Risk Sharing in Private Information Models with Asset Accumulation:

Explaining the Excess Smoothness of Consumption*

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Abstract

We derive testable implications of model in which first best allocations are not achieved because of a moral hazard problem with hidden saving. We show that in this environment agents typically achieve more insurance than that obtained under autarchy via saving, and that consumption allocation gives rise to 'excess smoothness of consumption', as found and defined by Campbell and Deaton (1987). We argue that the evidence on excess smoothness is consistent with a violation of the simple intertemporal budget constraint considered in a Bewley economy (with a single asset) and use techniques proposed by Hansen et al. (1991) to test the intertemporal budget constraint. We also construct a closed form example where the excess smoothness parameter has a structural interpretation in terms of the severity of the moral hazard problem. Evidence from the UK on the dynamic properties of consumption and income in micro data is consistent with the implications of the model.

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1 Introduction

This paper attempts to characterize the behavior of consumption and income in a class of private information models with asset accumulation and use this characterization to test these models empirically. Interest in this class of models is motivated by the failure of other workhorses in consumption theory: the complete market and the permanent income models.

The complete insurance hypothesis is soundly rejected by the data (e.g. Attanasio and Davis, 1996). On the other hand, the permanent income model assumes that the only mechanism available to the agents to smooth consumption is self insurance, which operates through personal savings, possibly with a single asset. In between these two characterizations, there are other possibilities were individuals have access to some state contingent mechanisms that provide insurance over and above self insurance. These models include models were the mechanisms through which insurance is obtained are left unspecified (that is models were one potentially enriches the set of securities available to an individual) to models were the mechanism is specified as arising from specific imperfections, such as lack of enforceability of contracts or private information.

The evidence on the PIH is mixed. Starting with Hall (1978), many authors have reported violations of some of the implications of the Life-Cycle PIH models. These violations often took the form of 'excess sensitivity' of consumption growth to expected changes in income. Some authors (see Attanasio 2000) argue that the so-called excess sensitivity of consumption can be explained away by non-separability between leisure and consumption and demographic effects. Others, such as Campbell and Deaton (1989), point out that consumption also seems to be 'excessively smooth' to be consistent with the PIH: that is consumption does not seem to react 'enough' to permanent innovation to income.

In a life-cycle model (or in a Permanent Income model with infinite horizon), consumption is basically pinned down by two equations. An intertemporal Euler equation, which relates expected changes in consumption over time to intertemporal prices, and an intertemporal budget constraint. The former equation is valid under a variety of circumstances: in particular, one does not need to specify the complete set of assets (contingent and not) that are available to a consumer. As long as one considers an asset for which the consumer is not at a corner, then there is an Euler equation holding for that particular asset, regardless of what the consumer is doing in other markets. Notice, however, that such an equation cannot be used, on its own, to solve, in closed form or numerically, for the level of consumption. Instead, to even write down the intertemporal budget constraint, which ultimately pins down the level of consumption, one has to specify the entire set of assets (state contingent and not) available to an individual.

If one assumes that consumers have only access to an asset with a specified interest rate to borrow and save, then, Campbell and Deaton (1989) show how excess sensitivity and excess smoothness can be related and, conditional on the intertemporal budget constraint holding, are essentially equivalent. Campbell and Deaton (1989), developing the ideas in Flavin (1981) and Campbell (1987), essentially use the fact that the Euler equation and the intertemporal budget constraint imply a set of cross-equation restrictions on a VAR for consumption and disposable income.

If, in the spirit of the Euler equation approach, we want to be agnostic about the set of assets available to a consumer, we can have situations where the Euler equation for a given asset is not violated and yet, the intertemporal budget constraint based on that single asset is violated in a way to imply 'excess smoothness' of consumption.

In this paper, we show how a model with imperfect risk sharing because of moral hazard can reconcile these facts. In particular, consumption will not exhibit excess sensitivity but, because of the additional insurance provided to consumers relative to a Bewley economy, gives rise to 'excess smoothness' in the spirit of Campbell and Deaton.

We develop a common framework that allow us to compare two dynamic models of asymmetric information with asset accumulation, which virtually exhaust the existing literature in dynamic contracting: the hidden income (adverse selection) model and the action moral hazard framework. When asset accumulation is fully observable the two models have essentially the same prediction on income and consumption data. Instead, when the agent has secrete access to credit markets Allen (1985) and Cole and Kocherlakota (2001) (ACK) show that in the adverse selection model the optimal allocation of consumption coincides with the one the agents would get by insuring themselves through borrowing and lending at a given interest rate. Abraham and Pavoni (2004) (AP) in contrast show that, in the action moral hazard model, the efficient allocation of consumption generically differs from that arising from self insurance.[1] Our empirical strategy exploits this marked discrepancy

[1] AP also show that, in its general formulation, the action moral hazard model actually nests the ACK

between the two allocations under hidden assets to disentangle the nature of the information imperfection most relevant in reality.

The fact that individual consumption satisfies an Euler equation is a key distinguishing feature of models with hidden assets respect to models of asymmetric information where the social planner has information on assets and, effectively, controls intertemporal trades (Rogerson, 1985; and Ligon, 1998). Because of incentive compatibility on saving decisions, both in ACK and AP the time series of individual consumption satisfies the usual Euler equation. This implies that, in both models, conditional on the past marginal utility of consumption, the current marginal utility of consumption should not react to predictable changes in known variables and therefore to predictable changes in income. Under certain assumptions on preferences, such as separability with leisure, the (log) marginal utility can be approximated by (log) consumption and therefore one obtains the standard excess sensitivity test (Flavin, 1981).

The difference between the two models arises in terms of the degree of insurance agents can achieve. In ACK, agents cannot insure more than in a standard PIH model with a fixed asset. In AP, agents can get some additional insurance. This additional insurance, while still maintaining the Euler equation, can only be achieved by violating the intertemporal budget constraint with a single asset. Another way of saying the same thing, is that the allocations in AP are equivalent to those that would occur if the agents had access to a certain set of state contingent trades, rather than only to a single asset with a fixed interest rate.

In the single asset version of the self-insurance/ACK model consumption moves one to one with permanent income. Hence, it should fully react to unexpected shocks to permanent income.[2] In terms of Campbell and Deaton (1989), consumption should not display excess smoothness. Since in the AP model consumers obtain some additional insurance relative to what they get by self insuring with saving, consumption moves only partially to innovations to permanent income, therefore exhibiting excess sensitivity.

In this paper we propose a couple of closed form specifications of the model (one with

model of adverse selection. This is at the essence of the common framework we propose here.

[2]When the agent has quadratic utility this result holds exactly for the onsumption and income in levels. If agent's utility is isoelastic the same implication can be derived for consumption and income in logs, using standard approximations (e.g., Deaton, 1992)

quadratic preferences, another with logarithmic utility) where the magnitude of the excess smoothness of consumption can be directly related (with only one parameter) to the degrees of control the agent has on public outcomes i.e., to the degree of private information the agent has. We also discuss how the model can be used to provide a structural interpretation of recent empirical evidence of Blundell et al. (2003).

To perform the empirical test we propose we use synthetic cohort data constructed from the UK Family Expenditure Survey (FES). Since we are addressing questions which are essentially dynamic in nature it is important to have a long time series. By aggregating at the quarterly level, we work with almost 120 observations per cohort. With this pseudo panel of cohort aggregated data on consumption and income we estimate the parameters of a time series model for individual income and consumption processes that can be used to perform the test proposed by Hansen, Roberds and Sargent (1991). We estimate this model using a bias corrected OLS estimator whose standard errors are computed by block bootstrap. Our findings are supportive of the implications we derive for the model with moral hazard and hidden assets. In particular, we find 'excess smoothness' of consumption, indicating that consumption does not fully react to permanent innovations to income.

2 Model

Consider an economy consisting of a large number of agents that are ex-ante identical, and who each live $T \leq \infty$ periods. The individual income follows the process

$$y_t = x_t + \xi_t$$

where x_t and ξ_t summarize respectively the permanent and temporary components of income shocks. We allow for moral hazard problems to the innovations to income. We assume that each agent is endowed with a private stochastic production technology which takes the following form:

$$x_t = f(\theta_t, e_t). \tag{1}$$

That is, the individual income shock $x_t \in X$ can be affected by the agent's effort level $e_t \in E \subset \Re$ and the shock $\theta_t \in \Theta \subset \Re$ which, consistently with previous empirical studies,[3]

[3]See, for example, Abowd and Card (1989), and Meghir and Pistaferri (2004).

is assumed to follow a martingale process of the form

$$\theta_t = \theta_{t-1} + v_t^p.$$

The $i.i.d$ shock v_t^p can be interpreted as a permanent shock on agent's skill level. In each period, the effort e_t is taken after having observed the shock θ_t. The function f is assumed to be continuous, and increasing in both arguments. Both the effort e and the shocks θ (hence ν) will be considered private information, while x_t is publicly observable. Similarly, we assume $\xi_t = g\left(v_t^T, l_t\right)$ where v_t^T is a iid shock[4] only observable by the individual, l_t is private effort (which is taken after the realization of v_t^T), and ξ_t is publicly observable.

Below, we provide a closed form where optimal effort is time constant, delivering an equilibrium individual income process of the form:

$$y_t = y_{t-1} + v_t^p + \Delta v_t^T.$$

This characterization of the income process is reasonably general and in line with the permanent/transitory representation of income often used in permanent income models. For expositional simplicity, below we focus on moral hazard problems in the innovation to permanent income. We hence assume $g \equiv 0$ and normalize l_t to zero. Later on, we will consider the general case.

The history of income up to period t will be denoted by $x^t = (x_1, ..., x_t)$, while the agent's private history of shocks is $\theta^t = (\theta_1, ..., \theta_t)$.

Agents are born with no wealth, have von Neumann-Morgenstern preferences, and rank deterministic sequences according to

$$\sum_{t=1}^{T} \delta^{t-1}\, u\left(c_t, e_t\right),$$

with $c_t \in C$ and $\delta \in (0,1)$. We assume u to be real valued, continuous, strictly concave, and smooth. Moreover, we require u to be strictly increasing in c and decreasing in e. Notice that, given a plan for effort levels there is a deterministic and one-to-one mapping between histories of the private shocks θ^t and x^t, as a consequence we are entitled to use θ^t alone. Denote by μ^t the probability measure on Θ^t and assume that the law of large numbers applies so that $\mu^t\left(A\right)$ is also the fraction of agents with histories $\theta^t \in A$ at time t.

[4] We could easily allow for v_t^T to follow an $MA(d)$ process.

Since θ^t are unobservable, we make use of the revelation principle and define a reporting strategy $\sigma = \{\boldsymbol{\sigma}_t\}_{t=1}^T$ as a sequence of θ^t-measurable functions such that $\boldsymbol{\sigma}_t : \Theta^t \to \Theta$ and $\boldsymbol{\sigma}_t(\theta^t) = \hat{\theta}_t$ for some $\hat{\theta}_t \in \Theta$. A truthful reporting strategy σ^* is such that $\boldsymbol{\sigma}_t^*(\theta^t) = \theta_t$ a.s. for all θ_t. Let Σ be the set of all possible reporting strategies. A reporting strategy essentially generates publicly observable histories according to $h^t = \sigma(\theta^t) = (\sigma_1(\theta_1), \ ..., \ \sigma_t(\theta^t))$, with $h^t = \theta^t$ when $\sigma = \sigma^*$.

An allocation (α, c, x) consists in a triplet $\{\mathbf{e}_t, \mathbf{c}_t, \mathbf{x}_t\}_{t=1}^T$ of θ^t-measurable functions for effort, consumption and income growths (production) such that they are 'technically' attainable

$$\Omega = \left\{(\alpha, c, x) : \forall\, t \geq 1,\ \theta^t,\ \mathbf{e}_t(\theta^t) \in E,\ \mathbf{c}_t(\theta^t) \in C \text{ and } \mathbf{x}_t(\theta^t) = f\left(\theta_t, \mathbf{e}_t(\theta^t)\right)\right\}.$$

The idea behind this notation is that incentive compatibility will guarantee that the agent announces truthfully his endowments (i.e. uses σ^*) so that in equilibrium private histories are public information. Resources feasibility implies that by the law of large numbers

$$\mathbf{E}[c_t] \leq \mathbf{E}[y_t], \text{ for all } t, \tag{2}$$

where the expectation is taken with respect to the measure μ^t on θ^t. Equation (2) pins down the interest rate in this economy. For simplicity we disregard aggregate shocks, however since we consider a production economy, the aggregate income level may change with t. We could have considered alternative set ups: we could have assumed an exogenously given level of the interest rate (considering a small open economy or a village economy where a money lender has access to an external market) or even a production economy where savings takes the form of capital.[5] For what we do below, how we close the model in this dimension is not particularly important.

3 Market Arrangements

Having specified agents' tastes and the technological environment they face, to characterize intertemporal allocations we need to specify the market arrangements in which they operate. We consider three different environments. The first two are exogenously given, while in the

[5]For a similar model in a small open economy see Abraham and Pavoni, 2004. For a simple analysis in a closed economy with capital see Golosov *et al.*, 2003.

third the type of trades that are feasible is derived from the details of the specific information imperfection we consider. The first environment we consider is that of complete contingent markets. We then move on to the opposite extreme and assume that the only asset available to agents is a bond with certain return. Finally, we consider two types of endogenously incomplete markets where trades satisfy incentive compatibility constraints.

3.1 Full Information

In the complete market model there is no private information problem. In this economy, the representative agent solves

$$\max_{(\alpha,c,x)\in\Omega} \mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1}\, u\left(c_t, e_t\right)\right],$$

$$\text{s.t. } \sum_t \int_{\Theta^t} \mathbf{p}_t(\theta^t)\left(\mathbf{c}_t(\theta^t) - \mathbf{y}_t(\theta^t)\right) d\theta^t \leq 0, \tag{3}$$

where $\mathbf{p}_t(\theta^t)$ is the (Arrow-Debreu) price of consumption (and income) in state θ^t, and for all $A \subset \Theta^t$, we have $\int_A \mathbf{p}_t(\theta^t)d\theta^t = q_t^0 \mu^t(A)$.[6] This is the price at which the agent both buys rights to $\mathbf{c}_t\left(\theta^t\right)$ units of consumption goods and sells (and commits to supply) $\mathbf{y}_t(\theta^t)$ units of the same good. q_t^0 is the period one price of a bond with maturity t, and $q_1^0 = 1$. The budget constraint faced by agents under complete markets makes it clear that they have available a very wide set of securities whose return is state contingent. This richness in available assets imply that agents' marginal utilities are equated across histories

$$u'\left(c_t(\theta^t), e_t(\theta^t)\right) = u'\left(c_s(\theta^s), e_s(\theta^s)\right) \text{ for all } t, s \text{ and } \theta^t, \theta^s. \tag{4}$$

Since there are no aggregate shocks one can restrict attention to equilibria where q_t^0 are deterministic function of time, and determined so that the resources feasibility condition

$$\int_{\Theta^t} \left(\mathbf{c}_t(\theta^t) - \mathbf{y}_t(\theta^t)\right) d\mu^t = 0$$

holds in each period.

[6] We have hence guessed the 'fair price' equilibrium. Under standard conditions such equilibrium always exists and it is robust to the presence of asymmetric information (e.g., Bisin and Gottardi, 1999). We will disregard all technical complications associated to the fact that we have allow for a continuum of values for θ, and assume that C and E are such that expectations are always well defined.

3.2 Permanent Income (Self Insurance)

We call permanent income or self-insurance the allocation derived from autarchy by allowing the agents to participate to a simple credit market. They do not have access to any asset other than a risk free bond. Let $\{q_t\}$ the sequence of one period bonds prices and $b = \{\mathbf{b}_{t+1}\}_{t=1}^{T}$ the plan of asset holding, where $\mathbf{b}_t$ is a $\theta^{t-1}-$measurable functions. We have

$$\sup_{b,(\alpha,c,x)\in\Omega} \mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1}\, u\left(c_t, e_t\right)\right] \tag{5}$$

subject to

$$\mathbf{c}_t(\theta^t) + q_t \mathbf{b}_{t+1}(\theta^t) \leq \mathbf{b}_t(\theta^{t-1}) + \mathbf{y}_t(\theta^t), \tag{6}$$

where $b_0 = 0$. As usual we rule out Ponzi games by requiring that $\lim_{t\to T} q_t^0 \mathbf{b}_t(\theta^t) \geq 0$. The constraint (6) is the budget restriction typically used in Permanent Income models when the agent has only access to a risk free bond market. For future reference, notice that this problem can be seen as an extension of the permanent-income model studied by Bewley (1977) which allows for endogenous labor supply and non stationary income. If one wants to consider the case in which the return on the bond is constant (may be because equation (2) is substituted by an open economy assumption), one can consider equation (6) with a fixed q, and $q_{t+1}^0 = (q)^t$.[7]

It is well known that one of the main implications of the self insurance model can be obtained by considering the following perturbation of the agent's consumption plan: reduce consumption infinitesimally at date t (node θ^t after $\mathbf{e}_t(\theta^t)$ has been taken), invest $\frac{1}{q_t}$ this amount for one period, then consume the proceeds of the investment at date $t+1$. If the consumption plan is optimal, this perturbation must not affect the agent's utility level. The first-order necessary condition for his utility not to be affected is the Euler equation:

$$u_c'(c_t, e_t) \geq \frac{\delta}{q_t} \mathbf{E}_t\left[u_c'(c_{t+1}, e_{t+1})\right], \tag{7}$$

where $\mathbf{E}_t\left[\cdot\right]$ is the conditional expectation operator on future histories given θ^t.

Another necessary condition that individual intertemporal allocations have to satisfy in this model can be derived by repeatedly applying the budget constraint (6) starting from any period $t \geq 1$ asset holding level b_t we have that almost surely (a.s.) for each given history

[7] Recall indeed that by arbitrage $q_{t+1}^0 = \prod_{n=1}^{t} q_n = q_t^0 q_t$.

$\bar{\theta}^T$ implying the asset level $b_t = \mathbf{b}_t(\bar{\theta}^{t-1})$ the following net present value condition (NPVC) must be satisfied

$$\sum_{n=t}^{T} \frac{q_{n-1}^0}{q_{t-1}^0} \left(\mathbf{c}_n(\bar{\theta}^n) - \mathbf{y}_n(\bar{\theta}^n) \right) \leq b_t. \tag{8}$$

Notice that, given the income process and the sequence $\{q_t\}$ equations (7) and (8) define (even when a closed form solution does not exist), consumption. It is interesting to compare equation (8) for $t = 1$ with equation (3). In the complete market case, the agent has available a wide array of state contingent securities that are linked in an individual budget constraint that sums over time and across histories, as all trades can be made at time 1. In the permanent income model, the agent has a single asset. Hence this restriction on trade is reflected in the presence of a different intertemporal budget constraint for each history of shocks. This structure also implies that in the complete market case, we will have a first order condition for each state contingent security which will hold exactly (see conditions (4)). Instead, in the permanent income model there is a single first order condition, equation (7), that holds in expectations.

3.3 Endogenously incomplete Markets

We will now consider a series of complete market economies with different assumptions on the degree of private information.

We have in mind an equilibrium concept à la Prescott and Townsend (1984a-b), and Kehoe and Levine (2001). We will therefore use the following definition of equilibrium.

Definition 1 *An equilibrium for economy i is an allocation (α, c, x) and a set of prices $p = \{\mathbf{p}_t(\theta^t)\}_{t=1}^T$ such that - given p - the agent maximizes his expected discounted utility subject to the (Arrow-Debreu) budget constraint, and incentive compatibility constraint IC_i, and all markets clear.*

One might interpret the different degree of asymmetric information as different market arrangements, i.e. as endogenous limitation on the 'set' of available assets. It is however important to notice that all equilibria will be (almost by construction) constrained efficient. The full information model can also be seen as a special case of this model, when the incentive compatibility constraints are not restrictive since effort is fully observable. It is easy to see indeed that in this case the only announcement strategy consistent with (α, c, x)

is σ^*. Below we show that the Permanent Income allocation can also be generated as the equilibrium outcome of a special case of the moral hazard model with hidden assets.

We assume that effort is not observable. Within this environment with imperfect information we consider two cases. In the first, private assets are observable. This is equivalent to considering a situation where there are no private assets and all savings are done by the planner. In the second, instead, private assets are hidden.

3.3.1 Moral Hazard with Monitorable Asset Holdings

Consider the case where each agent has private information on his/her effort level e, but there is full information on consumption and asset decisions, and trade contracts can be made conditional on these decisions. We define the expected utility from reporting strategy $\sigma \in \Sigma$, given the allocation $(\alpha, c, x) \in \Omega$ as

$$\mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1} u\left(c_t, e_t\right) \ \backslash \ (\alpha, c, x)\,; \sigma\right] = \sum_{t=1}^{T} \delta^{t-1} \int_{\Theta^t} u\left(\mathbf{c}_t\left(\sigma(\theta^t)\right), g\left(\theta_t, \mathbf{x}_t\left(\sigma(\theta^t)\right)\right)\right) d\mu^t(\theta^t)$$

where $g(x, \theta)$ represents the effort level needed to generate x when shock is θ, i.e., g is the inverse of f with respect to e keeping fixed θ. Since x is observable, the misreporting agent must adjust his/her effort level so that the lie is not detected.

The equilibrium allocation solves the following problem for the agent

$$\max_{(\alpha,c,x)\in\Omega} \mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1}\, u\left(c_t, e_t\right)\right],$$

$$\text{s.t. } \sum_t \int_{\Theta^t} \mathbf{p}_t(\theta^t)\left(\mathbf{c}_t(\theta^t) - \mathbf{y}_t(\theta^t)\right) d\theta^t \le 0, \tag{9}$$

together with the incentive compatibility constraint

$$\mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1}\, u\left(c_t, e_t\right) \ \backslash \ (\alpha, c, x)\,;\ \sigma^*\right] \ge \mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1}\, u\left(c_t, e_t\right) \ \backslash \ (\alpha, c, x)\,;\ \sigma\right] \tag{10}$$

for all $\sigma \in \Sigma$. The key difference between this problem and that of full information is the incentive constraint (10), which defines the set of allocations for which the agent will be induced to tell the truth and supplying the effort plan α.

In the additive separable case, $u(c, e) = u(c) - v(e)$, the key characteristic of the equilibrium allocation is summarized by (e.g., Rogerson, 1985; and Golosov *et al.*, 2003)

$$q_t \mathbf{E}_t\left[\frac{1}{u'(c_{t+1})}\right] = \delta \frac{1}{u'(c_t)}. \tag{11}$$

In order to relate (11) to the Euler equation (7), notice that the inverse $(1/x)$ is a strictly convex transformation. As a consequence, Jensen inequality implies

$$u'(c_t) \leq \frac{\delta}{q_t} \mathbf{E}_t \left[u'(c_{t+1}) \right],$$

with strict inequality if c_{t+1} is not constant with positive probability. That is, the optimality condition (11) is incompatible with the Euler equation (7). The optimal pure moral hazard contract tends to frontload transfers, and agents' consumption process behaves as it the agent were saving constrained. This consideration may play an important role in distinguishing this allocation from permanent income (see Ligon, 1998; and Section 4.6 below).

3.3.2 Moral Hazard with Hidden Asset Accumulation

Assume now that in addition to the moral hazard problem, agents have hidden access to a simple credit market and consumption is not observable (and/or contractable). The agents do not have private access to any other asset market. The equilibrium allocation must solve the following problem

$$\max_{(\alpha,c,x)\in\Omega} \mathbf{E} \left[\sum_{t=1}^{T} \delta^{t-1} \; u\left(c_t, e_t\right) \right],$$

$$\text{s.t.} \; \sum_t \int_{\Theta^t} \mathbf{p}_t(\theta^t) \left(\mathbf{c}_t(\theta^t) - \mathbf{y}_t(\theta^t) \right) d\theta^t \leq 0,$$

and the incentive compatibility constraint:

$$\mathbf{E} \left[\sum_{t=1}^{T} \delta^{t-1} \; u\left(c_t, e_t\right) \; \backslash \; (\alpha, c, x) \, ; \; \sigma^* \right] \geq \mathbf{E} \left[\sum_{t=1}^{T} \delta^{t-1} \; u\left(\hat{c}_t, e_t\right) \; \backslash \; (\alpha, c, x) \, ; \; \sigma \right] \text{ for all } \sigma \in \Sigma, \tag{12}$$

where the deviation for consumption $\hat{c}$ must be such that the new path of consumption can be replicated by a risk free bond, hence satisfy the self insurance budget constraint (6).[8] It is straightforward to see that the incentive constraint (12) (considered at σ^*) implies that the allocation (α, c, x) must satisfy

$$u_c'(c_t, e_t) \geq \frac{\delta}{q_t} \mathbf{E}_t \left[u_c' \left(c_{t+1}, e_{t+1}\right) \right], \tag{14}$$

[8] Formally, for any σ, a deviation $\hat{c}^\sigma$ is admissible if there is a plan of bond holdings $\hat{b}^\sigma$ such that for all t and a.s. for all histories θ^t we have

$$\hat{\mathbf{c}}_t^\sigma \left(\theta^t\right) = \mathbf{c}_t \left(\sigma(\theta^t)\right) + \hat{\mathbf{b}}_t^\sigma(\theta^{t-1}) - q_t \hat{\mathbf{b}}_{t+1}^\sigma(\theta^t), \tag{13}$$

and $\lim_{t\to T} q_t^0 \hat{\mathbf{b}}_{t+1}^\sigma(\theta^t) \geq 0$.

where the marginal utilities are evaluated at the equilibrium values dictated by (α, c, x). Notice that condition (14) essentially replicates (7). Clearly the Euler equation is consistent with many stochastic processes for consumption. In particular it does not says anything about the variance of c_t. For example, the full information model satisfies this conditions as well, and when preferences are separable c_t has zero variance. The key distinguishing feature between this allocation and the permanent income model is the fact that the former does not satisfy the (NPVC) (8). That is, the intertemporal budget constraint based on that *single asset* is violated.

4 Characterizing equilibria

In this section we consider the properties of the different market environments we considered and, in particular, that of endogenously incomplete markets. We have mentioned that the equilibrium allocations we consider are constrained Pareto efficient. This means that the equilibrium allocation (α, c, x) can be replicated by an incentive compatible plan of lump sum transfers $\tau = \{\boldsymbol{\tau}_t(\theta^t)\}_{t=1}^T$ that solves the constrained welfare maximization problem of a benevolent social planner who can transfers resources intertemporally at a rate q_t (dictated by the aggregate feasibility constraint). The optimal transfer scheme τ solves

$$\max_{\tau,(\alpha,c,x)\in\Omega} \mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1}\, u\left(c_t, e_t\right)\right]$$

s.t.

$$\mathbf{c}_t(\theta^t) = \mathbf{y}_t(\theta^t) + \boldsymbol{\tau}_t(\theta^t), \tag{15}$$

the incentive constraint (12), and the planner intertemporal budget constraint

$$\mathbf{E}\left[\sum_{t=1}^{T} q_t^0 \tau_t\right] \leq 0.$$

From condition (15) it is easy to see that the optimal allocation implies that the agents do not trade intertemporally ($b_t \equiv 0$). This is just a normalization. Alternatively, τ could be chosen so that the transfer $\tau_t = \boldsymbol{\tau}_t(\theta^t)$ represents the net trade on state contingent assets the agent implements at each date t node θ^t. In this case, resources feasibility would require $\mathbf{E}\left[\tau_t\right] = 0$.

4.1 Implementing the efficient allocation with income taxes

We now show that under some conditions the optimal allocation (α, c, x) can be 'implemented' using a transfer scheme τ which is function of income histories x^t alone. This will simplify the analysis and allow us to describe the consumption allocation in terms of observables.

Notice first, that through $x_t = \mathbf{x}_t(\theta^t)$ the x component of the optimal allocation generates histories of income levels x^t. Let's denote by $x^t(\theta^t) = (\mathbf{x}_1(\theta^1), ..., \mathbf{x}_t(\theta^t))$ this mapping. Clearly, in general this mapping is not invertible, as it might be the case that for a positive measure of histories of shocks θ^t we get the same history of incomes x^t. A generalization of the argument used in Kocherlakota (2006) however shows that it suffices to assume that the optimal plan of consumption c alone is x^t-measurable.[9] This is what we will assume thereafter.

Now notice that $\mathbf{y}_t$ is x^t-measurable by construction. As a consequence, from (15) is easy to see that the x^t-measurability of c implies that τ is x^t-measurable as well. From the transfer scheme τ, we can hence obtain the new x^t-measurable scheme τ^* as follows: $\boldsymbol{\tau}_t^*(x^t(\theta^t)) = \boldsymbol{\tau}_t(\theta^t)$. Given τ^*, let

$$\mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1}\, u(c_t^*, \hat{e}_t) \ \backslash\ \hat{\alpha}\right] = \sum_{t=1}^{T} \delta^{t-1} \int_{\Theta^t} u\left(\mathbf{c}_t^*\left(\hat{x}^t(\theta^t)\right), \hat{\mathbf{e}}_t\left(\theta^t\right)\right) d\mu^t(\theta^t)$$

where $\mathbf{c}_t^*(\hat{x}^t(\theta^t)) = \boldsymbol{\tau}_t^*(\hat{x}^t(\theta^t)) + \mathbf{y}_t^*(\hat{x}^t(\theta^t))$, and the new mapping is induced by $\hat{\alpha}$ as follows: $\hat{\mathbf{x}}_t(\theta^t) = f(\theta_t, \hat{\mathbf{e}}_t(\theta^t))$ for all t, θ^t. For any history of shocks θ^t a plan $\hat{\alpha}$ not only entails different effort costs, it also generates a different distribution over income histories x^t hence on transfers and consumption. This justifies our notation for the conditional expectation.

We say that the optimal allocation (α, c, x) can be *implemented* with x^t-measurable transfers if the agent does not have incentive to deviate from c^*, α^* given τ^*. The incentive constraint in this case is as follows:

$$\mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1}\, u(c_t^*, e_t^*) \ \backslash\ \alpha^*\right] \geq \mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1}\, u(\hat{c}_t, \hat{e}_t) \ \backslash\ \hat{\alpha}\right], \tag{16}$$

[9] That is, that there exists a sequence of x^t-measurable functions c^* such that for all t, θ^t we have $\mathbf{c}_t^*(x^t(\theta^t)) = \mathbf{c}_t(\theta^t)$. We will see below that under fairly general conditions the implementation idea of Kocherlakota (2006) extends to the general case with hidden savings. A more extensive formal proof of our argument is also available upon request.

where, as usual, the deviation path of consumption $\hat{c}$ mast be replicated by a plan of risk free bonds $\hat{b}$. An important restriction in the deviations $\hat{\alpha}$ contemplated in constraint (16) is that they are required to generate 'attainable' histories of $x's$, i.e. histories of $x's$ that can happen in an optimal allocation. The idea is that any off-the-equilibrium value for x^t will detect a deviation with certainty. One can hence set planner transfers to a very low value (perhaps minus infinity) in these cases, so that the agent will never have incentive to generate such off-the-equilibrium histories.

Now consider the agent choosing an effort plan $\hat{\alpha}$ so that the realized history $\hat{x}^t$ is attainable in equilibrium. This means both that there is a reporting strategy $\hat{\sigma}$ so that $\hat{x}^t = (\mathbf{x}_1(\hat{\sigma}), \mathbf{x}_2(\hat{\sigma}), ..., \mathbf{x}_t(\hat{\sigma}))$ and that given a consumption plan $\hat{c}$ the utility the agent gets is $\mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1} u(\hat{c}_t, e_t) \setminus (\alpha, c, x); \hat{\sigma}\right]$.[10] But then we are done since the incentive constraint (12) guarantees that the agent will chose the truth-telling strategy which implies the equilibrium plans α and c as optimal.

Below, we first present a specific economy where we get the 'Allen-Cole-Kocherlakota' (ACK) result. As stressed in AP, the crucial restrictions to obtain the 'self-insurance' result are on the way effort is converted into output. We then move on to relax these restrictions. Within the more general case, we consider a specific parametrization of the income process that allows us to obtain a closed form solution for the optimal transfers. While this example is useful because it gives very sharp predictions, some of the properties of the allocations we discuss generalize to the more general case and inform our empirical specification.

[10]More in detail, notice that by definition we have $\hat{\mathbf{e}}_t(\theta^t) = g(\theta_t, \hat{\mathbf{x}}_t(\theta^t))$. Since $\hat{\mathbf{x}}_t(\theta^t)$ is 'attainable' it can be induced from x by a 'lie', i.e., there exists a $\hat{\sigma}$ such that $\hat{\mathbf{x}}_t(\theta^t) = \mathbf{x}_t(\hat{\sigma}(\theta^t))$. But then $\hat{\mathbf{e}}_t(\theta^t) = g(\theta_t, \mathbf{x}_t(\hat{\sigma}(\theta^t)))$ and from the definition of τ^* we have $\mathbf{c}_t^*(\hat{\mathbf{x}}_t(\theta^t)) = \mathbf{c}_t^*(\mathbf{x}_t(\hat{\sigma}(\theta^t))) = \mathbf{c}_t(\hat{\sigma}(\theta^t))$, which implies that

$$\mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1} u(c_t^*, \hat{e}_t) \setminus \hat{\alpha}\right] = \mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1} u(c_t, e_t) \setminus (\alpha, c, x); \hat{\sigma}\right]$$

for some $\hat{\sigma} \in \Sigma$. Finally, to see that constraint (13) guarantees incentive compatibility of c^* is straightforward.

A final remark. One can easily show that under the same conditions, α must also be x^t-measurable, implying that there is no efficiency loss. The intuition is a follows. If α is not x^t-measurable it means that for at least two some θ^t, and $\bar{\theta}^t$ we have $\mathbf{e}_t(\theta^t) \neq \mathbf{e}_t(\bar{\theta}^t)$ while $f(\theta_t, \mathbf{e}_t(\theta^t)) = f(\theta_t, \mathbf{e}_t(\bar{\theta}^t))$. However, since u is decreasing in e, (static) effort incentive compatibility (together with $b_t \equiv 0$) implies that $\boldsymbol{\tau}_s(\theta^s) \neq \boldsymbol{\tau}_s(\bar{\theta}^s)$ for some $s \geq t$ with τ not x^s-measurable. A contradiction to the fact that the optimal transfer scheme τ is x^t-measurable.

4.2 The ACK Economy

It seems intuitive that our model nests that of Allen (1985). In order to clarify this link and introduce to our closed form, we derive the ACK result within our framework. Perhaps the analysis that follows also clarifies further the nature the ACK self-insurance result in terms of a the degree of asymmetric information in the economy. The following model builds on Allen's.

The first specification regards preferences: $u(c, e) = u(c - e)$. Since consumption and effort enter the utility function in a liner fashion, effectively they can be considered as essentially the same good. The second specification, crucial in order to obtains the self-insurance result, is the use of a production function which is linear in effort and separable in the shock (the linearity in θ is obviously irrelevant)

$$x_t = f(e_t, \theta_t) = \theta_t + e_t,$$

with $\Theta = (-\infty, \theta_{\max}]$ and $E = (-\infty, e_{\max}]$. Obviously, in this environment the plan α of effort levels will be indeterminate. We can hence set, without loss of generality, $e_t \equiv 0$. This normalization has two advantages. First, since e_t does not change with θ^t while $f(\theta_t, e_t)$ is strictly increasing in θ_t, all variations in θ_t will induce variations in x_t, automatically guaranteeing the x^t-measurability of c. Second, an added bonus of the constant effort is that we can focus on the risk sharing dimension of the optimal allocation. This last argument also motivates the modeling choice for our closed form solution below.

We saw above that incentive compatibility according to (16) with x^t-measurable transfers are necessary condition for optimality.[11] We now show that, in this case, incentive compatibility fully characterizes the efficient allocation.

Assume $T < \infty$ and consider the last period of the program. Using the budget constraint, given any history x^{T-1} we have

$$u(c_T - e_T) = u\left(\theta_T + \boldsymbol{\tau}_T^*\left(x^{T-1}, x_T\right) + b_T\right) = u\left(\theta_T + \boldsymbol{\tau}_T^*\left(x^{T-1}, \theta_T + e_T\right) + b_T\right).$$

The key aspect to notice here is that since utility depends on the effort choice only through the transfer, the agent will make to happen the income level x_T delivering the maximal

[11] In fact, since in this example x^t histories fully span θ^t histories the two incentive compatibility constraints define the same set of allocations.

transfer. Since this structure of production allows the agent to obtain any x_T from any θ_T this will always be possible. In addition, the preferences allow him to do it at no cost. In order to be incentive compatible, the transfer scheme must hence be invariant across x_T's.[12]

Now consider the problem in period $T-1$. Taking into account this invariance of τ_T from x_T, the incentive compatibility constraint for e_{T-1} becomes (we set $b_{T-1} = b_T = 0$ for expositional simplicity)

$$\begin{aligned} & u\left(\theta_{T-1} + \boldsymbol{\tau}^*_{T-1}(x^{T-2}, x_{T-1})\right) + \delta \mathbf{E}_{T-1} u\left(\theta_T + \boldsymbol{\tau}^*_T(x^{T-2}, x_{T-1})\right) \\ \geq\, & u\left(\theta_{T-1} + \boldsymbol{\tau}^*_{T-1}\left(x^{T-2}, \hat{x}_{T-1}\right)\right) + \delta \mathbf{E}_{T-1} u\left(\theta_T + \boldsymbol{\tau}^*_T\left(x^{T-2}, \hat{x}_{T-1}\right)\right) \text{ for all } \hat{x}_{T-1} \end{aligned}$$

This constraint says that given x^{T-2} and θ_{T-1}, the planner can only transfer deterministically across time. When the agent can save and borrow he/she will induce the x_{T-1} realization generating the largest $T-1$ net present value of transfers. In order to see it more easily assume the transfer scheme is differentiable and write the agent's first order conditions with respect to e_{T-1} evaluated at $e^*_T = e^*_{T-1} = 0:$

$$\frac{\partial \boldsymbol{\tau}^*_{T-1}\left(x^{T-1}\right)}{\partial x_{T-1}} + \delta \frac{\partial \boldsymbol{\tau}^*_T\left(x^T\right)}{\partial x_{T-1}} \mathbf{E}_{T-1}\left[\frac{u'\left(c^*_T\right)}{u'\left(c^*_{T-1}\right)}\right] = 0.$$

Notice that $\frac{\partial \boldsymbol{\tau}^*_T\left(x^T\right)}{\partial x_{T-1}}$ has been taken out from the expectation operator as we saw that $\boldsymbol{\tau}_T$ is constant in x_T shocks. From the Euler's equation - i.e. the incentive constraint for bond holding - we get

$$\mathbf{E}_{T-1}\left[\frac{u'\left(c^*_T\right)}{u'\left(c^*_{T-1}\right)}\right] = \frac{q_{T-1}}{\delta}$$

which implies $\frac{\partial \boldsymbol{\tau}^*_{T-1}(x^{T-1})}{\partial x_{T-1}} + q_{T-1} \frac{\partial \boldsymbol{\tau}^*_T(x^T)}{\partial x_{T-1}} = 0$, the discounted value of transfer must be constant across x_{T-1} as well, that is $\boldsymbol{\tau}^*_{T-1}(x^{T-1}) + q_T \boldsymbol{\tau}^*_T(x^T)$ is x^{T-2}-measurable. The fact that the agent faces the same interest rate of the planner implies that for any x^{T-1} incentive compatibility requires that the net present value of the transfers must be the same across x^{T-1}

[12] The incentive constraint for $e^*_T = 0$ is

$$\begin{aligned} & u\left(\theta_T + \boldsymbol{\tau}^*_T\left(x^{T-1}, \theta_T\right) + b_T\right) \\ \geq\, & u\left(\theta_T + \boldsymbol{\tau}^*_T\left(x^{T-1}, \theta_T + \hat{e}_T\right) + b_T\right) \text{ for all } \hat{e}_T \in E. \end{aligned}$$

histories, hence as a sole function of x^{T-2}. Going backward, we get that $\sum_{n=1}^{T-t} q_{t+n}^{0} \boldsymbol{\tau}_{t+n}^{*}(x^{t+n})$ is x^{t-1} measurable since it is a constant number for all history continuation $x_t, x_{t+1}, ..., x_T$.

Now recall that self insurance has two defining properties: first, it must satisfy the Euler equation. Second, it must satisfy the intertemporal budget constraint with one bond, i.e., the period zero net present value must be zero for all x^T.[13] Since the Euler equation is automatically satisfied here, the only way of obtaining a different allocation is that the transfers scheme τ^* permits to violate the agent's period zero self insurance intertemporal budget constraint for some history x^T. The previous argument demonstrates that it cannot be the case. If we normalize asset holding to zero, we indeed have $\tau_T^* = c_T^* - y_T^*$, $\tau_{T-1}^* = c_{T-1}^* - y_{T-1}^*$, and so on. The above argument hence implies that for all t the quantity

$$NPV_t = \sum_{n=1}^{T-t} q_{t+n}^{0} \boldsymbol{\tau}_{t+n}^{*}(x^{t+n}) = \sum_{n=1}^{T-t} q_{t+n}^{0} \left(\mathbf{c}_{t+n}^{*}(x^{t+n}) - \mathbf{y}_{t+n}^{*}(x^{t+n}) \right)$$

is x^{t-1} measurable. If we consider the $t = 1$ case, incentive compatibility implies that $NPV_1 \left(x^T \right) = \sum_{n=1}^{T} q_n^0 \left(c_n - y_n \right)$ is one number a.s. for any history x^T. Hence, in order to satisfy resources feasibility the planner is forced to set this number to zero.[14] This implies that the optimal contract derived using a relaxed first order version of the incentive constraint corresponds to the bond economy allocation. Since this allocation is obviously incentive compatible, it must be the optimal one.

The intuition for this result is as follows. First, since the agent can span the whole real line with his/her effort level implies that he/she has *full control* over the publicly observable outcome x. Moreover, the perfect substitutability between consumption and effort in the utility function on one side and between income and effort in production on the other side imply that the agent can substitute effort for income at *no cost*. Agent's preferences hence only depend on the planner transfers. In particular, as emphasized by ACK the free access to the credit market implies that what the agent cares about the transfers is their net present value, which must hence be constant across histories. Making the self insurance allocation

[13]Recall conditions (7) and (8) and the fact that there is a one-to-one onto mapping between x^t and θ^t histories in this model.

[14]Fornally, if for a positive measure set A of histories we have $NPV_1 \left(x^T \right) > 0 \; \forall x^T \in A$ there must be at least another positive measure set $\bar{A}$ of histories for which $NPV_1 \left(\bar{x}^T \right) < 0 \; \forall \bar{x}^T \in \bar{A}$ which contradicts the fact that NPV_1 is one number a.s. .

the only incentive feasible allocation. The result can hence be summarized as follows (see also AP).

Proposition 1 *Assume $T < \infty$ and that agents have perfect and costless control over publicly observable income histories. Then the efficient allocation coincides with self insurance. However, in more general specifications of the income process, the efficient allocation differs from self insurance.*

The first part of the proposition has been shown above. Notice that we never used the time series properties of θ_t. Indeed this result is pretty general, and as we will see below it also applies to the case with two type of shocks.[15] In order to see an example of what is stated in the second part of the proposition we will now generalize the production function to allow for nonlinearities.

4.3 The General case

Consider now the general case where the function f is left unspecified. Recall that at period t the agent objective function is:

$$u\left(c_t - e_t\right) = u\left(f\left(\theta_t, e_t\right) - e_t + \boldsymbol{\tau}_t\left(x^{t-1}, f\left(\theta_t, e_t\right)\right)\right)$$

To gain the most basic intuition, we start by considering the final period (T) of the model. The incentive constraint (IC) in the last period is

$$\begin{aligned} & u\left(f\left(\theta_T, e_T^*\right) - e_T^* + \boldsymbol{\tau}_T^*\left(x^{T-1}, f\left(\theta_T, e_T^*\right)\right)\right) \\ \geq\, & u\left(f\left(\theta_T, \hat{e}_T\right) - \hat{e}_T + \boldsymbol{\tau}_T^*\left(x^{T-1}, f\left(\theta_T, \hat{e}_T\right)\right)\right) \end{aligned}$$

which, in its first order condition form becomes

$$1 + \frac{\partial \boldsymbol{\tau}_T^*\left(x^{T-1}, x_T\right)}{\partial x_T} = \frac{1}{f_e'}. \tag{17}$$

Recall that in the ACK model we had $f_e' = \frac{\partial f\left(\theta_T, e_T^*\right)}{\partial e_T} = 1$. Since risk sharing requires $\frac{\partial \boldsymbol{\tau}_T^*\left(x^{T-1}, x_T\right)}{\partial x_T} < 0$ no insurance is possible in this environment. However, in general, f_e'

[15] ACK, they all study the case with *iid* shocks. It is worth noticing, however, that in the presence of liquidity constraints (the case considered by Cole and Kocherlakota), the extension to any persistence of shocks does not apply.

might be greater than one, and this is compatible with some risk sharing. We will see that this argument translates into a multiperiod setting as well.

What is the intuition for this fact? If the planner's aim is to make agents share risk, the key margin for an optimal scheme is to guarantee that the agent does not shirk. That is that she does not *reduce* effort. The value $\frac{1}{f'_e}$ in the right hand side of (17) represents the return (in terms of consumption) the agent derives by shirking so that to reduce output by one (marginal) unit. The left hand side is the net consumption loss: when the marginal tax/transfer is negative the direct reduction of one unit of consumption is mitigated by the increase in tax revenues. A large f'_e reduces shirking returns making easier for the planner to satisfy the incentive compatibility, hence to provide insurance.[16]

We now consider a model that uses this intuition heavily to deliver a closed form for the transfers, both in the static and dynamic environments. Consider first the last two periods. It is easy to see that the first order version of the effort incentive compatibility becomes

$$1+\frac{\partial \boldsymbol{\tau}^{*}_{T-1}\left(x^{T-1}\right)}{\partial x_{T-1}}+q\frac{\partial \boldsymbol{\tau}^{*}_{T}\left(x^{T}\right)}{\partial x_{T-1}}=\frac{1}{f'_e}.$$

The intuition is the same. The left had side represents the net cost of shirking while in the right hand side we have the agent's return from misbehaving. For the same reasons we explained above, when $f'_e>1$, the optimal scheme permits the net present value of transfers to decrease with x_T, allowing for some additional risk sharing on top of self insurance.

4.4 Closed Forms

Now consider a very special specification for f. Income x_t depends on exogenous shocks θ_t and effort e_t as follows:

$$x_t=f\left(\theta_t,e_t\right)=\theta_t+a\min\left\{e_t,0\right\}+b\max\left\{e_t,0\right\}, \tag{18}$$

[16] An equivalent intuition, based on the revelation game, suggests that f'_e may be related to the marginal cost of lying. Recall that in this model x_T is observable by the planner, and notice that when the agent gets the realization θ_T and declares $\hat{\theta}_T$ instead, it must adjust effort so that x_T is consistent with the declaration. If we denote $x_T\left(\hat{\theta}_T\right)-x_T\left(\theta_T\right)=\hat{\Delta}$. The agent obliviously enjoy $\frac{\partial \boldsymbol{\tau}^{*}_{T}\left(x^{T-1},x_T\right)}{\partial x_T}\hat{\Delta}$ but he is forced to reduce 'consumption' (c_T-e_T) by $\frac{f'_e-1}{f'_e}\hat{\Delta}$ in order to make $x_T\left(\hat{\theta}_T\right)$ to appear in place of $x_T\left(\theta_T\right)$. The quantity $\frac{f'_e-1}{f'_e}$ can hence be seen as the 'net cost' of lying.

with $a \geq 1 \geq b$. In Figure 1 we represent graphically the production function f in this case. Notice that when $a = b = 1$, one obtains the linear specification used to obtain the ACK result. Preferences are as in the previous section

$$u(c_t, e_t) = u(c_t - e_t)$$

with u quadratic. The budget constraint obviously does not changes

$$c_t = y_t + b_t + \tau_t - qb_{t+1}$$

where all t subscript variables are x^t measurable but b_t which is x^{t-1} measurable.

In Appendix A we show that in the case of a constant $q = \delta$, we get that the following simple closed form for transfers:[17]

$$(\mathbf{E}_{t+1} - \mathbf{E}_t)\left[\sum_{n=0}^{T-t-1} q^n \tau_{t+1+n}\right] = \left(\frac{1}{a} - 1\right)(\mathbf{E}_{t+1} - \mathbf{E}_t)\left[\sum_{n=0}^{T-t-1} q^n y_{t+1+n}\right]$$

Given the assumed process for skills, since $e_t \equiv 0$ we have

$$(\mathbf{E}_{t+1} - \mathbf{E}_t)\left[\sum_{n=0}^{T-t-1} q^n y_{t+1+n}\right] = \frac{1 - q^{T-t}}{1 - q}(\theta_{t+1} - \theta_t) = \frac{1 - q^{T-t}}{1 - q}\nu_{t+1}^p,$$

where ν_{t+1}^p is the permanent shock on income. When u is quadratic, the Euler's equation implies:

$$c_t = \mathbf{E}_t c_{t+s}$$

for all $s \geq 1$. From the budget constraint together with $b_t \equiv 0$, we get

$$\Delta c_{t+1} = \frac{(1-q)}{1 - q^{T-t}}(\mathbf{E}_{t+1} - \mathbf{E}_t)\left[\sum_{n=0}^{T-t-1} q^n y_{t+n+1}\right] + \frac{(1-q)}{1 - q^{T-t}}(\mathbf{E}_{t+1} - \mathbf{E}_t)\left[\sum_{n=0}^{T-t-1} q^n \tau_{t+n+1}\right].$$

The above expression for taxes hence delivers

$$\Delta c_{t+1} = \frac{1}{a}\frac{(1-q)}{1 - q^{T-t}}(\mathbf{E}_{t+1} - \mathbf{E}_t)\left[\sum_{n=0}^{T-t-1} q^n x_{t+1}\right] = \frac{1}{a}\nu_{t+1}^p.$$

Hence for $a = 1$ we are back to the PIH, for larger a we get some more risk sharing over and above self-insurance, with full insurance obtainable as a limit case for $a \to \infty$.

[17] It should be clear from the above analysis, that a very similar generalization of this formulation can still be obtained also when $q_t \neq \delta$ and q_t is changing deterministically with time

In this example, our ability to derive a closed form solution is driven by two factors. The assumption of quadratic utility (which, as is well known, allows one to derive closed form solution in a standard life cycle model) and the simple concavity assumed on the income process that takes the form of a piece-wise linear function. Such a simple function makes zero effort the optimal level the planner is trying to implement so that one can clearly separate the incentive and the risk sharing margins. The amount of risk sharing the planner can give the agent is that amount that does not induce the agent to shirk. If the planner tries to guarantee a bit more risk sharing, the agent will set effort equal to $-\infty$. For values of a close to one, the amount of risk sharing that can be implemented is not much and, in the limit, when a approaches unity, the planner cannot do better than self insurance. a is the consumption cost of one unit of effort: when is very large, that is when is very important to guarantee that effort is set to the optimal value, the planner can actually achieve first best.

The income process we have used is less restrictive than it looks at first sight. First, the fact that zero is the optimal level of effort, can be interpreted as a normalization. Second, at the optimal level of effort, the income process is actually identical to the standard income process used in the permanent income literature, with the innovation to permanent income equal to the random variable θ_t.

4.5 Introducing Temporary Shocks

We now allow for temporary shocks. Recall that $\xi_t = g\left(v_t^T, l_t\right)$. Let specify a production function for g of the form

$$\xi_t = g\left(v_t^T, l_t\right) = v_t^T + a^T \min\{l_t, 0\} + b^T \max\{l_t, 0\} \text{ with } a^T > 1 > b^T,$$

f as in (18), and the following agent's preferences over c_t, l_t and $e_t : u(c_t - l_t - e_t)$. We can now follow a similar line of proof than that adopted for the permanent shock (see Appendix A), and show that the reaction of consumption to the different shocks can be written as

$$\begin{aligned}\Delta c_{t+1} &= \frac{1}{a^p}\Delta x_{t+1} + \frac{1-q}{a^T}\Delta \xi_{t+1} \\ &= \frac{1}{a^p} v_{t+1}^p + \frac{1-q}{a^T} v_{t+1}^T,\end{aligned}$$

where, for consistency, we denoted by a^p the slope of f for $e_t \leq 0$.

Interestingly, our closed form with temporary shocks provides a structural interpretation of recent empirical evidence. Using the evolution of the cross sectional variance and covariance of consumption and income, Blundell et al. (2003) estimate two parameters representing respectively the fraction of permanent and temporary shocks reflected into consumption. Within our model such estimated parameters can be interpreted as the severity of informational problems to income shocks of different persistence.

4.6 The Case with Logarithmic Utility

Let us now go back to the simpler version of the model with only permanent shocks. For details on this case see Appendix B.

Assume that the agent has CRRA preferences of the following form:

$$\mathbf{E}_0 \sum_{t=0} \delta^t \frac{\left(C_t \cdot N_t^{-1}\right)^{1-\gamma}}{1-\gamma} \text{ for } \gamma > 1; \quad \text{and} \quad \mathbf{E}_0 \sum_{t=0} \delta^t \left(\ln C_t - \ln N_t\right) \text{ for } \gamma = 1.$$

and that the production function is represented by a small departure from the Cobb-Douglas:

$$X_t = \Theta_t e_t^a \text{ for } e_t \leq 1; \quad \text{and} \quad X_t = \Theta_t e_t^b \text{ for } e_t \geq 1, \quad \text{with} \quad a > 1 > b,$$

where

$$\ln \Theta_t \equiv \theta_t = \theta_{t-1} + v_t^p,$$

and v_t^p is normally distributed with zero mean and variance $\sigma^2_{v^p}$. We are able to obtain a very similar closed form for discounted taxes, which lead to the following permanent income formulation[18]

$$\Delta \ln c_{t+1} = \frac{1}{a} \Delta v_{t+1}^p + \frac{\gamma}{2a} \sigma^2_{v^p}.$$

The only difference it is that it is all expressed in logs. The presence of precautionary saving motive implies that the efficient allocation to displays increasing (log) consumption. Notice interestingly, that now $a > 1$ not only permits to reduce the cross sectional dispersion of consumption, it also mitigates the precautionary saving motives, hence the steepness of consumption (i.e., 'intertemporal dispersion').

[18]The derivation follows closely that in levels, and uses the log normality of shocks to θ_t in order to get a closed form expression for the Euler equation (in logs) and the discounted value of taxes. For $\delta = q$, when consumption is log normally distributed, we get

$$\mathbf{E}_t \Delta \ln c_{t+1} = \frac{\gamma}{2} \sigma_c^2,$$

where σ_c^2 is the conditional variance of $\Delta \ln c_{t+1}$ and γ is the coefficient of risk aversion.

Observable Assets. The further interesting aspect of this formulation in logs is that, under the same assumptions on preferences and technology, when assets are *monitorable,* for $\delta = q$ we get the following expression for expected consumption growth (see equation (11)):

$$\mathbf{E}_t \Delta \ln c_{t+1} = -\frac{\gamma}{2}\sigma_c^2,$$

i.e., the trend in log consumption is now *negatively* affected by consumption dispersion. On discrepancies on consumption patterns between our model and that with observable assets also see AP.

Before moving to the empirical specification, we note that even when the income process is more general than the one that yields a closed form solution, the general results that the planner can provide agents with more insurance than the Permanent Income model holds. This fact is important for our empirical approach.

5 Empirical Implications of the Model

The main intuition that comes from the ideas discussed in the previous sections is that, relatively to the standard permanent income model in which individuals can transfer resources over time using a single asset with a given interest rate, in a model with moral hazard and hidden saving the typical consumer is able to diversify some of its idiosyncratic risk even though the presence of the information asymmetries prevents first best allocations to be achievable. We also noticed that with hidden assets the Euler equation for consumption will always hold. As in the standard Permanent Income model allocations are completely pinned down by the Euler equation and the intertemporal budget constraint, if the consumer has to get some additional insurance while the Euler equation has to hold, it follows that the IBC has to be violated. And it has to be violated in a way that gives additional insurance: this implies having consumption higher than under the PIH when shocks are 'bad' and lower when shocks are 'good'. Notice also that this implies that the consumer reacts less than under the PIH to innovation to permanent income, that is consumption is 'excessively' smooth. In this sense our model can explain a result in the empirical literature on consumption: Campbell and Deaton (1989) and West (1988) had stressed the fact that consumption did not seem to be reactive enough to innovations to income.

The Euler equation (7) has been used extensively in testing the PIH. It can be derived and holds under a variety of situations. In particular,the Euler equation holds whenever the agent is not at a corner with respect to the decision of holding the asset whose return is considered in the Euler equation. It is important to stress that this results holds regardless of the availability of other assets or of the presence of imperfections in the trading of various assets. Situations where the equation does not hold include the presence of exogenous liquidity constraints (which effectively means that the agent is at a corner as she would like to borrow more than what is allowed on the market) and the model with asymmetric information and observed savings. In the latter situation, all saving is effectively done by the social planner who then allocates consumption to agents with a system of transfers that guarantees that incentive compatibility is satisfied. We discussed above that the efficient transfer scheme is such that the agent is willing to save. When assets are hidden, the set of incentive compatibility constraints will have to include the Euler equation with the interest rate provided by the hidden saving technology because such a margin is always available to the agents.

With a fixed interest rate, the Euler equation can be used to derive the standard random walk result that states that changes in (log) marginal utility are a random walk with drift. This result has been in turn used to derive the so-called excess sensitivity tests: consumption, conditional on lagged consumption should not be related on any other lagged variable, including those that help to predict income.

The quadratic utility case with a fixed interest rate has been extensively studied because this model yields a closed form solution. In such a situation, given a time series process for income, the intertemporal budget constraint and the Euler equation for consumption will induce a set of cross equation restriction son the joint representation of consumption and income. This is the strategy behind Flavin (1981) and, subsequently, Campbell (1987).[19]

Campbell and Deaton (1989) and West (1988), explored a different implication of the PIH hypothesis. In particular, they pointed out that, if current income has a permanent component, then consumption should fully adjust to innovations to such component, because they are fully reflected in permanent income. These studies then go on to show that, at least in aggregate data, consumption seems to be excessively smooth, in that it does not react to

[19]If one uses approximations, one can use similar ideas with more general utility functions (e.g., see Deaton, 1992).

innovations in permanent income. Campbell and Deaton (1989) also point out that, if one *takes for granted* the intertemporal budget constraint, excess smoothness can be related to excess sensitivity, in that the failure of consumption to react to innovations to permanent income can be recasted, if one imposes the intertemporal budget constraint, in terms of predictability of consumption changes with lagged information.

Our approach is obviously related to these papers. However, in our theoretical model, the intertemporal budget constraint with a single asset does not necessarily hold. Indeed, the additional insurance consumers get in our model relatively to a Bewley model is obtained by violating such intertemporal budget constraint: consumers with positive innovations to permanent income would consume quantities that are below what would be predicted by the PIH, while consumers with negative innovations would consume in excess of what would be predicted by the PIH. Notice that these deviations effectively imply what has been defined as excess smoothness of consumption. At the same time, however, consumption allocations would satisfy the Euler equation for consumption and therefore would not show 'excess sensitivity'. Notice also that the deviation of consumption allocations from the predictions of the standard PIH would be very different from the directions observed under binding liquidity constraints. There consumers that received negative innovations would consumer less than what is implied by the permanent income model.[20]

A study that explicitly tests the empirical implications of intertemporal budget constraints is the one by Hansen, Roberds and Sargent (1991) (HRS). HRS show two important results. First, given a consumption and income process and an asset that pays a fixed interest rate, it is not possible to test the implications of the intertemporal budget constraints without imposing additional structure. Second, and more importantly, they show that if the income process is exogenous and the consumption process is determined by a linear-quadratic model, so that consumption follows a martingale, the intertemporal budget constraint imposes additional restrictions on the joint process of income and consumption. In this sense the implications of the Euler equation (which informs many of the so-called excess sensitiv-

[20]Here we are beeing vague about the nature of innovations. In a standard PIH, if the end of life is distant enough, temporary shocks are completely absorbed and permanent shocks are completely reflected in consumption. With binding liquidity constraints it is possible that negative temporary shocks find their way into consumption. In a finite live context the issues are slighly more complicated.

ity tests), and the additional restriction implied by the intertemporal budget constraint are distinct.

Using the notation above, if preferences are quadratic, the test suggested by HRS amounts to estimating the following regression:

$$\gamma(L)y_t = \beta(L)(c_t - c_{t-1}) + \varepsilon_t \tag{19}$$

where $\beta(L)$ and $\gamma(L)$ are polynomials in the lag operator. The test, which, as HRS note implies the West (1988) variance bound test, consists in testing:

$$(1-q)\frac{\beta(q)}{\gamma(q)} = 1 \tag{20}$$

where $q = 1/(1+r)$. Clearly the test in (20) is very simple to implement and has recently been used by Nalewaik (2004) as a test of the PIH.[21]

The example we presented at the end of last section, in which we could derive a closed form solution for consumption is particularly interesting because it gives rise to time series representation for consumption and income that is substantially identical to that in equation (19). Indeed, in our case, if we consider the case with $\gamma(q) = 1 - q$, then we have that the PIH implies $\beta(q) = 1$, while the model we present implies $\beta(q) = a$. The extent of 'excess smoothness' has in our context, at least for this example, a structural interpretation. It represents the severity of the incentive problem. As we discuss above, when a is much larger than 1, one gets a considerable amount of risk sharing, and, in the limit case, one gets full insurance. This is because the cost of not implementing the optimal effort level is very high and, as the social planner finds it relatively easy to motivate agents to work hard, can provide more insurance. On the other extreme, when a is close to unity, the allocations are similar to those that one would observe under the PIH.

[21] Another paper that is related to our empirical strategy is Gali' (1991), who imposes the intertemporal budget constraint on top of the Euler equation for consumption so to derive a test that uses only data on consumption. The intertermporal budget constraint is used to derive the relationship between the spectral density at zero frequency of consumption and the variance of innovations to permanent income. While Gali' (1991), as Campbell and Deaton (1989), interpret his test as a test of the PIH (in various incarnations) this is because he takes for granted the intertemporal budget constraint with a single asset.

While the structural interpretation we have just given only holds for the particular example we have considered, the intuition about the relationship between excess smoothness and the trade-off between incentives and insurance is more general. Unless we are in a situation in which the optimal effort is at higher levels than what would prevail under the PIH (maybe because of some type of complementarity between shocks and effort) the intuition will be valid. It should also be noticed that while the specific income process we consider seems special, the income process we get *in equilibrium* from such an example is identical to the income process that is typically used in the PIH literature.

The fact that assets are unobservable makes the moral hazard models we are considering very close to the PIH, which is one aspect of the ACK results. As we mentioned above we can re-write the intertemporal budget constraint faced by a consumer as a sequence of period to period budget constraints where the consumer is given the possibility of buying state contingent assets or in terms of the planners made by the social planner:

$$\tau_t(x^t) + y_t - c_t + b_t \geq q_t b_{t+1} \tag{21}$$

where b_t is the amount held in the hidden saving technology, which pays an exogenously given return r, and $\tau_t(x^t)$ is the state contingent transfer that the consumer receives from the social planner. Such sequence of transfers and the corresponding consumption allocations will satisfy the incentive compatibility constraint (14). The advantage of re-writing the intertemporal budget constraint as (21) are two. First, this formulation makes it clear that the Euler equation

$$u_c'(c_t, e_t) \geq \frac{\delta}{q_t} \mathbf{E}_t \left[u_c'(c_{t+1}, e_{t+1}) \right], \tag{22}$$

must be part of the incentive compatibility constraints. Moreover, many of the standard results one gets for the standard Permanent Income model can be applied here, considering the transfer as a part of the income process. Notice that, as long as the utility function is additively separable in effort and consumption, the fact that income and the transfer are endogenously determined does not matter.[22] For instance, suppose that utility is additively separable and that utility in consumption is quadratic. Define with $\widetilde{y}_t$ total income, including the (net) transfer $\tau_t(x^t)$, $\widetilde{y}_t = \tau_t(x^t) + y_t$. Suppose also that the return on the (hidden) storage

[22] Of course, the same is true for our closed form specification, since effort is constant in equilibrium.

technology is constant and equal to $\frac{1}{q} - 1$. Then consumption at time t will be given by:

$$c_t = \phi \mathbf{E}_t \left[\sum_{j=1}^{\infty} q^j \widetilde{y}_{t+j} \right]. \tag{23}$$

with $\phi = (1-q)$. Equation (23) is derived using the Euler equation for quadratic utility and the intertemporal budget constraint (21) and implies that changes in consumption are equal to the present discounted value of revised expectations about future values of $\widetilde{y}_t$. This implies an important property of the optimal incentive compatible transfer. Even though the transfer itself might depend in a complex way on the past history of shocks, the change in the transfer from one period to another reflects only the properties of changes in income. For instance, if income is i.i.d. (so that changes in income are an MA(1)), the changes in transfer are an MA(1) as well (given by the sum of a white noise process - the innovation to consumption- and an MA(1)). Equation (23) also makes it clear why the test of the intertemporal budget constraint is informative in this context. One should not get excess smoothness or a violation of the intertemporal budget constraint if one were to use income *net of the planner transfers* $\widetilde{y}_t = \tau_t(x^t) + y_t$ as the definition of income.

6 Empirical strategy

To implement the strategy sketched above, we start from a specification of the individual income process:

$$\Delta y_t^h = \beta(L) \Delta c_t^h + \varepsilon_t^h \tag{24}$$

where the index h refers to the household, and the index t to time periods. y is income, c is consumption and ε is the innovation to the process. $\beta(L)$ is a polynomial in the lag operator of order p. An equation such as (24) cannot be estimated on individual data using time series of independent cross sections, because we do not observe the same individuals over time. The lack of longitudinal individual data implies that we cannot compute either the changes in consumption nor the changes in income at the individual level. However, following Deaton (1985) and Browning, Deaton and Irish (1985), we can aggregate equation (24) over individuals belonging to a certain group with fixed membership. This gives us:

$$\Delta y_t^g = \beta(L) \Delta c_t^g + \varepsilon_t^g \tag{25}$$

where the subscript g indicates the mean of a certain variable among the individuals belonging to group g. While the variables in equation (25) are not observable, we can estimate them. In particular, given a time series of repeated cross sections, we can, at each date, compute sample average income and consumption for each group at each point in time. We can then substitutes these estimates in equation (25) to obtain:

$$\Delta\overline{y}_t^g = \beta(L)\Delta\overline{c}_t^g + \overline{\varepsilon}_t^g + \Delta v_t^{yg} - \beta(L)\Delta v_t^{cg} \tag{26}$$

where a bar indicates the sample mean of a variable and v_t^{yg} and v_t^{cg} represent the difference between the sample and population mean for income and consumption, respectively, at time t for group g. Notice that, while we obviously do not observe v_t^{cg}or v_t^{yg} we know that they have zero mean and a variance that can be estimated using the variability within each group-year cell. In particular, if consumption (income) for group g at time t has a variance of σ_{cgt}^2 (σ_{ygt}^2),then the variance of v_t^{cg} (v_t^{yg}) will be σ_{cgt}^2/N_{gt} (σ_{ygt}^2/N_{gt}).As the error is driven by the fact that the sample changes over time, it is likely that v_t^{cg} and v_t^{yg} are correlated. As in the case of the variances, the covariance between consumption and income errors can be estimated from the variability within cell and will be denoted with σ_{cygt} and will be scaled by the cell size. Notice also that the presence of 'measurement error' on the population means in levels, creates a complicated $MA(p+1)$ structure for the residuals of equation (26) in which first difference data enter. Finally, given that repeated cross sections are constituted by independent samples, it is realistic to assume that the errors in the estimation of the means at t are independent from the errors at other times.

While equation (24) could be estimated (if individual panel data were available) by OLS, estimation of equation (26) needs to take into account the presence of measurement error both on the left and on the right hand side. There are two possible strategies. On the one hand, we could try to use an instrumental variable approach that would try to find instruments for $\Delta\overline{c}_t^g$ that are uncorrelated with $\Delta v_t^{yg} - \beta(L)\Delta v_t^{cg}$. For instance, if we had two independent samples from the same population, one could use the information from one to instrument the other. Alternatively, as we have information on the relevant variances and covariances, one can try to use it to correctly directly the biases in the OLS estimation.

If we have information on G groups, let's denote with θ a $p+1$ vector that contains the coefficients of the polynomial $\beta(L)$, with Y the $TGx1$ vector containing changes in income,

Z the $TGx(p+1)$ matrix that contains the current changes in consumption and with u the $TGx1$ vector containing the composite error of equation (26), we can write such an equation as[23]:

$$Y = Z\theta + u$$

The OLS estimator will be equal to:

$$\widetilde{\theta} = \theta + (Z'Z)^{-1}Z'u.$$

Such an estimator is inconsistent, of course, because of the correlation between Z and u. If we consider the covariance between a generic z_t of Z and u_t,we have:

$$Cov(z_t, u_t) = \left\{ \begin{array}{c} Cov(\Delta v_t^{cg}, \Delta v_t^{yg}) - \beta_0 \; Var(\Delta v_t^{cg}) - \beta_1 \; Cov(\Delta v_t^{cg} \Delta v_{t-1}^{cg}) \\ Cov(\Delta v_{t-1}^{cg}, \Delta v_t^{yg}) - \beta_0 \; Cov(\Delta v_t^{cg}, \Delta v_{t-1}^{cg}) - \beta_1 \; Var(\Delta v_{t-1}^{cg}) \\ -\beta_1 Cov(\Delta v_{t-2}^{cg}, \Delta v_{t-1}^{cg}) - \beta_2 \; Var(\Delta v_{t-2}^{cg}) - \beta_3 \; Cov(\Delta v_{t-2}^{cg} \Delta v_{t-3}^{cg}) \\ -\beta_2 Cov(\Delta v_{t-3}^{cg}, \Delta v_{t-2}^{cg}) - \beta_3 \; Var(\Delta v_{t-3}^{cg}) - \beta_4 \; Cov(\Delta v_{t-3}^{cg} \Delta v_{t-4}^{cg}) \\ \ldots \\ -\beta_{p-2} Cov(\Delta v_{t-p+2}^{cg}, \Delta v_{t-p+1}^{cg}) - \beta_{p-1} \; Var(\Delta v_{t-p+1}^{cg}) - \beta_p \; Cov(\Delta v_{t-p+1}^{cg} \Delta v_{t-p}^{cg}) \\ -\beta_{p-1} Cov(\Delta v_{t-p+1}^{cg}, \Delta v_{t-p}^{cg}) - \beta_p \; Var(\Delta v_{t-p}^{cg}) \end{array} \right\} \tag{27}$$

Notice that all the elements on the right-hand side of equation (27) can be estimated using the information on within cell variability. In particular, given that samples from different time periods are different, we have:

$$\begin{aligned} Var(\Delta v_t^{cg}) &= Var(v_t^{cg}) + Var(v_{t-1}^{cg}), \quad t = 2, ..., T \\ Cov(\Delta v_t^{cg}, \Delta v_{t-1}^{cg}) &= -Var(v_{t-1}^{cg}), \quad t = 2, ..., T \\ Cov(\Delta v_t^{cg}, \Delta v_t^{yg}) &= Cov(v_t^{cg}, v_t^{yg}) + Cov(v_{t-1}^{cg}, v_{t-1}^{yg}) \\ Cov(\Delta v_{t-1}^{cg}, \Delta v_t^{yg}) &= -Cov(v_{t-1}^{cg}, v_{t-1}^{yg}) \end{aligned}$$

Therefore, we can use these formulae to modify the OLS estimator and obtain a bias corrected version:

$$\widehat{\theta} = A^{-1}[\widetilde{\theta} - B] \tag{28}$$

[23] We are assuming a balanced pseudo panel. The formulae we derive below can be easily generalized to the case in which the pseudo panel is unbalanced.

where:

$$B = (Z'Z)^{-1}\Gamma = (Z'Z)^{-1}\left\{\begin{array}{c} \frac{1}{T-1}\sum_{t=2}^{T}\frac{\sigma_{cygt}}{N_{gt}} + \frac{\sigma_{cygt-1}}{N_{gt-1}} \\ -\frac{1}{T-1}\sum_{t=2}^{T}\frac{\sigma_{cygt-1}}{N_{gt-1}} \\ 0 \\ ... \\ 0 \end{array}\right\}_{p+1x1}$$

and

$$A = \left[I - (Z'Z)^{-1}\Omega\right]$$

$$\Omega = \left\{\begin{array}{cccccc} a & b & 0 & ... & 0 & 0 \\ b & a & b & ... & 0 & 0 \\ 0 & b & a & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & a & b \\ 0 & 0 & 0 & ... & b & a \end{array}\right\}_{p+1xp+1}$$

where $a = \frac{1}{T-1}\sum_{t=2}^{T}\left(\frac{\sigma^2_{cgt}}{N_{gt}} + \frac{\sigma^2_{cgt-1}}{N_{gt-1}}\right)$ and $b = -\frac{1}{T-1}\sum_{t=2}^{T}\left(\frac{\sigma^2_{cgt-1}}{N_{gt-1}}\right)$.

So far we have considered the case where equation (24) does not have any regressors in addition to consumption changes and its lags. However, it is straightforward to consider the case in which there are such variables, possibly unaffected by measurement error. An example of such variables one might want to consider are seasonal effects, which are likely to be important determinants of consumption. In this case, one would simply partition the Z matrix as well as the matrices that contain the variances and covariances of measurement error, inserting the appropriate number of zeros.

Given the expression for the biased corrected estimator, one can work out an expression for its asymptotic variance-covariance matrix. This will take into account that the residuals u are correlated over time (and, in the case of multiple cohorts, across cohorts for contemporaneous time periods). If we denote with $\Theta = E[u\ u']$, then the variance covariance of the estimator in (28).is given by:

$$TV(\widehat{\theta} - \theta) = (Z'Z)^{-1}\left[Z'\Theta Z - (\Gamma' + \theta'\Omega)(\Gamma + \Omega\theta)\right](Z'Z)^{-1}$$

While this expression for the asymptotic variance-covariance matrix is relatively straightforward, given its complex structure (and in particular the complex MA framework), there is no guarantee that in small sample the estimated matrix is positive definite. Rather than implementing some arbitrary filter, we decided to compute the standard errors by bootstrapping the sample used in estimation. In doing so we use a block-bootstrap procedure to take into account the time dependence of the residuals (and the fact that, presumably, contemporaneous observations are correlated across cohorts). The mechanics of the bootstrapping is slightly complicated by the fact that the pseudo panel we consider is unbalanced. The bootstrap is performed so to maintain the structure of the sample in this respect.

7 Data and empirical results.

To perform the exercise described above we use the UK Family Expenditure Survey from 1974 to 2000. In particular, we use data on households headed by individuals born in the 1930s, 1940s, 1950s and 1960s to form pseudo panels for 4 year of birth cohorts. As we truncate the samples so to have individuals aged between 25 and 60, the four cohorts form an unbalanced sample. The 1930s cohort is observed over later periods of its life cycle and exits before the end of our sample , while the opposite is true for the 1960s cohort.

The data, which has been used in many studies of consumption (see, for instance, Attanasio and Weber, 1993), contains detailed information on consumption, income and various demographic and economic variables. We report results obtained as 'consumption' our measured of expenditure on non durable items and services, in real terms and divided by the number of adult equivalents in the households (where for the latter we use the McClemens definition of adult equivalents). Income is after tax and cum benefits total household income. More details on the data are available upon request. The FES data are used to construct quarterly time series.

The results we obtain are reported in Table 1, which reports, for each coefficient of the polynomial in the lag operator, its estimate and the corresponding standard errors. The Table also reports a test of the hypothesis that $\beta(q) = 1$.

Table 1

	estimate	standard error
β_0	1.304	0.158
β_1	0.042	0.184
β_2	0.172	0.137
β_3	0.299	0.161
β_4	0.350	0.182
β_5	0.115	0.181
$\beta(q)$	2.3329	0.6087

Seasonal dummies included in the regressions

The test of the hypothesis that $\beta(q) = 1$ is rejected with a p-value of 0.025. On the other hand, we do not reject standard excess sensitivity tests using the same data. Our estimations are based on an annual interest rate of 2.5%. For quarterly data this implies a discount factor of about $q = .99385$. The punctual estimation for $\beta(q)$ are hence consistent with a production parameter $a = \beta(q) = 2.3329$.

8 Conclusions

In in this paper we discuss the empirical implications of a model where perfect risk sharing is not achieved because of information problems. In particular, we consider a combination of moral hazard and information problems on assets (and therefore consumption). Developing results by Abraham and Pavoni (2004), we show that in such a model we obtain 'excess smoothness' in the sense of Campbell and Deaton (1989), while we do not get excess sensitivity. The latter result follows from the fact that an individual Euler equation always holds because of the presence of hidden assets. The presence of excess smoothness follows from the fact that, even in the presence of moral hazard and hidden assets, in general, an efficient competitive equilibrium is able to provide some insurance over and above what individuals achieve on their own by self insurance. The constrained efficient allocations that prevail in the environment we consider differ from those one would obtain in a Bewley model with no insurance, in that the agents are able to share some risk and are not forced to rely only on self-insurance. This additional insurance is what generates excess smoothness in consumption, which can then be interpreted as a violation of the intertemporal budget constraint with a single asset. What such an equation is neglecting is the state contingent Arrow Debreu security that the agent can purchase, in addition to the riskless asset, in a constrained efficient equilibrium. Or, if one prefers the metaphor of the social planner, to get the standard PIH results, one should be considering income net of transfers, rather than the standard concept used in the PIH literature.

In the empirical application we provide some evidence in favor of this class of models. We exploit the intuition above and implement the test of the intertemporal budget constraint suggested by Hansen, Roberds and Sargent (1991). We also propose a simple example in which one can obtain a closed form solution for the social planner transfer and in which the excess smoothness parameter estimated in our empirical exercise can be interpreted as being related to the severity of the moral hazard problem.

The empirical application, which is one of the first exercises that estimates a time series model for consumption and income on micro data, obtains estimates that indicate a substantial amount of excess smoothness.

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9 Appendix A: Closed Form

The outcome of this section will be a structural interpretation, in terms of the marginal cost/return of effort, of the coefficient ϕ coming from a generalized permanent income equation of the form:

$$\Delta c_t = \phi \Delta y_t^p.$$

We will have

$$\phi = \frac{1}{a}$$

where $a \geq 1$ and $\frac{1}{a}$ is the marginal return to shirking. Since in our model wealth effects are absent, it will deliver a constant effort level in any period, which will be normalized to zero. Zero effort will also be the first best level of effort. So the whole margin in welfare will derive from risk sharing. The incentive compatibility constraint will hence dictate the degree of such insurance as a function of the marginal cost of effort. A lower effort cost/return will allow the planner to insure a lot the agent without inducing him to shirk. And the planner will use the whole available margin to impose transfers and obtain consumption smoothing.

For didactical reasons, we will now solve the model is steps, with increasing degree of complication. Upon request we have available the proof for a more general degree of persistence of income shocks.

9.1 The Model

Consider the following specification of the technology[24]

$$y_t = f(\theta_t, e_t) = \theta_t + a \min\{e_t, 0\} + b \max\{e_t, 0\}, \quad \text{with} \quad a \geq 1 \geq b, \tag{29}$$

and $e \in (-\infty, e_{\max}]$. In other words, the production function has a kink at zero.[25] Interestingly, as we have seen in Section (4.2) for $a = 1$ we are in the standard ACK case (this is

[24] It will be clear soon that the linearity of f for $e > 0$ is not crucial for the closed form, as long as the slope of f is uniformly bounded above by one in this reagion.

[25] None of the results change if we choose the kink at any other $\bar{e} > -\infty$.

true even when $b < 1$) hence there is no room for risk sharing at all (on top of self insurance) and the model replicates the Bewley model.

Finally, notice that as long as $a > 1$ (and $b < 1$) the first best effort level would be zero. However, the first best allocation would also imply a constant consumption. This allocation can only be obtained by imposing a constant tax rate such that $\tau_t' = -1$. Obviously, this allocation is not incentive feasible in a world where effort (and θ) is private information of the agent.

9.2 Static Conditions

We are looking for optimal transfer schemes among that class of continuously differentiable transfer schemes.

Consider first the static problem. When the production function is of the form (29), the incentive compatibility constraint in its differentiable form is as follows

$$\begin{aligned} b\,(1+\tau') &\leq 1 \quad \text{if } e^* \geq 0 \text{ and } e^* < e_{\max} \\ a\,(1+\tau') &\geq 1 \quad \text{if } e^* \leq 0 \;\text{ and } e^* > -\infty, \end{aligned}$$

with equalities if $e \neq 0$. As long as $\tau' \leq 0$ the relevant deviation would always be to reduce effort, i.e. only the second constraint is relevant. The great advantage of this formulation is that, since production efficiency is not an issue,[26] the planner will always use the whole margin to give risk sharing to the agent. Hence the optimal tax rate is exactly $\tau' = \frac{1}{a} - 1$. That is, τ' increases with a, and it approaches -1 (the first best level) for $a \to \infty$. Recall that the intuition is as follows: when a is large the agent does not find very attractive to deviate from the optimal level $e = 0$ and this reduces incentive costs, allowing more risk sharing. Clearly, a simple normalization $z = ae$ induces preferences of the form $u\left(c - \frac{z}{a}\right)$ hence $\frac{1}{a}$ is the marginal cost/return of effort. When the marginal cost/return of effort is low the agent is easier to convince not to shirk.

Notice two important things that are very evident in this static case, but that will be verified in the general case as well. First, since $\tau' = \frac{1}{a} - 1$ must hold for all income levels,

[26] We will normalize $e^* = 0$. Notice that as long as as long as $a > 1 > b$, $e^* = 0$ is the unique efficient effort level under full information. With asymmetric information, incentive costs make $e^* > 0$ even less attractive. Moreover, the linearity for $e < 0$ implies that a negaive effort level cannot be optimal since it requires the same consumption dispersion as $e = 0$ and it implies lower net welfare compared to $e^* = 0$.

the tax schedule must be linear in x_t. Second, that since agent's utility and the production function are concave, when facing a linear tax schedule the agent problem is concave. Hence the incentive compatibility can be substituted by the agent's first order conditions. So, a linear tax is indeed optimal.

9.3 Two Periods

To get an idea of the mechanism in a dynamic framework, let's now consider the two period version of this model for the agent. They can obviously be seen as the last two periods of a general $T < \infty$ horizon model. If we normalize to zero the initial level of assets and neglect the notation for previous history x^{T-2}, the agent objective function is (recall that $\delta = q$)

$$u\left(c_{T-1} - e_{T-1}\right) + q\mathbf{E}_{T-1} u\left(c_T - e_T\right)$$

where for a given tax schedule $\boldsymbol{\tau}$ we have

$$\begin{aligned} c_{T-1} &= y_{T-1} + \tau_{T-1}\left(x_{T-1}\right) - qb_T; \\ c_T &= y_T + \tau_T\left(x_{T-1}, x_T\right) + b_T; \end{aligned}$$

and, as before,

$$y_t = \theta_t + a \min\left\{e_t, 0\right\} + \max\left\{e_t, 0\right\}.$$

The Euler's equation is the usual one (as optimality for b_T)

$$1 = \mathbf{E}_{T-1} \frac{u'\left(c_T - e_T\right)}{u'\left(c_{T-1} - e_{T-1}\right)},$$

while the optimal effort choice e_{T-1} solves

$$1 + \frac{\partial \tau_T\left(x_{T-1}\right)}{\partial x_{T-1}} + q\mathbf{E}_{T-1} \frac{\partial \tau_T\left(x_{T-1}, x_T\right)}{\partial x_{T-1}} \frac{u'\left(c_T - e_T\right)}{u'\left(c_{T-1} - e_{T-1}\right)} = \frac{1}{a}. \tag{30}$$

The static conditions we derived above imply that

$$1 + \frac{\partial \tau_T\left(x_{T-1}, x_T\right)}{\partial x_T} = \frac{1}{a}.$$

We will see more in detail below that since $\frac{\partial \tau_T(x_{T-1}, x_T)}{\partial x_T}$ does not depend on x_{T-1}, whenever the scheme is differentiable, we have that $\frac{\partial \tau_T(x_{T-1}, x_T)}{\partial x_{T-1}}$ is constant in x_T. The incentive constraint for e_{T-1} hence takes the simpler form

$$\frac{\partial \tau_T\left(x_{T-1}\right)}{\partial x_{T-1}} + q\frac{\partial \tau_T\left(x_{T-1}, x_T\right)}{\partial x_{T-1}} = \frac{1}{a} - 1, \text{ for all } x_{T-1}, x_T.$$

This implies that the discounted sum of the last two taxes is a linear function of x_{T-1}, whose slope does not depend on x_T. We hence get

$$\begin{aligned}\Delta\tau_T \equiv \tau_T - \mathbf{E}_{T-1}\tau_T &= \left(\frac{1}{a}-1\right)x_T - \mathbf{E}_{t-1}x_T\left(\frac{1}{a}-1\right)\\ &= \left(\frac{1}{a}-1\right)\Delta x_T.\end{aligned}$$

And, similarly for $\Delta\left(\tau_{T-1}+q\tau_T\right)$, we get

$$\tau_{T-1}+q\tau_T - \mathbf{E}_{T-2}\left[\tau_{T-1}+q\tau_T\right] = \left(\frac{1}{a}-1\right)\left(x_{T-1}+qx_T - \mathbf{E}_{T-2}\left[x_{T-1}+qx_T\right]\right).$$

This expression does not depend on the process on θ_t, and it provides a very simple (linear) relationship between the innovation on the expected discounted value of taxes and the innovation in the permanent income.

9.4 Generic Time Horizon

We are now ready to derive the results for a generic finite horizon model. The analogous to (30), for all $s,n,t\geq 0$, is

$$\mathbf{E}_{t-s}\sum_{n=0}^{T-t}q^n\left[\frac{\partial\tau_{t+n}\left(x^{t+n}\right)}{\partial x_t}\frac{u'\left(c_{t+n}-e_{t+n}\right)}{u'\left(c_t-e_t\right)}\right] = \mathbf{E}_{t-s}\mathbf{E}_t\left[\sum_{n=0}^{T-t}q^n\frac{\partial\tau_{t+n}\left(x^{t+n}\right)}{\partial x_t}\frac{u'\left(c_{t+n}-e_{t+n}\right)}{u'\left(c_t-e_t\right)}\right] = \frac{1}{a}-1.$$

In order to complete the derivation, we first show the following result.

Lemma 1. *Within the class of continuous differentiable transfer schemes the discounted value of marginal taxes* $\sum_{n=0}^{T-t}q^n\frac{\partial\boldsymbol{\tau}_{t+n}\left(x^{t+n}\right)}{\partial x_s}$ *does not depend on* $(x_t,...,x_T)$ *for all* s*. They are hence linear functions of* x_s *given* x^t*.*

The proof is by induction. Recall the discussion made for the static case, and notice that the static model corresponds to the last period of a finite period model. We argued above that the optimal tax satisfies $\frac{\partial\tau_T\left(x^{T-1},x_T\right)}{\partial x_T}=\frac{1}{a}-1$ for all x^{T-1} and x_T. This implies that the cross derivative $\frac{\partial\tau_T\left(x^{T-1},x_T\right)}{\partial x_T\partial x_t}=0$. Since τ_T is continuously differentiable, it must be that $\frac{\partial\tau_T\left(x^T\right)}{\partial x_t}$ is constant in x_T for all $t<T$ as claimed above.[27]

Now consider τ_{T-1}. Since $\frac{\partial\tau_T\left(x^T\right)}{\partial x_{T-1}}$ does not depend on x_T, the effort incentive compatibility can be written as follows:

$$\frac{\partial\boldsymbol{\tau}_{T-1}\left(x^{T-1}\right)}{\partial x_{T-1}}+q\frac{\partial\boldsymbol{\tau}_T\left(x^T\right)}{\partial x_{T-1}}\mathbf{E}_{T-1}\left[\frac{u'\left(c_T\right)}{u'\left(c_{T-1}\right)}\right]=\frac{1}{a}-1,\qquad\text{for all } x^{T-2} \text{ and } x_{T-1}.$$

[27]The tax/transfer function can hence be written as $\tau_T\left(x^{T-1},x_T\right)=h\left(x^{T-1}\right)+\left(\frac{1}{a}-1\right)x_T$ with h differentiable.

Since $\mathbf{E}_{T-1}\left[\frac{u'(c_T)}{u'(c_{T-1})}\right]=1$, we have that $\frac{\partial\boldsymbol{\tau}_{T-1}\left(x^{T-1}\right)}{\partial x_{T-1}}+q\frac{\partial\boldsymbol{\tau}_T\left(x^T\right)}{\partial x_{T-1}}$ is a constant for all all x^{T-2} and x_{T-1}. Since the tax scheme is assumed to be differentiable, this property implies that $\frac{\partial\boldsymbol{\tau}_{T-1}\left(x^{T-1}\right)}{\partial x_t}+q\frac{\partial\boldsymbol{\tau}_T\left(x^T\right)}{\partial x_t}$ is also constant in x_{T-1} (and x_T) for all t.[28] Going backward, we have our result: $\sum_{n=t}^T q^{n-t}\frac{\partial\boldsymbol{\tau}_n(x^n)}{\partial x_s}$ is constant in $x_t,...x_T$ for all s. **Q.E.D.**

Given the above results we can apply the law of iterated expectations and get

$$
\begin{aligned}
&\mathbf{E}_t\left[\sum_{n=0}^{T-t}q^n\frac{\partial\tau_{t+n}\left(x^{t+n}\right)}{\partial x_t}\frac{u'\left(c_{t+n}-e_{t+n}\right)}{u'\left(c_t-e_t\right)}\right]\\
=&\mathbf{E}_t\left[\sum_{n=0}^{T-t-1}q^n\frac{\partial\tau_{t+n}\left(x^{t+n}\right)}{\partial x_t}\frac{u'\left(c_{t+n}-e_{t+n}\right)}{u'\left(c_t-e_t\right)}+q^{T-t}\mathbf{E}_{T-1}\frac{\partial\tau_T\left(x^T\right)}{\partial x_t}\frac{u'\left(c_T-e_T\right)}{u'\left(c_t-e_t\right)}\right]\\
=&\mathbf{E}_t\left[\sum_{n=0}^{T-t-1}q^n\frac{\partial\tau_{t+n}\left(x^{t+n}\right)}{\partial x_t}\frac{u'\left(c_{t+n}-e_{t+n}\right)}{u'\left(c_t-e_t\right)}+q^{T-t}\mathbf{E}_{T-1}\frac{\partial\tau_T\left(x^T\right)}{\partial x_t}\frac{u'\left(c_T-e_T\right)}{u'\left(c_t-e_t\right)}\right]\\
=&\mathbf{E}_t\left[\sum_{n=0}^{T-t-1}q^n\frac{\partial\tau_{t+n}\left(x^{t+n}\right)}{\partial x_t}\frac{u'\left(c_{t+n}-e_{t+n}\right)}{u'\left(c_t-e_t\right)}+q^{T-t}\frac{\partial\tau_T\left(x^T\right)}{\partial x_t}\mathbf{E}_{T-1}\frac{u'\left(c_T-e_T\right)}{u'\left(c_{T-1}-e_{T-1}\right)}\frac{u'\left(c_{T-1}-e_{T-1}\right)}{u'\left(c_t-e_t\right)}\right]\\
=&\mathbf{E}_t\left[\sum_{n=0}^{T-t-1}q^n\frac{\partial\tau_{t+n}\left(x^{t+n}\right)}{\partial x_t}\frac{u'\left(c_{t+n}-e_{t+n}\right)}{u'\left(c_t-e_t\right)}+q^{T-t}\frac{\partial\tau_T\left(x^T\right)}{\partial x_t}\frac{u'\left(c_{T-1}-e_{T-1}\right)}{u'\left(c_t-e_t\right)}\right]\\
=&\mathbf{E}_t\left[\sum_{n=0}^{T-t-2}q^n\frac{\partial\tau_{t+n}\left(x^{t+n}\right)}{\partial x_t}\frac{u'\left(c_{t+n}-e_{t+n}\right)}{u'\left(c_t-e_t\right)}+q^{T-t-1}\mathbf{E}_{T-2}\left(\frac{\partial\tau_{T-1}\left(x^{T-1}\right)}{\partial x_t}+q\frac{\partial\tau_T\left(x^T\right)}{\partial x_t}\right)\frac{u'\left(c_{T-1}-e_{T-1}\right)}{u'\left(c_t-e_t\right)}\right]\\
=&\mathbf{E}_t\left[\sum_{n=0}^{T-t-2}q^n\frac{\partial\tau_{t+n}\left(x^{t+n}\right)}{\partial x_t}\frac{u'\left(c_{t+n}-e_{t+n}\right)}{u'\left(c_t-e_t\right)}+q^{T-t-1}\left(\frac{\partial\tau_{T-1}\left(x^{T-1}\right)}{\partial x_t}+q\frac{\partial\tau_T\left(x^T\right)}{\partial x_t}\right)\frac{u'\left(c_{T-2}-e_{T-2}\right)}{u'\left(c_t-e_t\right)}\right]\\
&...\\
=&\mathbf{E}_t\left[\sum_{n=0}^{T-t}q^n\frac{\partial\tau_{t+n}\left(x^{t+n}\right)}{\partial x_t}\right].
\end{aligned}
$$

where we repeatedly used the linearity of expectations and the Euler equation. We can hence easily get

$$
\left(\mathbf{E}_{t+1}-\mathbf{E}_t\right)\left[\sum_{n=0}^{T-t-1}q^n\tau_{t+1+n}\left(x^{t+1+n}\right)\right]=\left(\frac{1}{a}-1\right)\left(\mathbf{E}_{t+1}-\mathbf{E}_t\right)\left[\sum_{n=0}^{T-t-1}q^nx_{t+1+n}\right]. \tag{31}
$$

Now we are ready to show the global concavity of the agent problem when facing the optimal tax scheme. Intuitively, this will be true since with linear taxes, what matters is the

[28] Using the result for τ_T we get that the discounted *sum* takes it takes the following from

$$
\tau_{T-1}\left(x^{T-1}\right)+q\tau_T\left(x^T\right)=g\left(x^{T-2}\right)+\left(\frac{1}{a}-1\right)x_{T-1}+q\left(\frac{1}{a}-1\right)x_T.
$$

concavity of utility and the production function. It turns out that this result also depends on the agent's preferences over consumption.

Lemma 2. *If the agent has quadratic preferences, within the class of differentiable schemes, taxes are in fact linear with* $\frac{\partial \boldsymbol{\tau}_T(x^t)}{\partial x_{t-s}} \equiv 0$ for all $s > 0$.

When the agent has quadratic preferences, the Euler equation in each period is (recall that we want to implement $b_t \equiv 0$, i.e. $c_t = y_t + \tau_t$)

$$x_t + \tau_t\left(x^t\right) = \mathbf{E}_t\left[x_{t+1} + \tau_{t+1}\left(x^{t+1}\right)\right] \tag{32}$$

Consider first period $T-1$, since τ_T is linear in x_T with constant slope $\left(\frac{1}{a} - 1\right)$, (32) becomes

$$x_{T-1} + \tau_{T-1}\left(x^{T-1}\right) = \mathbf{E}_{T-1}\left[\frac{1}{a}x_T + \tau_T\left(x^{T-1}\right)\right] = \frac{1}{a}x_{T-1} + \tau_T\left(x^{T-1}\right),$$

where we used the fact that in equilibrium $\mathbf{E}_t x_T = x_{T-1}$, and we abused in notation denoting by $\tau_T\left(x^{T-1}\right)$ the x^{T-1} part of τ_T. For this condition to hold for all x_{T-1} given any x^{T-2}, it must be that $\frac{\partial \boldsymbol{\tau}_{T-1}(x^{T-1})}{\partial x_{T-1}} - \frac{\partial \boldsymbol{\tau}_T(x^{T-1})}{\partial x_{T-1}} \equiv \frac{1}{a} - 1$. If we subtract this condition to the incentive constraint for e_{T-1} : $\frac{\partial \boldsymbol{\tau}_{T-1}(x^{T-1})}{\partial x_{T-1}} + q\frac{\partial \boldsymbol{\tau}_T(x^T)}{\partial x_{T-1}} = \frac{1}{a} - 1$, we get $(1+q)\frac{\partial \boldsymbol{\tau}_T(x^{T-1})}{\partial x_{T-1}} \equiv 0$. Given this result, the situation for $T-2$ is very similar to that for $T-1$, since we now know that $\frac{\partial \boldsymbol{\tau}_{T-1}(x^{T-1})}{\partial x_{T-1}} = \frac{\partial \boldsymbol{\tau}_T(x^T)}{\partial x_T} = \frac{1}{a} - 1$. Going backward, we hence get the desired result.

Lemma 3. *If the agent has quadratic preferences, when facing the above tax, the agent's problem is concave, so the so derived tax scheme is optimal.*

We have to show that, when facing the optimal tax scheme, the agent's problem is jointly concave in $\{e_t(x_t)\}_{t=0}^T$ and $\{b_{t+1}(x_t)\}_{t=0}^T$. Consider two contingent plans $\mathbf{e}^1, \mathbf{b}^1, \mathbf{c}^1$ and $\mathbf{e}^2, \mathbf{b}^2, \mathbf{c}^2$. Now consider the plan $\mathbf{e}^\alpha, \mathbf{b}^\alpha, \mathbf{c}^\alpha$ where for all x^t, $e_t^\alpha(x^t) \equiv \alpha e_t^1(x^t) + (1-\alpha)e_t^2(x^t)$ and similarly for b_t and c_t. First of all, since both effort and bonds enter linearly in the utility function, the concavity of the agent's utility and the additive separability over time and states imply that if we show that $\mathbf{c}^\alpha$ is attainable we are done.

Notice that $\frac{\partial \boldsymbol{\tau}_T(x^t)}{\partial x_{t-s}} = 0$ implies that τ_t is linear with slope $\frac{\partial \boldsymbol{\tau}_T(x^t)}{\partial x_t} \equiv \tau_t^{(t)} = \frac{1}{a} - 1$ for all t. The final part of the proof is hence very simple. Period t consumption is

$$\begin{aligned} x_t^\alpha + \tau_t\left(x_0^\alpha, x_1^\alpha, ..., x_t^\alpha\right) &= \left(1 + \tau_t^{(t)}\right)x_t^\alpha + k_t \\ &= \frac{1}{a}x_t^\alpha + k_t \\ &= \frac{1}{a}f\left(\theta_t, e_t^\alpha\right) + k_t \end{aligned}$$

$$\geq \frac{1}{a}\left[\alpha f\left(\theta_t, e_t^1\right) + (1-\alpha) f\left(\theta_t, e_t^2\right)\right] + k_t$$
$$= \frac{1}{a}\left[\alpha x_t^1 + (1-\alpha) x_t^2\right] + k_t,$$

where the inequality in the penultimate row comes from the concavity of f in e and $a \geq 1 > 0$. The last line is definitional. **Q.E.D.**

The derivation is now very simple, since we know that only contemporaneous taxes are positive, hence for all t we have

$$c_t\left(x^t\right) = x_t + \tau_t\left(x^t\right) = x_t + k_t + \left(\frac{1}{a} - 1\right) x_t = k_t + \frac{1}{a} x_t.$$

This, together with the Euler equation implies $k_{t+1} = k_t$ and

$$\Delta c_{t+1} = c_{t+1} - c_t = \frac{1}{a}\left(x_{t+1} - x_t\right) = \frac{1}{a}\nu_{t+1}^p.$$

Notice that given our process of income, the result is independent of the time horizon, although we tend to consider long horizons.

Infinite Horizon We consider the infinite horizon case as a limit case, for T very large. So, what we will derive is in fact the limit of the finite horizon case. Since Θ is bounded above, a finite horizon also guarantees that we can always choose a quadratic utility specification such that the bliss point is never reached. It suffices assuming

$$u\left(c-e\right) = -\frac{1}{2}\left(\bar{B} - \left(c-e\right)\right)^2 \qquad \text{with } \bar{B} >> T\theta_{\max}.$$

The closed form for taxes is

$$\left(\mathbf{E}_{t+1} - \mathbf{E}_t\right)\left[\sum_{n=0}^{\infty} q^n \tau_{t+1+n}\left(x^{t+1+n}\right)\right] = \left(\frac{1}{a} - 1\right)\left(\mathbf{E}_{t+1} - \mathbf{E}_t\right)\left[\sum_{n=0}^{\infty} q^n y_{t+1+n}\right]. \tag{33}$$

and then using quadratic utility and the agent's budget constraint, we get

$$\Delta c_{t+1} = \frac{1}{a} v_{t+1}^p.$$

9.5 Temporary shocks

In presence of temporary shocks the planner should obviously condition on $\xi_t = g\left(v_t^T, l_t\right)$ realizations as well. Denote by $h^t = (x^t, \xi^t)$ the combined public history. If we specify a production function of the form

$$\xi_t = g\left(v_t^T, l_t\right) = v_t^T + a^T \min\left\{l_t, 0\right\} + b^T \max\left\{l_t, 0\right\} \text{ with } a^T > 1 > b^T,$$

and the following agent's preferences over c_t, l_t and e_t : $u\left(c_t - l_t - e_t\right)$.

The analysis is now performed separately for the two type of shocks, and we get

$$\mathbf{E}_t \sum_{n=0}^{T-t} q^n \left[\frac{\partial \tau_{t+n}\left(h^{t+n}\right)}{\partial \xi_t} \frac{u'\left(c_{t+n} - e_{t+n}\right)}{u'\left(c_t - e_t\right)} \right] = \frac{1}{a^T} - 1$$

and

$$\mathbf{E}_t \sum_{n=0}^{T-t} q^n \left[\frac{\partial \tau_{t+n}\left(h^{t+n}\right)}{\partial x_t} \frac{u'\left(c_{t+n} - e_{t+n}\right)}{u'\left(c_t - e_t\right)} \right] = \frac{1}{a^p} - 1,$$

where, for consistency, we denoted by a^p the slope of f for $e_t \leq 0$. If $\delta = q$, using the Euler equation, the sum of taxes can be written as

$$\mathbf{E}_t \sum_{n=0}^{T-t} q^n \frac{\partial \tau_{t+n}\left(h^{t+n}\right)}{\partial \xi_t} = \frac{1}{a^T} - 1$$

and

$$\mathbf{E}_t \sum_{n=0}^{T-t} q^n \frac{\partial \tau_{t+n}\left(h^{t+n}\right)}{\partial x_t} = \frac{1}{a^p} - 1.$$

Assuming quadratic preferences, the Euler equation furthermore implies for temporary shocks

$$\frac{\partial \tau_t\left(h^t\right)}{\partial \xi_s} = \frac{\partial \tau_t\left(h^t\right)}{\partial \xi_k} \equiv \tau_\xi^{(k)}, \text{ for all } \; k, s < t,$$

which together with the incentive constraint, in the infinite horizon specification one obtains

$$1 + \frac{\partial \tau_t\left(h^t\right)}{\partial \xi_t} = \frac{\partial \tau_{t+1}\left(h^{t+1}\right)}{\partial \xi_t} = \tau_\xi^{(t)}. \tag{34}$$

For permanent shocks, we obtain exactly the same conditions as before:

$$1 + \frac{\partial \tau_t\left(h^t\right)}{\partial x_t} = 1 + \frac{\partial \tau_{t+1}\left(h^{t+1}\right)}{\partial x_{t+1}} + \frac{\partial \tau_{t+1}\left(h^{t+1}\right)}{\partial x_t} \tag{35}$$

with $\frac{\partial \tau_{t+1}\left(h^{t+1}\right)}{\partial x_{t-s}} = 0$ for $s > 0$. In the infinite horizon specification, the analysis hence becomes very simple and we obtain

$$1 + \tau_x = \frac{1}{a^p} \quad \text{and} \quad \tau_\xi = \frac{1-q}{a^T},$$

where $1 + \tau_x = 1 + \frac{\partial \tau_t\left(h^t\right)}{\partial x_t}$ and $\tau_\xi = \tau_\xi^{(k)} = 1 + \frac{\partial \tau_t\left(h^t\right)}{\partial \xi_t}$, and taxes are time invariant. The agent's consumption reaction to the different shocks to income is

$$\begin{aligned}\Delta c_{t+1} &= \frac{1}{a^p} \Delta x_{t+1} + \frac{1-q}{a^T} \Delta \xi_{t+1} \\ &= \frac{1}{a^p} v_{t+1}^p + \frac{1-q}{a^T} v_{t+1}^T.\end{aligned}$$

10 Appendix B: The Case with Iso-elastic Utility[29]

We will only consider the infinite horizon case. The outcome of this section will be an expression for innovation in *log* consumption of the form

$$\ln c_{t+1} - \ln c_t = \frac{1}{a}\Delta v^p_{t+1} - \frac{\ln \frac{q}{\delta}}{\gamma} + \frac{\gamma}{2a}\sigma^2_{v^p},$$

where Δv^p_{t+1} is the innovation to (log) permanent income and $\frac{1}{\gamma}$ is the intertemporal elasticity of substitution of consumption.

The Model Assume that the income process is that used in most of the literature. If we let $y_t = \ln Y_t$ we have the following process for income

$$y_t = x_t + \xi_t$$

where

$$x_t = f(\theta_t, e_t) = \theta_t + a \min\{0, e\} + b \max\{0, e\}, \qquad \text{with} \quad a > \lambda > b,$$

where $e \equiv \ln N$. This is precisely the same formulation as above, but in logs. Notice that the production function corresponds to a modification of the the standard Cobb-Douglas: $X_t = \Theta_t N_t^a$, and $X_t = \Theta_t N_t^b$ for $N_t \leq 1$ and $N_t \geq 1$ respectively.

We specify the following process for skills:

$$\ln \Theta_t \equiv \theta_t = \theta_{t-1} + v^p_t.$$

An additional assumption, which will be crucial for us to get an exact closed form, is that the shocks v^p_t are normally distributed with mean μ_{v^p} and variance $\sigma^2_{v^p}$.

Moreover, assume that the agent has Cobb-Douglas/CRRA preferences of the following form:

$$\mathbf{E}_0 \sum_{t=0} \delta^t \frac{\left(C_t \cdot N_t^{-1}\right)^{1-\gamma}}{1-\gamma} \text{ for } \gamma > 1; \quad \text{and} \quad \mathbf{E}_0 \sum_{t=0} \delta^t \left(\ln C_t - \ln N_t\right) \text{ for } \gamma = 1.$$

For future use, notice that we can write:

$$\frac{\left(C_t \cdot N_t^{-\lambda}\right)^{1-\gamma}}{1-\gamma} = \frac{1}{1-\gamma} \exp\left\{(1-\gamma)\left(c_t - e_t\right)\right\}$$

[29] In this section, we change a bit notation hoping that it will be all clear since we define any new variable.

where $c_t \equiv \ln C_t$.

It will be convenient to write the problem in logs so that we can use the analogies to the case in levels. The budget constraint can be written as follows:

$$\exp\{c_t\} + qb_t = \exp\left\{x_t + \tau_t\left(x^t\right)\right\} + b_{t-1}.$$

Since in equilibrium we will have $N_t^* \equiv 1$, the Euler equation is the usual one

$$\mathbf{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right] = \mathbf{E}_t\left[\exp\left(-\gamma\frac{c_{t+1}}{c_t}\right)\right] = \exp\left(-\gamma\mu_t + \gamma^2\frac{1}{2}\sigma_t^2\right) = \frac{q}{\delta},$$

where we used the fact that C_{t+1} is log normally distributed, with Δc_{t+1} having conditional mean μ_t and conditional variance σ_t^2.

Since we will implement $b_t^* \equiv 0$, $c_t\left(x^t\right) = x_t + \tau_t\left(x^t\right)$ and the objective function becomes

$$\mathbf{E}_0\sum_{t=0}\delta^t\frac{1}{1-\gamma}\exp\left\{(1-\gamma)\left(x_t + \tau_t\left(x^t\right)\right) + e_t\right\}.$$

Given the specification of f, the objective function is all in logs. It is now easy to see the strong analogy to the case in levels considered above. The first order condition for (log) effort e_t is

$$\mathbf{E}_t\sum_{n=0}^{\infty}\delta^n\left(\frac{C_{t+n}}{C_t}\right)^{1-\gamma}\frac{\partial\tau_{t+n}\left(x^{t+n}\right)}{\partial x_t} = \frac{1}{a} - 1$$

One can again show that conditional expectations can be decomposed since $\frac{\partial\tau_{t+n}\left(x^{t+n}\right)}{\partial x_t}$ does not depend on x_{t+n}. This is exactly as above for the model in levels. Moreover, since C_t is log normally distributed, we have

$$\mathbf{E}_t\left[\left(\frac{C_{t+n}}{C_t}\right)^{1-\gamma}\right] = \mathbf{E}_t\left[\exp\left((1-\gamma)\frac{c_{t+1}}{c_t}\right)\right] = \exp\left((1-\gamma)\mu_t + \frac{1}{2}(1-\gamma)^2\sigma_t^2\right).$$

From the Euler equation we have:

$$\begin{aligned}\exp\left((1-\gamma)\mu_t + \frac{1}{2}(1-\gamma)^2\sigma_t^2\right) &= \exp\left(-\gamma\mu_t + \gamma^2\frac{1}{2}\sigma_t^2\right)\exp\left(\mu_t + \frac{1}{2}(1-2\gamma)\sigma_t^2\right)\\ &= \frac{q}{\delta}\exp\left(\mu_t + \frac{1}{2}\sigma_t^2 - \gamma\sigma_t^2\right)\\ &\equiv \frac{q}{\delta}s_t.\end{aligned}$$

Similarly, since by the law of iterated expectations, we have

$$\mathbf{E}_t\left[\left(\frac{C_{t+n}}{C_t}\right)^{-\gamma}\right] = \mathbf{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\mathbf{E}_{t+1}\left(\frac{C_{t+2}}{C_{t+1}}\right)^{-\gamma}\cdot\ldots\cdot\mathbf{E}_{t+n-1}\left(\frac{C_{t+n}}{C_{t+n-1}}\right)^{-\gamma}\right] = \left(\frac{q}{\delta}\right)^n$$

Using the properties of the normality, and assuming that log consumption innovation conditional variance and conditional mean are constant and equal to μ and σ^2 respectively, and denoting $s \equiv \exp\left(\mu_t + \frac{1}{2}\sigma_t^2 - \gamma\sigma_t^2\right) > 0$, we have

$$\begin{aligned}\mathbf{E}_t\left[\exp\left((1-\gamma)\frac{c_{t+n}}{c_t}\right)\right] &= \mathbf{E}_t\left[\exp\left\{-\gamma\frac{c_{t+1}}{c_t}\right\} s\mathbf{E}_{t+1}\exp\left\{-\gamma\frac{c_{t+2}}{c_{t+1}}\right\} s\cdot ..\cdot \mathbf{E}_{t+n-1}\exp\left\{-\gamma\frac{c_{t+n}}{c_{t+n-1}}\right\} s\right] \\ &= \left(\frac{qs}{\delta}\right)^n\end{aligned}$$

By the law of iterated expectations, the incentive constraint hence becomes

$$\mathbf{E}_t\sum_{n=0}^{\infty}(qs)^n\frac{\partial\tau_{t+n}\left(x^{t+n}\right)}{\partial x_t} = \frac{1}{a} - 1$$

From the Euler equation, one can easily see that future marginal taxes must be zero, so that

$$\frac{\partial\tau_t\left(x^t\right)}{\partial x_t} = \frac{1}{a} - 1,$$

and

$$\Delta c_{t+1} = c_{t+1} - c_t = \frac{1}{a}\Delta v_{t+1}^p + \mu,$$

where $\mu = \mathbf{E}_t\Delta c_{t+1}$. Since, from the above expression, the variance of log consumption is $\sigma_c^2 = \frac{1}{a^2}\sigma_{v^p}^2$, from the Euler equation, we have

$$\mu = -\frac{\ln\frac{q}{\delta}}{\gamma} + \gamma\frac{1}{2}\sigma_c^2 = -\frac{\ln\frac{q}{\delta}}{\gamma} + \frac{\gamma}{2a^2}\sigma_{v^p}^2.$$

In order to get the expression for log taxes (not in levels) notice that the *log* of tax must display a deterministic drift:

$$\tau_t\left(x^t\right) = \left(\frac{1}{a} - 1\right)x_t + t\left[-\frac{\ln\frac{q}{\delta}}{\gamma} + \frac{\gamma}{2a^2}\sigma_{v^p}^2\right].$$

Why consumption is log normally distributed? Well, this can be shown by using a similar derivation to that of the quadratic case.

From the Euler equation, we have

$$c_t = x_t + \tau_t\left(x^t\right) = -\ln\mathbf{E}_t\exp\left\{-x_{t+1} - \tau_{t+1}\left(x^{t+1}\right)\right\}$$

Going backward we obtain that τ_{t+1} only depends on x_{t+1}, and it is actually linear in x_{t+1} (with constant slope), hence consumption is log normally distributed. Start by the Euler equation for b_{T-1}, since we know that the tax in the last period is linear in x_T, we have

$$\begin{aligned}
x_{T-1} + \tau_{T-1}\left(x^{T-1}\right) &= -\frac{1}{\gamma}\ln \mathbf{E}_{T-1}\exp\left\{-\gamma\frac{1}{a}x_T - \gamma\tau_T\left(x^{T-1}\right)\right\} \\
&= -\frac{1}{\gamma}\ln\exp\left\{-\gamma\frac{1}{a}\mathbf{E}_{T-1}x_T - \gamma\tau_T\left(x^{T-1}\right) + \gamma^2\frac{1}{2}\frac{1}{a^2}\sigma_{v^p}^2\right\} \\
&= \frac{1}{a}x_{T-1} - \tau_T\left(x^{T-1}\right) + \frac{\gamma}{2a^2}\sigma_{v^p}^2.
\end{aligned}$$

where we used the fact that in equilibrium $\mathbf{E}_t x_T = x_{T-1}$. And so on until period 1.

Figure 1: The Production Function