi)\lambda(hi) - \hat{\theta}_{t-1}(s^{t-1}, lo)\lambda(lo)\right]}{[\lambda(hi) - \lambda(lo)]}$$

If we impose that $\hat{\theta}_{t-1}(s^{t-1}, lo) = \hat{\theta}_{t-1}(s^{t-1}, hi) = \hat{\theta}_{t-1}(s^{t-1})$, we simply obtain the following expression for the stock position:

$$\sigma_{t-1}^{IC}(s^{t-1}) = \frac{\hat{\theta}_{t-1}(s^{t-1})}{[\hat{v}_t + \alpha]}$$

where I have assumed that $\hat{v}_t(z^t) = \hat{v}_t$, while the bond position is given by;

$$\hat{b}_{t-1}(s^{t-1}) + \frac{\hat{\theta}_{t-1}(s^{t-1})}{[\hat{v}_t(z^t) + \alpha]} \left[\hat{v}_t(z^t) + \alpha\right] \lambda(z_t) = \hat{\theta}_{t-1}(s^{t-1})\lambda(z_t),$$

which implies the bond position is $\hat{b}_{t-1}(s^{t-1}) = 0$ for all $s^{t-1}$. So all of the state-contingency is obtained by going long or short in the stock, depending on whether you're borrowing or lending in the deflated economy. In the two-state case, the bond is redundant, it seems. The only way to get some trade in bonds is by imposing a short-sales constraint on the stock. This argument generalizes to the case of $N$ aggregate states quite easily. We just set the stock position equal to:

$$\sigma_{t-1}^{IC}(s^{t-1}) = \frac{\hat{\theta}_{t-1}(s^{t-1})}{[\hat{v}_t + \alpha]}$$

and we simply let $\hat{b}_{t-1}(s^{t-1}) = 0$. This immediately implies that the wealth position at the start of $t$ in the $IC$ economy coincides with that in the Arrow economy for all $z_t$ :

$$b_{t-1}(s^{t-1}) + \sigma_{t-1}^{IC}(s^{t-1}) \left[v_t(z^t) + \alpha e_t(z^t)\right] = \theta_{t-1}^{A}(s^{t-1}, z_t),$$

■

Proof of Corollary 5.1

*Proof.* We constructed the same net wealth positions at the start of each period in the *IC* economy as in the Arrow economy. As a result, the solvency constraints imposed in the Arrow economy will be satisfied in the *IC* economy. In the stationary Arrow economy the constraints were

$$\begin{aligned} \frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t^A(y^t)\hat{v}_t &\geq \hat{K}_t(y^t) \\ \hat{a}_t(y^t) + \hat{\sigma}_t(y^t)\left[\hat{v}_{t+1} + \alpha\right] &\geq \hat{M}_t(y^t) \end{aligned}$$

But from (**??**) we have

$$\begin{aligned} \frac{\hat{b}_t(s^t)}{R_t(z^t)} + \hat{\sigma}_t^{IC}(y^t)\hat{v}_t &= \frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t^A(y^t)\hat{v}_t \geq \hat{K}_t(y^t) \\ \frac{\hat{b}_t(s^t)}{\lambda(z_{t+1})} + \sigma_t^{IC}(y^t)\left[\hat{v}_{t+1} + \alpha\right] &= \hat{a}_t(y^t) + \hat{\sigma}_t^A(y^t)\left[\hat{v}_{t+1} + \alpha\right] \geq \hat{M}_t(y^t) \end{aligned}$$

and thus we know that the asset position for the bond economy satisfy the shortsale constraint, which bind if and only if they bind in the stationary Arrow (or Bewley) economy. So we can indeed use the same Lagrange multipliers. ■

Proof of :A.1

*Proof.* The first order conditions for the representative agent are given by:

$$\begin{aligned} 1 &= \frac{\hat{\beta}\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t, z_{t+1})} \frac{u'(\hat{c}_{t+1}(z^t, z_{t+1}))}{u'(\hat{c}_t(z^t))} \; \forall z_{t+1} \\ 1 &= \hat{\beta} \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)}\right] \frac{u'(\hat{c}_{t+1}(z^t, z_{t+1}))}{u'(\hat{c}_t(z^t))} \end{aligned} \tag{52}$$

■

Proof of Lemma 6.1:

*Proof.* First, consider a household whose borrowing constraints do not bind. The Euler equation of the Arrow/$IC$ economy for the stock respectively reads as follows:

$$1 = \hat{\beta} \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left[ \frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)} \right] * \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))}.$$

The Euler equation in the Arrow/$IC$ economy for the bond reads as follows:

$$1 = \hat{\beta} \hat{R}_t \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))}$$

This implies that

$$\hat{\beta} \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left[ \frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)} - \hat{R}_t \right] = 0$$

which then also holds if we replace $\hat{\beta}$ by $\beta$. This immediately implies that

$$E_t[\left(R^s_{t+1} - R_t\right) \beta \left( \frac{e_{t+1}}{e_t} \right)^{-\gamma} ] = 0.$$

In the Arrow economy, the Consumption-CAPM prices any excess return $R^i_{t+1} - R_t$ as long as the returns only depend on the aggregate state $z_{t+1}$. In the IC model, it only prices a claim to aggregate consumption.

■

Proof of Proposition 6.1:

*Proof.* Let us define

$$1/\hat{R}^{RE}_\tau = 1/\hat{R}^{RE} = \sum_{z_{t+1}} \hat{q}_t(z^t, z_{t+1}) = \sum_{z_{t+1}} \hat{\beta} \hat{\phi}(z_{t+1}) = \hat{\beta}.$$

We know that in the Arrow economy, the equilibrium Arrow prices are given by:

$$q^A_t(z^t, z_{t+1}) = \frac{\hat{q}^A_t(z^t, z_{t+1})}{\lambda(z_{t+1})} = \frac{\hat{\phi}(z_{t+1})}{\lambda(z_{t+1}) \hat{R}^A_t(z^t)}$$

whereas in the representative agent economy Arrow prices are given as

$$q_t(z^t, z_{t+1}) = \frac{\hat{q}_t(z^t, z_{t+1})}{\lambda(z_{t+1})} = \hat{\beta} \frac{\hat{\phi}(z_{t+1})}{\lambda(z_{t+1})}$$

This implies that the stochastic discount factor in the Arrow economy equals the SDF in the representative agent economy multiplied by a non-random number $\kappa_t$ :

$$m_{t+1}^{A} = m_{t+1}^{RE}\kappa_t$$

with $\kappa_t = \frac{\hat{R}_t^E}{\hat{R}_t^A}$ and $\left(\hat{R}^E\right)^{-1} = \hat{\beta}$. ■

Proof of Theorem 6.1:

*Proof.* First, note that the multiplicative risk premium can be stated as a weighted sum of risk premia on strips ((Alvarez and Jermann 2001)):

$$\begin{aligned} 1+\nu_t &= E_t m_{t+1} E_t\left(\frac{\sum_{k=1}^{\infty} E_{t+1} m_{t+1,t+k}\alpha e_{t+k}}{\sum_{k=1}^{\infty} E_t m_{t,t+k}\alpha e_{t+k}}\right) \\ &= \sum_{k=1}^{\infty} \frac{\frac{E_{t+1} m_{t,t+k} e_{t+k}}{E_t m_{t,t+k} e_{t+k}}}{1/E_t m_{t+1}} \frac{E_t m_{t,t+k} e_{t+k}}{\sum_{k=1}^{\infty} E_t m_{t,t+k} e_{t+k}} \\ &= \sum_{k=1}^{\infty} \omega_k \frac{E_t R_{t,1}\left[e_{t+k}\right]}{R_{t,1}\left[1\right]}, \end{aligned}$$

where the weights are

$$\omega_k = \frac{E_t m_{t,t+k} e_{t+k}(z^{t+k})}{\sum_{l=1}^{\infty} E_t m_{t,t+l} e_{t+l}(z^{t+k})} \tag{53}$$

The multiplicative risk premium on a one-period strip ( a claim to the Lucas tree's dividend next period only, not the entire stream) is the same in the Arrow economy as in the representative agent economy. First, we show that the one-period ahead conditional strip risk premia are identical:

$$\frac{E_t \frac{\alpha e_{t+1}(z^{t+1})}{E_t[m_{t+1}^{A}\alpha e_{t+1}(z^{t+1})]}}{\frac{1}{E_t[m_{t+1}^{A}]}} = \frac{E_t \frac{\lambda_{t+1}(z^{t+1})}{E_t[m_{t+1}^{A}\lambda_{t+1}(z^{t+1})]}}{\frac{1}{E_t[m_{t+1}^{A}]}} = \frac{E_t \frac{\lambda_{t+1}(z^{t+1})}{E_t[m_{t+1}^{RE}\lambda_{t+1}(z^{t+1})]}}{\frac{1}{E_t[m_{t+1}^{RE}]}}$$

Next, we follow Alvarez and Jermann's proof strategy (in a different setting), and we show that the risk premia on $k$-period strips are identical. Here is the risk premium on a $k-$period strip:

$$\frac{E_t R_{t,t+1}\left[\alpha e_{t+k}\right]}{E_t R_{t,t+1}\left[1\right]} = \frac{E_t \frac{E_{t+1}[m_{t+1,t+k}^{A}\alpha e_{t+k}(z^{t+k})]}{E_t[m_{t,t+k}^{A}\alpha e_{t+k}(z^{t+k})]}}{\frac{1}{E_t[m_{t+1}^{A}]}}$$

Now, since the aggregate shocks are iid, the term structure of the risk premia in the representative agent economy is flat (i.e. the risk premia does not depend on $k$) :

$$\frac{E_t R^{RE}_{t,t+1}\left[\alpha e_{t+k}\right]}{E_t R^{RE}_{t,t+1}\left[1\right]} = \frac{E_t \frac{[E\lambda(z)^{1-\gamma}]^{k-1}}{[E\lambda(z)^{1-\gamma}]^{k}}}{\frac{1}{E_t[m^{RE}_{t+1}]}} = \frac{\frac{1}{E[\lambda(z)^{1-\gamma}]}}{\frac{1}{E[\lambda(z)^{-\gamma}]}} = \frac{E[\lambda(z)^{-\gamma}]}{E[\lambda(z)^{1-\gamma}]}$$

(same proof by Alvarez and Jermann on page 37.) Now, to keep things simple, assume the $R^A_t = R^{RE}$ for all $t, and\ \kappa_t$ is constant as a result, then this implies that then the term structure of risk premia in the Arrow economy is flat as well.

$$\frac{E_t R^{RE}_{t,t+1}\left[\alpha e_{t+k}\right]}{E_t R^{RE}_{t,t+1}\left[1\right]} = \frac{E_t \frac{[E\lambda(z)^{1-\gamma}\kappa]^{k-1}}{[E\lambda(z)^{1-\gamma}\kappa]^{k}}}{\frac{1}{E_t[m^{rep}_{t+1}]}} = \frac{\frac{1}{E[\lambda(z)^{1-\gamma}\kappa]}}{\frac{1}{E[\lambda(z)^{-\gamma}\kappa]}} = \frac{E[\lambda(z)^{-\gamma}]}{E[\lambda(z)^{1-\gamma}]}$$

But this implies that the multiplicative risk premia is unchanged since the risk premia on the consumption strips are invariant in $k$ and the one-period ahead .This means the multiplicative risk premium is unchanged. The proof goes through for time-varying $\kappa_t$, but the algebra is a little messier. ■

Proof of 7.1:

*Proof.* First, suppose the borrowing constraints are not binding, which is the easiest case. Suppose that the equilibrium allocations $\{c_t(y^t)\}$ only depend on $y^t$. Then, as before, conditions 2.2 and 2.3 imply that the Euler equations of the Arrow economy read as

$$\begin{aligned}
1 &= \frac{\hat{\beta}(z_t)\hat{\phi}(z_{t+1}|z_t)}{\hat{q}_t(z^t, z_{t+1})} \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \ \forall z_{t+1} \\
1 &= \hat{\beta}(z) \sum_{z_{t+1}} \hat{\phi}(z_{t+1}|z_t) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)}\right] \\
&\quad * \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))}
\end{aligned}$$

Nothing depends on $z^t$ in the first equation; the only part that depends on the current aggregate shock $z_t$ is $\frac{\hat{\beta}(z_t)\hat{\phi}(z_{t+1}|z_t)}{\hat{q}_t(z^t, z_{t+1})}$. It seems natural to conjecture an equilibrium Markov interest rate process : $\hat{R}^A_t(z_t)\hat{\beta}(z_t) = \rho_t\widetilde{\beta}$, where

$$\widetilde{\beta} = \sum_{z'} \Pi(z')\hat{\beta}(z'),$$

with Arrow prices are:

$$\hat{q}_t(z^t, z_{t+1}) = \frac{\hat{\phi}(z_{t+1}|z_t)}{\hat{R}^A_t(z_t)}$$

With this condition equation (39) becomes

$$1 = \widetilde{\beta}\rho_t \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))}$$

This is the standard Euler equation we obtained in the case of iid aggregate shocks, if you replace $\hat{R}_t$ by $\rho_t$. ■

Proof of Proposition 7.2:

*Proof.* Next ,we need to check that these allocations satisfy the budget constraint in the economy with time-varying discount factors. Let us take $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t) = 0,\ \hat{\sigma}_t(\theta_0, y^t)\}$ from the constant discount factor economy. Back out the implied state-contingent Arrow securities positions recursively, for each $y^t, z_t$, from the budget constraint:

$$\begin{aligned} \frac{\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)}{\lambda(z_t)} &= \hat{c}_t(y^t, \theta_0) - \eta(y_t) \\ &\quad + \sum_{\tau=t+1}^{\infty} \sum_{z^\tau, \eta^\tau} \hat{Q}_t(z^\tau|z_t) \left(\hat{c}_\tau(y^\tau, \theta_0) - \eta(y_\tau)\right) \\ &\quad - \sigma_{t-1}(y^{t-1}) \left[\hat{v}_t(z_t) + \alpha\right]. \end{aligned} \tag{54}$$

This makes sure that $\left\{\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0), \sigma_{t-1}(y^{t-1}, \theta_0)\right\}$ finance the consumption allocations $\{\hat{c}_t(\theta_0, y^t)\}$. Next, we need to check that the bond market clears:

$$\begin{aligned} \int \sum_{y^t-1} \pi(y^{t-1}|y_0) \frac{\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)}{\lambda(z_t)} d\Theta_0 &= \\ \frac{1}{\lambda(z_t)} \sum_{y^t|y^{t-1}} \pi(y_t|y_{t-1}) \int \sum_{y^{t-1}} \pi(y^{t-1}|y_0) \hat{a}_{t-1}(y^{t-1}, z_t, \theta_0) d\Theta_0 &= \\ \int \sum_{y^t} \pi(y^t|y_0) \hat{a}_{t-1}(y^{t-1}, z_t, \theta_0) d\Theta_0 &= 0 \end{aligned}$$

for each $z_t$. We now multiply equation (54) by $\pi(y^t|y_0)$ and then sum over all $y^t$. For the first term we obtain, using the aggregate resource constraint,

$$\begin{aligned} & \int \sum_{y^{t-1}} \pi(y^{t-1}|y_0) \sum_{y^t} \pi(y^t|y_{t-1}) \left(\hat{c}_t(y^t,\theta_0) - \eta(y_t)\right) d\Theta_0 \\ = & \int \sum_{y^t} \pi(y^t|y_0) \left(\hat{c}_t(y^t,\theta_0) - \eta(y_t)\right) d\Theta_0 = \alpha. \end{aligned} \tag{55}$$

By the same token we obtain

$$\begin{aligned} & \int \sum_{y^{t-1}} \pi(y^{t-1}|y_0) \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \hat{Q}_t(z^\tau|z_t) \left(\hat{c}_\tau(y^\tau,\theta_0) - \eta(y_\tau)\right) d\Theta_0 \\ = & \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \hat{Q}_t(z^\tau|z_t)\alpha. \end{aligned} \tag{56}$$

Furthermore we know from the market clearing in the market for shares that

$$\int \sum_{y^{t-1}} \pi(y^{t-1}|y_0)\sigma_{t-1}(y^{t-1},\theta_0) \left[\hat{v}_t(z_t) + \alpha\right] d\Theta_0 = \left[\hat{v}_t(z_t) + \alpha\right], \tag{57}$$

and finally we know that the share price can be written as follows:

$$\left[\hat{v}_t(z_t) + \alpha\right] = \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \hat{Q}_t(z^\tau|z_t)\alpha + \alpha, \tag{58}$$

which in turn implies, substituting (55), (56) and (57) into (54) that

$$\frac{1}{\lambda(z_t)} \int \sum_{y^{t-1}} \pi(y^{t-1}|y_0)\hat{a}_{t-1}(y^{t-1},z_t) d\Theta_0 = 0$$

Thus, each of the Arrow securities markets clears for the new trading strategies. ■

*Proof.* of Lemma 7.1: We know that we can set $\frac{\widetilde{a}_{t-1}(y^{t-1},z_t,\theta_0)}{\lambda(z_t)} = 0$ w.l.o.g in the ergodic economy.

Hence, we can subtract $\frac{\widetilde{a}_{t-1}(y^{t-1},z_t,\theta_0)}{\lambda(z_t)}$ from the expression for $\frac{\hat{a}_{t-1}(y^{t-1},z_t,\theta_0)}{\lambda(z_t)}$ :

$$\begin{aligned}\frac{\hat{a}_{t-1}(y^{t-1},z_t,\theta_0)}{\lambda(z_t)} &= \sum_{\tau=t+1}^{\infty}\sum_{z^\tau}\left(\hat{Q}_\tau(z^\tau|z_t)-\widetilde{Q}_\tau\right)\sum_{\eta^\tau}\left(\hat{c}_\tau(y^\tau,\theta_0)-\eta(y_\tau)\right)\\ &\quad -\sigma_{t-1}(y^{t-1})\left[\hat{v}_t(z_t)-\widetilde{\nu}_t(z^t)\right].\\ &= \sum_{\tau=t+1}^{\infty}\sum_{z^\tau}\left(\hat{Q}_\tau(z^\tau|z_t)-\widetilde{Q}_\tau\right)\sum_{\eta^\tau}\left(\hat{c}_\tau(y^\tau,\theta_0)-\eta(y_\tau)\right)\\ &\quad -\sigma_{t-1}(y^{t-1})\alpha\sum_{\tau=t+1}^{\infty}\sum_{z^\tau}\left(\hat{Q}_\tau(z^\tau|z_t)-\widetilde{Q}_\tau\right)\end{aligned}$$

■

**Lemma B.1.** *The taste shocks cause the price of a unit of consumption to be delivered at $\tau$ to depend on the current state $z_t$ and to deviate from those in the ergodic economy, $\widetilde{Q}_\tau$ :*

$$\sum_{z^\tau}\hat{Q}_\tau(z^\tau|z_t)\neq\widetilde{Q}_t,$$

*but, as a result, on average, these prices coincide:*

$$\sum_{z_t}\hat{\Pi}(z_t)\sum_{z^\tau}\hat{Q}_\tau(z^\tau|z_t)=\widetilde{Q}_t,$$

Proof of Lemma B.1:

*Proof.* ¿From the definition of a stationary distribution, it follows that:

$$\begin{aligned}&\sum_{z_t}\hat{\Pi}(z_t)\sum_{z^\tau}\hat{\pi}(z^\tau|z_t)\hat{\beta}_{t,\tau}(z^\tau|z_t)\\ =&\sum_{z^\tau}\hat{\Pi}(z^\tau)\hat{\beta}_{t,\tau}(z^\tau|z_t)=\hat{E}[\hat{\beta}_{t,\tau}]\\ =&\ \widetilde{\beta}^{\tau-t}\end{aligned}\tag{59}$$

where we have used $\hat{\beta}_{t,\tau}(z^\tau|z_t)$ to denote the product of time discount factors:

$$\hat{\beta}_{t,\tau}(z^\tau|z_t)=\hat{\beta}(z_t)\hat{\beta}(z_{t+1})\ldots\hat{\beta}(z_\tau)$$

The taste shocks cause the price of a unit of consumption to be delivered at $\tau$ to depend on the current state $z_t$ and to deviate from those in the ergodic economy, $\widetilde{Q}_\tau$ :

$$\sum_{z^\tau} \hat{Q}_\tau(z^\tau|z_t) \neq \widetilde{Q}_t,$$

but, as a result, on average, these prices coincide:

$$\sum_{z_t} \hat{\Pi}(z_t) \sum_{z^\tau} \hat{Q}_\tau(z^\tau|z_t) = \widetilde{Q}_t,$$

because:

$$\begin{aligned} & \sum_{z_t} \hat{\Pi}(z_t) \sum_{z^\tau} \hat{Q}_\tau(z^\tau|z_t) \\ = & \sum_{z^\tau} \hat{\pi}(z^\tau|z_t) \frac{\hat{\beta}_{t,\tau}(z^\tau|z_t)}{\widetilde{\beta}^{\tau-t}\widetilde{R}_{t,\tau}} = \frac{1}{\widetilde{R}_{t,\tau}}, \end{aligned}$$

where we have used the result in equation (59).Households take contingent bond positions to hedge against these interest rate/taste shocks. ■

**Corollary B.1.** *The average state-contingent bond portfolio is zero*

$$\sum \hat{\Pi}(z_{t-1}) \sum_{z_t} \hat{\pi}(z_t|z_{t-1}) \frac{\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)}{\lambda(z_t)} = 0$$

Proof of Corollary B.1:

*Proof.* Let $\widetilde{Q}_t$ denote the implied sequence of state prices in the constant discount factor economy.

$$
\begin{aligned}
\frac{\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)}{\lambda(z_t)} &= \hat{c}_t(y^t, \theta_0) - \eta(y_t) \\
&\quad + \sum_{\tau=t+1}^{\infty} \sum_{z^\tau, \eta^\tau} \hat{Q}_t(z^\tau | z_t) \left(\hat{c}_\tau(y^\tau, \theta_0) - \eta(y_\tau)\right) \\
&\quad - \sigma_{t-1}(y^{t-1}) \left[\hat{v}_t(z_t) + \alpha\right]. \\
&= \hat{c}_t(y^t, \theta_0) - \eta(y_t) + \\
&\quad \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \widetilde{Q}_\tau \sum_{\eta^\tau} \left(\hat{c}_\tau(y^\tau, \theta_0) - \eta(y_\tau)\right) \\
&\quad - \sigma_{t-1}(y^{t-1}) \left[\hat{v}_t(z_t) + \alpha\right] \qquad (60)
\end{aligned}
$$

Now, we know that the state prices in the ergodic economy and the actual economy satisfy:

$$\sum_{z_t} \hat{\Pi}(z_t) \sum_{z^\tau} \hat{Q}_t(z^\tau | z_t) = \widetilde{Q}_\tau$$

and, similarly, stock prices in the ergodic economy and the actual economy satisfy:–

$$\sum_{z^t} \hat{\Pi}(z_t) \left[\hat{v}_t(z_t) + \alpha\right] = \widetilde{\nu}_t$$

which implies that we can rewrite the following expressions as:

$$
\begin{aligned}
& \sum_{z_t} \hat{\Pi}(z_t) \sum_{\tau=t+1}^{\infty} \sum_{z^\tau, \eta^\tau} \hat{Q}_t(z^\tau | z_t) \left(\hat{c}_\tau(y^\tau, \theta_0) - \eta(y_\tau)\right) \\
=& \sum_{\tau=t+1}^{\infty} \sum_{\eta^\tau} \left(\hat{c}_\tau(y^\tau, \theta_0) - \eta(y_\tau)\right) \sum_{z_t} \hat{\Pi}(z_t) \sum_{z^\tau} \hat{Q}_t(z^\tau | z_t) \\
=& \sum_{\tau=t+1}^{\infty} \sum_{\eta^\tau} \left(\hat{c}_\tau(y^\tau, \theta_0) - \eta(y_\tau)\right) \widetilde{Q}_\tau \\
=& \sum_{\tau=t+1}^{\infty} \widetilde{Q}_\tau \sum_{\eta^\tau} \left(\hat{c}_\tau(y^\tau, \theta_0) - \eta(y_\tau)\right)
\end{aligned}
$$

and this in turn implies that:

$$\sum_{z^t} \hat{\Pi}(z_t) \frac{\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)}{\lambda(z_t)} = 0$$

because equation (60) reduces to the present discounted value formulation of the household budget constraint in the ergodic Bewley economy:

$$\begin{aligned}\sum_{z^t} \hat{\Pi}(z_t) \frac{\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)}{\lambda(z_t)} &= \hat{c}_t(y^t, \theta_0) - \eta(y_t) + \\ &\quad \sum_{\tau=t+1}^{\infty} \widetilde{Q}_\tau \sum_{\eta^\tau} (\hat{c}_\tau(y^\tau, \theta_0) - \eta(y_\tau)) \\ &\quad -\sigma_{t-1}(y^{t-1}) [\widetilde{\nu}_t + \alpha] = 0\end{aligned}$$

Note that if $\hat{\Pi}(z_t) = \hat{\pi}(z_t|z_{t-1})$, we are back in the i.i.d. case, in which case, the portfolio of contingent bonds has zero cost in all states $z_t$. This is what we expect, because in the i.i.d. case, all the state contingent bond positions are zero. Of course, this also implies that:

$$\hat{R}_{t-1}(z_{t-1}) \sum_{z^t} \hat{\Pi}(z_t) \frac{\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)}{\lambda(z_t)} = 0$$

Now, the actual cost of the state-contingent bond portfolio is given by:

$$\hat{R}_{t-1}(z_{t-1}) \sum_{z^t} \hat{\pi}(z_t|z_{t-1}) \frac{\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)}{\lambda(z_t)}$$

So, this implies that the actual cost of the state-contingent bond portfolio at time zero can be stated as:

$$\begin{aligned}& \hat{R}_{t-1}(z_{t-1}) \sum_{z^t} \hat{\pi}(z_t|z_{t-1}) \frac{\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)}{\lambda(z_t)} \\ = \; & \hat{R}_{t-1}(z_{t-1}) \sum_{z^t} \left[\hat{\pi}(z_t|z_{t-1}) - \hat{\Pi}(z_t)\right] \frac{\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)}{\lambda(z_t)}\end{aligned}$$

In the i.i.d case, obviously $\hat{\pi}(z_t|z_{t-1}) - \hat{\Pi}(z_t) = 0$ for all $z_t$ and the actual cost is zero. Consider the perfectly symmetric case in which $\hat{\Pi}(z_t) = \frac{1}{\#Z}$ and

$$\begin{aligned}\hat{\pi}(z_t|z_{t-1}) &= (1-\rho)\frac{1}{\#Z} \\ \hat{\pi}(z_t|z_{t-1}) &= (1-\rho)\frac{1}{\#Z} + \rho \text{ if } z_t = z_{t-1}\end{aligned}$$

Then the actual cost can be stated as follows:

$$\hat{R}_{t-1}(z_{t-1}) \rho \frac{\hat{a}_{t-1}(y^{t-1}, z_{t-1}, \theta_0)}{\lambda(z_{t-1})}$$

As the persistence of the aggregate shock $\rho$ decreases, the cost of the portfolio decreases in absolute value. $\blacksquare$

*Proof.* of Lemma 7.2: Suppose the stationary Bewley allocation $\{\widetilde{\hat{a}}_t(y^t), \hat{\sigma}_t(y^t)\}$ satisfies the constraint

$$\frac{\widetilde{\hat{a}}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t)\hat{v}_t \geq \hat{K}_t(y^t)$$

which seems the natural constraint to impose on the stationary Bewley economy. We want to show that the allocation for the stochastic Arrow economy satisfies the borrowing constraint if the allocation for the stationary Bewley economy does. Multiply both sides by $e_t(z^t)\hat{\phi}(z_{t+1})$ to obtain

$$\frac{\widetilde{a}_t(s^t, z_{t+1})\hat{\phi}(z_{t+1})}{\hat{R}_t\lambda(z_{t+1})} + \sigma_t(s^t)v_t(z^t)\hat{\phi}(z_{t+1}) \geq K_t(s^t)\hat{\phi}(z_{t+1})$$

Using the fact that

$$q_t(z^t, z_{t+1}) = \frac{\hat{\phi}(z_{t+1})}{\hat{R}_t\lambda(z_{t+1})}$$

and summing over all $z_{t+1}$ yields

$$\sum_{z_{t+1}} q_t(z^t, z_{t+1})\widetilde{a}_t(s^t, z_{t+1}) + \sigma_t(s^t)v_t(z^t) \geq K_t(s^t),$$

exactly the constraint of the stochastic Arrow economy, but the actual bond position differs from $\widetilde{a}_t(s^t, z_{t+1})$, because

$$\begin{aligned}\frac{\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)}{\lambda(z_t)} - \frac{\widetilde{a}_{t-1}(y^{t-1}, z_t, \theta_0)}{\lambda(z_t)} &= \sum_{\tau=t+1}^{\infty}\sum_{z^\tau}\left(\hat{Q}_\tau(z^\tau|z_t) - \widetilde{Q}_\tau\right)\sum_{\eta^\tau}(\hat{c}_\tau(y^\tau, \theta_0) - \eta(y_\tau)) \\ &\quad -\sigma_{t-1}(y^{t-1})\alpha\sum_{\tau=t+1}^{\infty}\sum_{z^\tau}\left(\hat{Q}_\tau(z^\tau|z_t) - \widetilde{Q}_\tau\right)\end{aligned}$$

So, we know that

$$\sum_{z_{t+1}} q_t(z^t, z_{t+1})a_t(s^t, z_{t+1}) + \sigma_t(s^t)v_t(z^t) \geq K_t(s^t) + \sum_{z_{t+1}} q_t(z^t, z_{t+1})\left(\begin{array}{c}\widetilde{a}_t(s^t, z_{t+1}) \\ -a_t(s^t, z_{t+1})\end{array}\right),$$

So, we can simply re-define the solvency constraints as :

$$K_t^*(s^t) = K_t(s^t) + \sum_{z_{t+1}} q_t(z^t, z_{t+1})\left(\begin{array}{c}\widetilde{a}_t(s^t, z_{t+1}) \\ -a_t(s^t, z_{t+1})\end{array}\right)$$

Next consider the state-by state wealth constraint. In the stationary Bewley economy we may impose

$$\hat{a}_t(y^t) + \hat{\sigma}_t(y^t)\left[\hat{v}_{t+1} + \alpha\right] \geq \hat{M}_t(y^t)$$

Multiplying by $e_{t+1}(z^{t+1})$ yields

$$\widetilde{a}_t(s^t, z_{t+1}) + \sigma_t(s^t)\left[v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z^{t+1})\right] \geq M_t(s^t, z_{t+1}) \text{ for all } z_{t+1}$$

By the same token, we can redefine:

$$M^*_{t+1}(s^{t+1}) = M_{t+1}(s^{t+1}) + \widetilde{a}_t(s^t, z_{t+1}) - a_t(s^t, z_{t+1})$$

Proof of Proposition 7.3: ■

*Proof.* We start by assuming the borrowing constraints do not bind, as before. We conjecture that the consumption allocations can be stated as

$$\frac{\hat{c}_{t+1}(y^t, y_{t+1})}{\mu(z_{t+1})}.$$

First, note that these allocations satisfy market clearing by construction because

$$\int \sum_{y^t} \widehat{\varphi}(y^t|y_0, z^t)\hat{c}_t(\theta_0, y^t)d\Theta_0 = \mu(z_t)$$

in each aggregate state, where

$$\widehat{\varphi}(y'|y) = \sum_{z'} \widetilde{\phi}(z')\varphi(y'|y, z').$$

and $\widetilde{\phi}(z') = \hat{\phi}(z')\frac{\mu(z')^\gamma}{\sum_{z'}\hat{\phi}(z')\mu(z')^\gamma}$

$$\widehat{\varphi}(y^t|y_0, z^t) = \widetilde{\phi}(z_t)\varphi(y_t|z_t, y_{t-1})\widehat{\varphi}(y^{t-1}|y_0, z^{t-1}),$$

Second, the Euler equations for the household problem, for the contingent claim are given by:

$$1 = \frac{\widehat{\beta}\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t, z_{t+1})} \sum_{y_{t+1}} \varphi(y_{t+1}|y_t, z_{t+1}) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \frac{\mu(z_{t+1})^\gamma}{\mu(z_t)^\gamma} \ \forall z_{t+1}.$$

We assume that the state prices can be written only as a function of the current aggregate shock $z_{t+1}$ and the last shock $z_t$ ::

$$\hat{q}_t(z_t, z_{t+1}) = \hat{\phi}(z_{t+1})\widehat{\beta} \sum_{y_{t+1}} \varphi(y_{t+1}|y_t, z_{t+1}) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \frac{\mu(z_{t+1})^\gamma}{\mu(z_t)^\gamma} \tag{61}$$

Then, in a second step, we check that these state prices together with the allocations satisfy the Euler equation. We know the interest rate $\hat{R}_t^{Be}$ in the Bewley economy is deterministic (i.e. it does not depend on the aggregate shocks $z$):

$$1 = \widetilde{\beta}\hat{R}_t^{Be} \sum_{z_{t+1}} \widehat{\varphi}(y_{t+1}|y_t) \sum_{y_{t+1}} \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))}$$

Now, we need to check that these conjectured allocations and the conjectured prices satisfy the Euler equations. Let us start with the Euler equation for the contingent claim, by summing across aggregate states $z_{t+1}$:

$$\begin{aligned}
\sum_{z_{t+1}} \hat{q}_t(z_t, z_{t+1}) &= \sum_{z_{t+1}} \hat{\phi}(z_{t+1})\widehat{\beta} \sum_{y_{t+1}} \varphi(y_{t+1}|y_t, z_{t+1}) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \frac{\mu(z_{t+1})^\gamma}{\mu(z_t)^\gamma} \\
\frac{\mu(z_t)^\gamma}{\hat{R}_t^A(z_t)} &= \sum_{z_{t+1}} \hat{\phi}(z_{t+1})\widehat{\beta} \sum_{y_{t+1}} \varphi(y_{t+1}|y_t, z_{t+1}) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \mu(z_{t+1})^\gamma \qquad (62) \\
\frac{\mu(z_t)^\gamma}{\hat{R}_t^A(z_t)} &= \widetilde{\beta} \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \frac{\mu(z_{t+1})^\gamma}{\sum_{z'} \hat{\phi}(z')\mu(z')^\gamma} \sum_{y_{t+1}} \varphi(y_{t+1}|y_t, z_{t+1}) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \\
\frac{1}{\hat{R}_t^{Be}} &= \widetilde{\beta} \sum_{y_{t+1}} \left[ \sum_{z_{t+1}} \hat{\phi}(z_{t+1})\varphi(y_{t+1}|y_t, z_{t+1}) \frac{\mu(z_{t+1})^\gamma}{\sum_{z'} \hat{\phi}(z')\mu(z')^\gamma} \right] \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \\
\frac{1}{\hat{R}_t^{Be}} &= \widetilde{\beta} \sum_{y_{t+1}} \widehat{\varphi}(y_{t+1}|y) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))}, \qquad (63)
\end{aligned}$$

where we have used the definition of the hatted transition probabilities:

$$\sum_{z_{t+1}} \hat{\phi}(z_{t+1})\mu(z_{t+1})^\gamma \varphi(y_{t+1}|y_t, z_{t+1}) = \widehat{\varphi}(y_{t+1}|y)$$

and the definition of the risk-free interest rate:

$$\widehat{\beta}\hat{R}_t^A(z_t) = \widetilde{\beta}\hat{R}_t^{Be}\mu(z_t)^\gamma$$

The expression in (**??**) is the Euler equation in the Bewley economy, which in turn means the Euler equation is satisfied. All that remains is to check that these Bewley allocations satisfy the budget constraint in the detrended Arrow economy. ■

Proof of Proposition 7.4:

*Proof.* Next ,we need to check that these allocations satisfy the budget constraint in the economy with time-varying discount factors. Let us take $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t) = 0,\ \hat{\sigma}_t(\theta_0, y^t)\}$ from the constant discount factor economy. Back out the implied state-contingent Arrow securities positions recursively, for each $y^t, z_t$, from the budget constraint:

$$\begin{aligned} \frac{\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)}{\lambda(z_t)} &= \frac{\hat{c}_t(y^t, \theta_0)}{\mu(z_t)} - \eta(y_t) \\ &+ \sum_{\tau=t+1}^{\infty} \sum_{z^\tau, \eta^\tau} \hat{Q}_t(z^\tau | z_t) \left( \frac{\hat{c}_\tau(y^\tau, \theta_0)}{\mu(z_\tau)} - \eta(y_\tau) \right) \\ &- \sigma_{t-1}(y^{t-1}) \left[ \hat{v}_t(z_t) + \alpha \right]. \end{aligned} \tag{64}$$

This makes sure that $\left\{\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0), \sigma_{t-1}(y^{t-1}, \theta_0)\right\}$ finance the consumption allocations $\{\frac{\hat{c}_t(\theta_0, y^t)}{\mu(z_\tau)}\}$. Next, we need to check that the bond market clears:

$$\begin{aligned} \int \sum_{y^t-1} \pi(y^{t-1}|y_0) \frac{\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)}{\lambda(z_t)} d\Theta_0 &= \\ \frac{1}{\lambda(z_t)} \sum_{y^t|y^{t-1}} \pi(y_t|y_{t-1}) \int \sum_{y^{t-1}} \pi(y^{t-1}|y_0) \hat{a}_{t-1}(y^{t-1}, z_t, \theta_0) d\Theta_0 &= \\ \int \sum_{y^t} \pi(y^t|y_0) \hat{a}_{t-1}(y^{t-1}, z_t, \theta_0) d\Theta_0 &= 0 \end{aligned}$$

for each $z_t$. We now multiply equation (54) by $\pi(y^t|y_0)$ and then sum over all $y^t$. For the first term we obtain, using the aggregate resource constraint,

$$\begin{aligned}
&\int \sum_{y^{t-1}} \pi(y^{t-1}|y_0) \sum_{y^t} \pi(y^t|y_{t-1}) \left( \frac{\hat{c}_t(y^t,\theta_0)}{\mu(z_t)} - \eta(y_t) \right) d\Theta_0 \\
= &\int \sum_{y^t} \pi(y^t|y_0) \left( \frac{\hat{c}_t(y^t,\theta_0)}{\mu(z_t)} - \eta(y_t) \right) d\Theta_0 = \alpha. \qquad (65)
\end{aligned}$$

By the same token we obtain

$$\begin{aligned}
&\int \sum_{y^{t-1}} \pi(y^{t-1}|y_0) \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \hat{Q}_t(z^\tau|z_t) \left( \frac{\hat{c}_\tau(y^\tau,\theta_0)}{\mu(z_\tau)} - \eta(y_\tau) \right) d\Theta_0 \qquad (66) \\
= &\sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \hat{Q}_t(z^\tau|z_t)\alpha.
\end{aligned}$$

Furthermore we know from the market clearing in the market for shares that

$$\int \sum_{y^{t-1}} \pi(y^{t-1}|y_0)\sigma_{t-1}(y^{t-1},\theta_0) \left[\hat{v}_t(z_t) + \alpha\right] d\Theta_0 = \left[\hat{v}_t(z_t) + \alpha\right], \qquad (67)$$

and finally we know that the share price can be written as follows:

$$[\hat{v}_t(z_t) + \alpha] = \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \hat{Q}_t(z^\tau|z_t)\alpha + \alpha, \qquad (68)$$

which in turn implies, substituting (55), (56) and (57) into (54) that

$$\frac{1}{\lambda(z_t)} \int \sum_{y^{t-1}} \pi(y^{t-1}|y_0)\hat{a}_{t-1}(y^{t-1}, z_t) d\Theta_0 = 0$$

Thus, each of the Arrow securities markets clears for the new trading strategies. ■

Proof of Proposition 7.5:

*Proof.* This follow immediately from

$$q_t(z^t, z_{t+1}) = \frac{1}{\hat{R}_t^{Be}} * \frac{\phi(z_{t+1})\mu(z_{t+1})^\gamma \lambda(z_{t+1})^{-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1})\lambda(z_{t+1})^{1-\gamma}} \frac{\sum_{y_{t+1}} \varphi(y_{t+1}|y_t, z_{t+1}) \frac{u'(\hat{c}_{t+1}(y^t,y_{t+1}))}{u'(\hat{c}_t(y^t))}}{\sum_{y_{t+1}} \widehat{\varphi}(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t,y_{t+1}))}{u'(\hat{c}_t(y^t))}}$$

Note that $q_t(z^t, z_{t+1})$ does not depend on $z^t$. ■

Proof of Proposition A.1:

*Proof.* First, we divided through by $e_t(z^t)$ on both sides in equation (37):

$$\begin{aligned}
\frac{V_t(s^t)}{e_t(z^t)} &= \left[(1-\beta)\frac{c_t^{1-\rho}}{e_t^{1-\rho}} + \beta\frac{(\mathcal{R}_t V_{t+1})^{1-\rho}}{e_t^{1-\rho}}\right]^{\frac{1}{1-\rho}} \\
\hat{V}_t(s^t) &= \left[(1-\beta)\hat{c}_t^{1-\rho} + \beta(\frac{\mathcal{R}_t V_{t+1}}{e_t})^{1-\rho}\right]^{\frac{1}{1-\rho}}.
\end{aligned} \tag{69}$$

Note that the risk-adjusted continuation utility can be stated as:

$$\begin{aligned}
\frac{\mathcal{R}_t V_{t+1}}{e_t(z^t)} &= \left(E_t\left(\frac{e_{t+1}}{e_t}\right)^{1-\alpha}\frac{V_{t+1}^{1-\alpha}}{e_{t+1}^{1-\alpha}}\right)^{1/1-\alpha} \\
&= \left(\sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}\hat{V}_{t+1}^{1-\alpha}(s_{t+1})\right)^{1/1-\alpha}
\end{aligned}$$

Next, we define *growth-adjusted* probabilities and the growth-adjusted discount factor as:

$$\hat{\pi}(s_{t+1}|s_t) = \frac{\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}}{\sum_{s_{t+1}}\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}} \text{ and } \hat{\beta}(s_t) = \beta\sum_{s_{t+1}}\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}.$$

and note that:

$$\begin{aligned}
\frac{\mathcal{R}_t V_{t+1}}{e_t(z^t)} &= \left(\sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}\hat{V}_{t+1}^{1-\alpha}(s_{t+1})\right)^{1/1-\alpha} \\
&= \left(\sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}\right)^{1/1-\alpha} \hat{\mathcal{R}}_t \hat{V}_{t+1}(s_{t+1})
\end{aligned}$$

Using the definition of $\hat{\beta}(s_t)$ :

$$\hat{\beta}(s_t) = \beta\left(\sum_{s_{t+1}}\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}\right)^{\frac{1-\rho}{1-\alpha}},$$

we finally obtain the desired result:

$$\hat{V}_t(s^t) = \left[(1-\beta)\hat{c}_t^{1-\rho} + \hat{\beta}(s_t)(\hat{\mathcal{R}}_t\hat{V}_{t+1}(s^{t+1}))^{1-\rho}\right]^{\frac{1}{1-\rho}}$$

As before, if the $z$ shocks are ii.d, then $\hat{\beta}$ is constant. ■

Proof of Proposition A.2:

*Proof.* First, we suppose the borrowing constraints are not binding, which is the easiest case. Assume the equilibrium allocations only depend on $y^t$, not on $z^t$. Then conditions 2.2 and 2.3 imply that the Euler equations of the Arrow economy, for the contingent claim and the stock respectively, read as follows:

$$1 = \frac{\hat{\beta}\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t, z_{t+1})}\sum_{y_{t+1}}\varphi(y_{t+1}|y_t)\left(\frac{\hat{c}_{t+1}(y^t, y_{t+1})}{\hat{c}_t(y^t)}\right)^{-\rho}\left(\frac{\hat{V}_{t+1}(y^{t+1})}{\hat{V}_t(y^t)}\right)^{\rho-\alpha} \quad \forall z_{t+1}$$

$$1 = \hat{\beta}\sum_{z_{t+1}}\hat{\phi}(z_{t+1})\left[\frac{\hat{v}_{t+1}(z^{t+1})+\alpha}{\hat{v}_t(z^t)}\right] \tag{70}$$

$$*\sum_{y_{t+1}}\varphi(y_{t+1}|y_t)\left(\frac{\hat{c}_{t+1}(y^t, y_{t+1})}{\hat{c}_t(y^t)}\right)^{-\rho}\left(\frac{\hat{V}_{t+1}(y^{t+1})}{\hat{V}_t(y^t)}\right)^{\rho-\alpha} \quad \forall z_{t+1} \tag{71}$$

In the first Euler equation, the only part that depends on $z_{t+1}$ is $\frac{\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t,z_{t+1})}$ which therefore implies that $\frac{\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t,z_{t+1})}$ cannot depend on $z_{t+1}$ : $\hat{q}_t(z^t, z_{t+1})$ is proportional to $\hat{\phi}(z_{t+1})$. Thus define $\hat{R}_t^A(z^t)$ by

$$\hat{q}_t(z^t, z_{t+1}) = \frac{\hat{\phi}(z_{t+1})}{\hat{R}_t^A(z^t)} \tag{72}$$

as the risk-free interest rate in the stationary Arrow economy. Using this condition, the Euler equation in (39) simplifies to the following expression:

$$1 = \hat{\beta}\hat{R}_t^A(z^t)\sum_{y_{t+1}}\varphi(y_{t+1}|y_t)\left(\frac{\hat{c}_{t+1}(y^t, y_{t+1})}{\hat{c}_t(y^t)}\right)^{-\rho} \tag{73}$$

$$\left(\frac{\hat{V}_{t+1}(y^{t+1})}{\hat{V}_t(y^t)}\right)^{\rho-\alpha} \tag{74}$$

First, notice that apart from $\hat{R}_t^A(z^t)$ noting in this condition depends on $z^t$, so we can choose $\hat{R}_t^A(z^t) = \hat{R}_t^A$. ■