Firing Costs, Misallocation, and Aggregate Productivity*

José-María Da-Rocha
ITAM and Universidade de Vigo†

Marina Mendes Tavares
ITAM and IMF‡

Diego Restuccia
University of Toronto§

February 2015

Abstract

We reassess the aggregate impact of firing taxes on aggregate total factor productivity (TFP) in a framework where the distribution of establishment-level productivity is not invariant to the policy. Whereas in the benchmark economy with no firing costs there is only one threshold productivity for which large establishments become small and small establishments become large, in economies with firing taxes there is an inaction zone—a range of productivity for which establishments remain either large or small. This inaction in establishment's decisions generates factor misallocation: relative to an undistorted economy, more employment is allocated in less productive establishments and less employment in more productive establishments with firing taxes. Importantly, the distribution of establishments across productivity also changes with firing taxes. We find that empirically plausible measures of firing taxes—such as 1 and 5 year's wages—generate large aggregate TFP loses of 0.5 and 0.16 relative to the benchmark economy with no distortions. These loses are much larger than those of comparable policies in the related literature when the distribution of establishment TFP is exogenous and independent of policy distortions.

Keywords: firing cost, inaction, misallocation, establishments, productivity.
JEL codes: O1, O4.

*Restuccia gratefully acknowledges the financial support from the Social Sciences and Humanities Research Council of Canada. Mendes Tavares gratefully acknowledges the financial support of DIFD.

†Calle Torrecedeira 105, 36208-Vigo, Spain. E-mail: jmrocha@uvigo.es.

‡Av. Camino Santa Teresa 930 C.P. 10700 Mexico, D.F. E-mail: marinamendestavares@gmail.com.

§150 St. George Street, Toronto, ON M5S 3G7, Canada. E-mail: diego.restuccia@utoronto.ca.

1 Introduction

A fundamental issue in economic growth and development is identifying the specific policies and institutions that account for the large cross-country differences in total factor productivity (TFP) and output per capita. A recent literature has studied the role of factor allocation across heterogenous production units as an important factor in explaining aggregate TFP differences. Some of the specific policies and institutions studied in accounting for factor misallocation and aggregate TFP losses include firing costs, size-dependent policies, financial frictions, among many others.[1] While each of these policies and institutions are shown to be important in explaining TFP differences across countries, the quantitative results are short of explaining the bulk of observed differences.[2] In this paper, we reassess the aggregate impact of firing taxes on TFP in a framework where the distribution of establishment-level productivity is not invariant to the policy. We find that empirically plausible measures of firing taxes—such as 1 and 5 year's wages—generate large aggregate TFP loses of 0.5 and 0.16 relative to the benchmark economy with no distortions.

We consider an otherwise standard model of producer heterogeneity following Hopenhayn (1992) and Hopenhayn and Rogerson (1993). The model is set up in continuous time for analytical tractability. Establishments are heterogeneous in their TFP that follow an exogenous process that depends on the establishment employment size. Crucially, and differently from the related previous literature, policies that distort the size of establishments such as firing costs, have an effect on the evolution of productivity for individual establishments and hence on the stationary distribution of productivity across establishments and aggregate TFP. Essentially, firing costs produce an inaction zone whereby small relatively more productive establishments remain small and large relatively less productive establishments remain large, generating both factor misallocation and lower average establishment-level

[1] See for instance, Hopenhayn and Rogerson (1993), Guner et al. (2008),Buera et al. (2011), and Greenwood et al. (2013).

[2] See for instance the surveys in Restuccia and Rogerson (2013), Restuccia (2013), and Hopenhayn (2014).

productivity.

We calibrate the model with no firing costs to micro and macro data for the United States and consider quantitative experiments that increase the size of the firing cost—the cost for an individual establishment to reduce employment—with a range from 1 month's wages to 5 year's wages. Relative to the benchmark economy with no firing costs, aggregate TFP in the economy with a firing cost of 1 year wages is 0.5 and in the economy with 5 year's wages is 0.16. These are very large TFP losses produced by empirically-plausible specific policy and much larger than the loses in the related literature with the same policies when the distribution of establishment TFP is exogenous and independent of policy distortions, e.g. Hopenhayn and Rogerson (1993), Moscoso-Boedo and Mukoyama (2012), and Hopenhayn (2014). Moreover, we show that the loses in our model when the distribution of productivity is kept constant to the one in the benchmark economy are similar in magnitude to the existing literature on firing taxes and aggregate productivity.

A desirable property of our framework is that we are able to provide direct analytical results on the main variables of interest following the seminal work of Dixit (1989). Whereas in the benchmark economy with no firing costs there is only one threshold productivity for which large establishments become small and small establishments become large, in economies with firing taxes there is an inaction zone, a range of productivity for which establishments remain either large or small. We show that the inaction zone becomes bigger and shifts to the left towards lower levels of productivity. This property of establishment decisions entails misallocation, large but less productive establishments remain large and small but more productive establishments remain small and, as a result, relative to an undistorted economy, more employment is allocated in less productive establishments and less employment in more productive establishments with firing taxes. Moreover, we show misallocation is aggravated with higher firing costs. We also show that job turnover is lower with firing taxes. Importantly, the distribution of establishments across productivity also changes with firing taxes, with lower productivity establishments being more likely in the economy with firing taxes.

As discussed earlier, our paper relates closely to the earlier literature assessing the aggregate productivity losses of firing taxes such as the seminal work of Hopenhayn and Rogerson (1993) and the more recent analysis in Moscoso-Boedo and Mukoyama (2012), Hopenhayn (2013), and Hopenhayn (2014). A critical distinction between our framework and these previous works is that firing costs affect the productivity process that establishments face which greatly contribute to amplify the negative aggregate productivity implications of the policy. In our framework, there is an equivalence between firing and hiring costs. We use this equivalence to relate our work to previous analysis of size-dependent policies such as that in Gourio and Roys (2014) where firing taxes in France only apply to establishments with 50 or more employees. In this sense our work is also closely related to the more general size-dependent policies studied in Guner et al. (2008). More broadly, our paper relates to a large literature studying the aggregate productivity implications of specific policies and institutions, a key distinction is that in our paper the policy distortion affects the distribution of establishment-level productivity. In generating differences in the distribution of establishment productivity, our paper shares with a small and growing set of papers in the misallocation literature such as financial frictions affecting the occupational chose of entrepreneurs in Buera et al. (2011) and models with endogenous productivity investment by establishments such as Hsieh and Klenow (2014), Da-Rocha et al. (2014), and the relevant references therein.

The remainder of the paper is organized as follows. In the next section we describe the economic environment in detail and characterize its main properties. Section 3 calibrates a benchmark economy with no distortions, i.e., no firing taxes, to data for the United States. In section 4 we perform a series of quantitative experiments to asses the aggregate implications of firing costs. We also show an equivalence result with hiring taxes with allows us to relate our results with a broader literature on size-dependent distortions. We conclude in Section

5. Appendix A contains the formal proofs of all the lemmas in the paper.

2 Model

Our framework builds from the work of Hopenhayn (1992) and Hopenhayn and Rogerson (1993). Establishments hire labor in a competitive market and their productivity follows an exogenous process. New entrants draw their productivity from an endogenous distribution that depends on establishment's hiring decisions. Time is continuous and the horizon is infinite. We focus on a stationary equilibrium of this model and study the impact of firing costs on aggregate measures of TFP and output.

2.1 General Description

The unit of production in the economy is the establishment. Establishments are heterogeneous in their productivity z. They are described by a production function $f(z, n)$ that uses labor to produce output. The function f is assumed to exhibit decreasing returns to scale in labor and to satisfy the usual Inada conditions. The production function is given by:

$$f(z, n) = zn^{\alpha}, \quad \alpha \in (0, 1).$$

We assume for simplicity that establishments can only hire two different amounts of labor n_1 and n_2, where n_2 is larger than n_1. Establishment's productivity z follows a Geometric Brownian motion, that depends on the establishment size, the Geometric Brownian motions are given by:

$$dz = \mu_1 z dt + \sigma_z z dw_z \quad \text{and} \quad dz = \mu_2 z dt + \sigma_z z dw_z,$$

where the drift of the brownian motion μ_i depends on the establishment's size and the standard deviation σ_z is the same for both sizes. We assume that small establishment's drift μ_1 is positive and large establishment's drift μ_2 is negative. This assumption implies that on average small establishments' productivity grow over time, while large establishments' productivity decreases over time so there is mean reversion in the productivity process of establishments. There is a large number of potential entrants. New establishments can be created of either size by paying an entry cost c_e in units of output.

There is a mass one of infinity-lived households with preferences over consumption goods described by the utility function,

$$\max \int_0^\infty e^{-\rho t} u(c) dt$$

where c is consumption and ρ is the discount rate. Households are endowed with one unit of productive time that they supply inelastic to large and small establishments. Households are the owner of establishments. Feasibility in the model requires:

$$C = Y - c_e(\lambda_1 + \lambda_2),$$

where C is aggregate consumption,Y is aggregate output, and c_e is the entry cost multiplied by the mass of small $\lambda_1$and large λ_2 establishments that enter. We next introduce firing cost policies that distort the establishment decision of adjust its size.

2.2 Policy Distortions

We study the quantitative impact of firing cost policies in a model where the distribution of TFP is endogenous. We assume that establishments have to pay a firing cost c_f per worker in units of labor to fire workers and reduce its size from large n_2 to small n_1. The firing cost policy creates inertia, because establishments delay their decisions of adjusting their size. We assume that the firing cost paid by establishments is redistributed to households in the form of a lump-sum transfer T.

2.3 Incumbents' problem

Incumbents maximize the present value of their profits. At each point in time, they observe their current TFP shock z and their size n. Establishments decide to keep its current size or to adjust. This is a standard optimal switching problem, described by Dixit (1989), the problem is characterized by the value function at the current state and by the value matching condition at the switching points. We first describe the dynamic problem of a small incumbent n_1 and then we describe the dynamic problem of a large incumbent n_2.

Small establishments observe their productivity and choose to keep their current size n_1 or to become larger n_2. They receive revenue from selling their output and they pay a wage bill every point in time. Formally, the dynamic problem of a small establishment is defined by:

$$\begin{aligned} \rho W_1(z) &= z n_1^{\alpha} - w n_1 + E_z \frac{dW_1(z)}{dt}, \\ & s.t. \quad dz = \mu_1 z dt + \sigma_z z dw_z, \end{aligned} \tag{1}$$

and by the value matching condition at the switching point z_1 where small establishments choose to become larger, $W_1(z_1) = W_2(z_1)$, and the smooth pasting condition at the switching, $W_1'(z_1) = W_2'(z_1)$.

Like small establishments, large establishments observe their current productivity z and choose to keep their current size n_2 or to pay a firing cost $c_f(n_2 - n_1)$ on wages w to reduce their size to n_1. Large establishments receive their revenue from selling output and they pay their wage bill. Formally, the dynamic problem of a large establishment is defined by:

$$\begin{aligned} \rho W_2(z) &= z n_2^{\alpha} - w n_2 + E_z \frac{dW_2(z)}{dt} \\ s.t. & \quad dz = \mu_2 z dt + \sigma_z z dw_z, \end{aligned} \tag{2}$$

and by the value matching condition at the switching point z_2, where large establishments decide to pay the firing cost c_f per worker to become smaller, $W_1(z_2) - c_f w(n_2 - n_1) = W_2(z_2)$, and the smooth pasting condition at the switching point, $W_1'(z_2) = W_2'(z_2)$. With the firing cost, the value matching condition guarantees that the establishment is indifferent between paying the firing cost $c_f w(n_2 - n_1)$ and switching size or keeping its current size. In Lemma 1 we characterize the value function of small $W^1(z)$ and large $W^2(z)$ establishments and the two switching points $\{z_1, z_2\}$.

Lemma 1. *Given a wage w, interest rates ρ and firing cost c_f, the value function of small establishments $W^1(z)$ and the value function of large establishments $W_2(z)$ that solve the small establishment's problem (1) and the large establishment's problem (2) are given by $W_i(z) = \frac{{n_i}^{\alpha}}{\rho - \mu_i} z - \frac{w n_i}{\rho} + B_i z^{\beta_i}$ where $\beta_i = -\left(\frac{\mu_i}{\sigma_z^2} - \frac{1}{2}\right) \pm \sqrt{\left(\frac{\mu_i}{\sigma_z^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma_z^2}}$ for $i \in \{1, 2\}$ and the constants $\{B_1, B_2\}$ together with the switching points $\{z_1, z_2\}$ solve the two value matching conditions and the two smoothing pasting conditions together.*

Proof See Appendix A.1.

We choose the positive root β_1 for small establishments and the negative root β_2 for large establishments. The positive root for small establishment has the desirable property that the option value of becoming larger increases when the productivity increases, while the negative root for large establishment has the desirable property that the option of becoming smaller decreases when the productivity increases. In the next Lemma 2, we show that B_1 and B_2 are positive.

Lemma 2. *If β_1 is the positive root and β_2 is the negative root, then B_1 and B_2 are positive.*

Proof See Appendix A.2.

The value function for large and small establishments have an intuitive interpretation, where the first two terms are the present value of being a small or a large establishment when switching is not allowed and the the third term is the present value of the switching option. Changes in the firing cost have two effects on the incumbent problem. It has a direct effect on the present value of being large and small trough the constants B_1 and B_2 and a general equilibrium effect trough changes in wages. In the next Lemma 3 we characterize these two effects.

Lemma 3. *Given a wage w, interest rates ρ, and firing cost c_f. The inaction zone rate, $\theta = z_2/z_1$, is the solution of the following non-linear equation:*

$$\varphi(\theta) = \frac{\left[\frac{(1-\beta_1)}{(1-\beta_2)}\left(\frac{1}{\Theta}-1\right)-1\right]}{\left[\frac{(1-\beta_1)}{(1-\beta_2)}\left(\frac{1}{\Theta}-1\right)\theta^{\beta_1}-\theta^{\beta_2}\right]} \quad = \quad \frac{1}{1-\rho c_f}, \tag{3}$$

and the entry point z_1 is given by:

$$z_1 = \frac{\frac{-1}{(1-\beta_2)}\left(\frac{n_2}{\rho}-\frac{n_1}{\rho}\right)}{\left(\frac{n_2^\alpha}{\rho-\mu_1}-\frac{n_1^\alpha}{\rho-\mu_2}\right)\left[\frac{(1-\beta_1)}{(1-\beta_2)}\left(\frac{1}{\Theta}-1\right)-1\right]\frac{\Theta}{-\beta_2}} w, \tag{4}$$

where $\Theta = \left(\frac{1-\theta^{1-\beta_1}}{1-\theta^{\beta_2-\beta_1}}\right)$.

Proof See Appendix A.3.

Lemma 3 is key to understand the dynamics of the model. The first part of the Lemma in equation (3) shows that the inaction zone rate is increasing in firing cost. The second part of the Lemma in equation (4) shows that the inaction zone moves with wages. An increase in the wage moves the inaction zone to the right, while a decrease in the wage moves the inaction zone to the left. As a result, the final effect of an increase in firing cost on the inaction zone will be a combination of the direct effect on the size of the inaction zone θ

and on the general equilibrium effect on wages. In the next section, we characterize the stationary distribution.

2.4 Stationary distribution of establishments

Given the optimal decision of incumbents, we can now characterize the stationary distribution of small establishments $g_1(z)$ and the stationary distribution of large establishments $g_2(z)$. In order to characterize these two distribution, we first define x as the logarithm of an establishment with productivity z and size i relative to the switching point z_i, that is $x = \log(z/z_i)$. The variable x is equal to zero at the switching points. Let $f_i(\cdot)$ be the distribution of the variable x normalized to the switching point z_i. The support of the small establishment distribution $f_1(\cdot)$ is the interval $(-\infty, 0]$, while the support of large establishments distribution $f_2(\cdot)$ is the interval $(0, +\infty]$. The integral over the distribution of small and the integral over the distribution of large establishments is the mass of each establishment. The distribution of small establishments and the distribution of large establishments satisfy the usual boundary conditions, where at the switching points both distributions have mass zero, $f_1(0) = 0$ and $f_2(0) = 0$, and at the boundaries both distributions have also mass zero, $f_1(-\infty) = 0$ and $f_2(\infty) = 0$.

These two distributions are also characterized by two entry points x_1 for small establishments and x_2 for large establishments. The entry points is where establishments that change size enter in the distribution. At the small establishments' entry point x_1, large establishments that decide to reduce their size enter in the small establishments distribution, while at the large establishments' entry point x_2, small establishments that increase their size enter in the large establishments distribution. Figure 3 demonstrates the dynamics of establishments, points x_1 and x_2 are the entry points and the two zeros are the switching points. The rectangular area between the entry points and zero are the inaction zone caused by the firing cost policy.

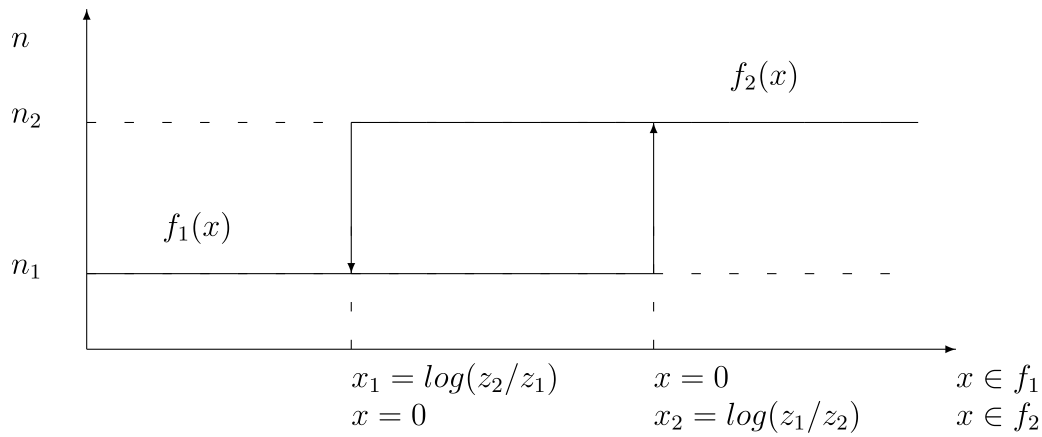


Figure 1: Establishments' Dynamics

From Figure 3 it is clear that at entry points the distribution of small and large establishments receive an inflow of establishments that change size, as a result, at the entry points, the stationary distribution of large and small establishments satisfy the following conditions:

$$f_1(x_1^-) = f_1(x_1^+) + \phi_2 \qquad \text{and} \qquad f_2(x_2^+) = \ f_2(x_2^-) + \phi_1. \tag{5}$$

These two conditions guarantee that the mass of small establishments after entry has taken place, at the point x_1^+, is equal to the mass of small establishments at point x_1^- plus the mass of establishments that enter into the small establishment distribution. The mass of establishment that change size is given by the left derivative of the distribution of large establishments at the switching point zero, $\frac{\partial f_2(0)}{\partial x}$. The same holds for the distribution of large establishments $f_2(\cdot)$ at the entry point x_2.

In equilibrium large and small establishments' distribution satisfy the Kolmogorov forward equation. The Kolmogorov forward equation characterizes the dynamics of the distribution of large and small establishments. New entrants enter by copying incumbents and they entry at rate λ_i. The Kolmogorov forward equation to each establishment outside of the entry points $\{x_1, x_2\}$ is given by the following equation:

$$0 = -\hat{\mu}_i \frac{\partial f_i(x)}{\partial x} + \frac{\hat{\sigma}^2}{2} \frac{\partial^2 f_i(x)}{\partial x} + \lambda_i f_i(x), \tag{6}$$

where the drift $\hat{\mu}_i$ is equal to $\mu_i - \frac{\sigma_z^2}{2}$ and the standard deviation $\hat{\sigma}_z$ is equal to σ_z.[3] In Lemma 4, we solve for the stationary distribution, using the boundaries conditions, the conditions at entry point, and the Kolmogorov forward equation.

Lemma 4. *Let $x_1 = \log(z_2/z_1)$ be the entry point, where large establishments that become small enter the small establishment distribution, and let $x_2 = \log(z_1/z_2)$ be the entry point, where small establishments that become large enter the large establishment distribution. The stationary distribution of small establishments $f_1(\cdot)$ and the stationary distribution of large establishments $f_2(\cdot)$ are given by:*

$$f_1(x) = \begin{cases} -xe^{\xi_1 x} & \forall x \in (-\infty, x_1^-] \\ \\ -\phi_1 x e^{\xi_1 x} & \forall x \in (x_1^+, 0] \end{cases} \qquad f_2(x) = \begin{cases} \phi_2 x e^{-\xi_2 x} & \forall x \in [0, x_2^-] \\ \\ xe^{-\xi_2 x} & \forall x \in (x_2^+, +\infty) \end{cases}$$

where $\xi_1 = \hat{\mu}_1/\hat{\sigma}^2$ and $\xi_2 = -\hat{\mu}_2/\hat{\sigma}^2$. The constant ϕ_1 and ϕ_2, after substituting that $x_1 = -x_2$, are equal to:

$$\phi_1 = \frac{x_1 e^{\xi_2 x_1}(1 - x_1 e^{\xi_1 x_1})}{1 - x_1^2 e^{(\xi_1+\xi_2)x_1}} \quad \textit{and} \quad \phi_2 = \frac{x_1 e^{\xi_1 x_1}(1 - x_1 e^{\xi_2 x_1})}{1 - x_1^2 e^{(\xi_1+\xi_2)x_1}}.$$

The entry rate of small establishments is $\lambda^1 = \hat{\mu}_1^2/2\hat{\sigma}^2$ and the entry rate of large establishments is $\lambda^2 = \hat{\mu}_2^2/2\hat{\sigma}^2$

Proof See Appendix A.4.

In Figure 2 we show the dynamics of small and large establishments' stationary distribution.

[3] We can rewrite the Geometrican Brownian motion of large and small establishments as a Brownian motion in the logarithm of the establishment productivity z_i as $dx_i = \hat{\mu}_i dt + \hat{\sigma}_z dw_z$.

The distribution at the top is the large establishments' distribution and at the bottom is the small establishments' distribution. The key points to understand the dynamics are the two zeros and the pair $\{x_1, x_2\}$. Starting from the large establishment distribution, at point zero, large establishment fire works and become small, they enter the small establishments distribution at point x_1. Then, establishments start to grow until they reach the zero of the small establishments' distribution. At this point establishments hire workers and become large and they enter the large establishments' distribution at x_2. Then, they start shrinking again until they reach the zero, and the cycle starts again. In the next section we describe the new entrants' problem.

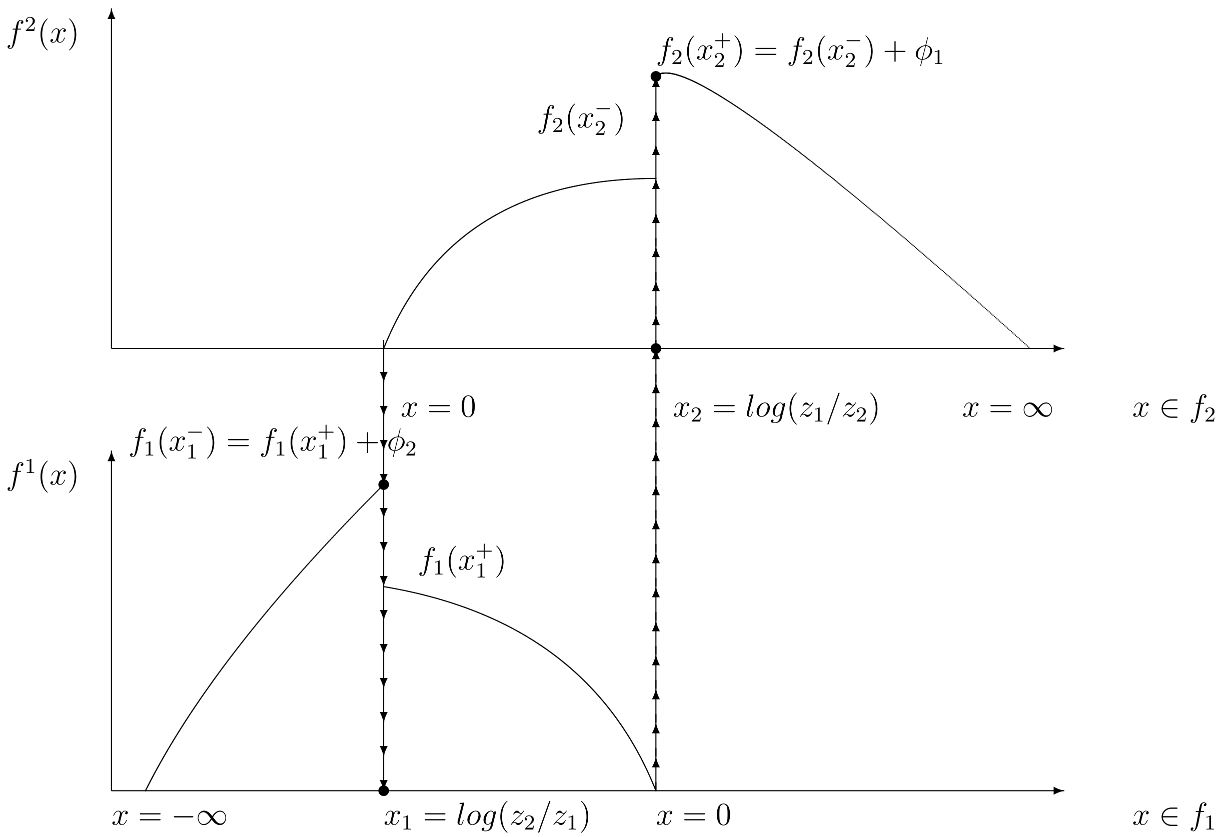


Figure 2: Stationary Distribution Dynamics

2.5 Entering establishment's problem

Potential new entrants pay an entry cost c_e in output and draw their productivity z from the endogenous distribution. The expected value of entering is equal to the sum of the expected value of being a small establishment and the expected value of being a large establishment minus the entry cost. The formal expression is found below:

$$W_e = \int_0^{z_1} W_1(z)g_1(z)dz + \int_{z_2}^{\infty} W_2(z)g_2(z)dz - c_e.$$

In an equilibrium with entry, W_e must be equal to zero, otherwise additional establishments would enter. The condition $W_e = 0$ is thus referred to as free-entry condition.

2.6 Household's problem

We assume that there is a measure one of identical households. They consume their entire income and supply labor inelastically to incumbents. We also assume full employment, consequently the proportion of workers fired must be the same as the proportion of workers hired every moment in time.

The proportion of workers hired is equal to the proportion of small incumbents that reach the switching point, normalized to zero, which is the right-derivative at zero ϕ_1, multiplied by the difference in the size of large and small establishments $(n_2 - n_1)$. The proportion of workers fired is equal to the proportion of large incumbents that reach the switching point zero, which is the left-derivative at zero ϕ_2 multiplied by the difference in the size of large and small establishments $(n_2 - n_1)$. The expression below summarizes the labor market clearing condition in flows:

$$\phi_1(n_2 - n_1) = \phi_2(n_2 - n_1).$$

In the next section, we characterize the stationary equilibrium

2.7 Stationary equilibrium

Definition The stationary equilibrium for this economy is an stationary distribution for small and large establishments $\{f_1(\cdot), f_2(\cdot)\}$, a value function for small and large establishments $\{W_1(\cdot), W_2(\cdot)\}$, a value function for entering establishments $\{W_e(\cdot)\}$, a policy function for small and large establishments $\{z_1, z_2\}$, a mass M of establishments, price $\{w, \rho\}$, and transfers T, such that:

i) Given prices, households consume their total income.

ii) Given prices, incumbents' policy functions $\{z_1, z_2\}$ and value functions $\{W_1(\cdot), W_2(\cdot)\}$ solve the incumbents' problem.

iii) The stationary distributions $\{f_1(\cdot), f_2(\cdot)\}$ solve the Kolmogorov forward equations.

iv) The entering establishments' value function $\{W_e(\cdot)\}$ solves the free-entry condition.

v) Labor market clears in levels: $M(n_1 \int_{-\infty}^{0} f_1(x)dx + n_2 \int_{0}^{+\infty} f_2(x)dx) = 1$.

vi) Labor market clears in flows: $\phi_1(n_2 - n_1) = \phi_2(n_2 - n_1)$.

vii) The government budget constraint is satisfied, $T = c_f(n_2 - n_1)\phi_2$.

Conditions (i) and (ii) are standard. Condition (iii) is the key condition to find the stationary distribution. Condition (iv) is the free-entry condition and conditions (v) and (vi) are the labor market clearing condition in levels and flows. Last, condition (*vii*) guarantees that the government budget constraint is satisfied. In the next section we characterize the stationary equilibrium.

2.8 Equilibrium properties

The model is very tractable and we can characterize key equilibrium properties in more detail. From the market clearing condition in flows, we know that in the stationary equilibrium the mass of establishments expanding must be equal to the mass of establishments shrinking to generate full employment. Lemma 5 formalizes this result.

Lemma 5. *In the stationary equilibrium the mass of small establishments switching from small to large must be equal to the mass of large establishments switching from large to small. This implies that in the stationary equilibrium ϕ_1 is equal to ϕ_2 and consequently ξ_1 is equal to ξ_2. The formal expression for ϕ is given by*

$$\phi(x_1) = \frac{-x_1 e^{\xi x_1}}{1 - x_1 e^{\xi x_1}},$$

where ξ is the same for small and large establishments and x_1 is the incumbents' decision rule.

Proof See Appendix A.5.

In Lemma 5 we characterize a key parameter of the model ϕ, which measures job turnover. From Lemma 3, we already know that an increase in the firing cost increases the inaction zone rate, in the next Lemma 6 we show that an increase in the inaction zone rate reduces job turnover.

Lemma 6. *Job turnover is decreasing in the inaction zone rate, i.e., $\phi'(\cdot) < 0$.*

Proof See Appendix A.6.

This key result has already appear quantitatively in the literature in **?**, but here emerges as a property of the model. Lemma 5 also have an implication to Geometric Brownian motions that solve the stationary equilibrium of the model. The Lemma 5 implies a linear relation between the drift of small establishments and the drift of large establishments, i.e. $\mu_1 = -\mu_2 + \sigma_z^2$. As a consequence of this equilibrium condition, in the stationary equilibrium, the mass of small and large establishment are the same and the entry rate of small and large establishments are also the same. Lemma 7 states these results.

Lemma 7. *In the stationary equilibrium the entry rate of small establishment λ_1 is equal to the entry rate of large establishments λ_2, and the mass of small establishments m_1 is equal to the mass of large establishments m_2, and is given by:*

$$m = (\phi(x_1) - 1)\left[\frac{x_1 e^{\xi x_1}}{\xi} - \frac{e^{\xi x_1}}{\xi^2}\right] + \frac{\phi(x_1)}{\xi^2}.$$

Proof See Appendix A.7.

After characterizing the stationary distributions, we can also solve output and TFP analytically . One desirable property of this economy is that both output and TFP are independent of the total mass of establishments M. In the following lemma 8 we formalize this property.

Lemma 8. *In the stationary equilibrium output Y and TFP are independent of the total mass of establishments M, and independent of the mass of large and small establishments m. The expression for output and TFP are given by:*

$$Y = \frac{n_1^\alpha E_1 z + n_2^\alpha E_2 z}{n_1 + n_2} \quad \text{and} \quad TFP = \frac{n_1^\alpha E_1 z + n_2^\alpha E_2 z}{n_1^\alpha + n_2^\alpha}.$$

and average productivity of small and large establishment is given by:

$$E_{n_1} z = z_1 \left\{ \frac{\phi + (1-\phi)\theta^{\xi+2}\left[1-(\xi+2)\ln\theta\right]}{\phi + (1-\phi)\theta^{\xi+1}\left[1-(\xi+1)\ln\theta\right]} \left(\frac{\xi+1}{\xi+2}\right)^2 \right\}$$

$$E_{n_2} z = z_1 \left\{ \frac{(1-\phi)\theta^2 + \phi\theta^{\xi}\left[(\xi-2)\ln\ \theta - 1\right]}{(1-\phi)\theta + \phi\theta^{\xi}\left[(\xi-1)\ln\ \theta - 1\right]} \left(\frac{\xi-1}{\xi-2}\right)^2 \right\}$$

Proof See Appendix A.8.

From Lemma 8 we can establish that the general equilibrium effect on wages affect both small and large establishments at the same rate through z_1, while changes on the inaction zone rate θ affect small and large establishment differently.

3 Calibration

We calibrate the benchmark economy, which is the economy without firing cost, to data for the United States. Our main objective is to study the quantitative impact of firing costs on the distribution of establishments and on aggregates relative to the undistorted economy in the same spirit of Restuccia and Rogerson (2008) and Hsieh and Klenow (2009). We start by defining briefly the undistorted economy that is our benchmark economy.

We consider our benchmark economy as an economy with with no firing costs (undistorted economy). In this economy, there is no inaction zone and large and small establishments change size at the same point (see Lemma 3). The invariant distribution of the undistorted economy is an application of Lemma 4, where now the entry point is equal to the switching point, $x_1 = x_2 = 0$. In Lemma 9 we solve for the stationary distribution of the undistorted economy.

Lemma 9. *Without firing cost, large and small establishments switch size at the same point.*

The stationary distribution of small and large establishments is given by:

$$f_1(x) = -xe^{\xi_1 x} \quad \text{and} \quad f_2(x) = xe^{-\xi_2 x},$$

where $\xi_1 = \hat{\mu}_1/\hat{\sigma}^2$ *and* $\xi_2 = -\hat{\mu}_2/\hat{\sigma}^2$*. The entry rate of small establishments is equal to* $\lambda^1 = \hat{\mu}_1^2/2\hat{\sigma}^2$ *and the entry rate of large establishments is equal to* $\lambda^2 = \hat{\mu}_2^2/2\hat{\sigma}^2$*.*

Proof Straightforward application of Lemma 4.

To calibrate this economy, we start by selecting a set of parameters that are standard in the literature, these parameters have either well-known targets which we match or the values have been well discussed in the literature. Following the literature we assume decreasing returns in the establishment-level production function and set α equal to 0.85 (e.g., Restuccia and Rogerson (2008)). We select the interest rate ρ to be equal to 0.05. We choose wages to normalize the entry cost c_e to 1. We normalize the size of small establishments n_1 to be equal 1 and we choose the size of large establishments n_2 to match the coefficient of variation for establishment with less than 50 employees, reported by Restuccia and Rogerson (2008), which is equal to 0.82.

The stochastic process has a direct implication to two moments in the data one is the Gini coefficient of firm size distribution, which is a function of ξ, and another one is the entry rate λ. We calibrate ξ and λ together to match the Gini coefficient reported by Luttmer (2010), which is equal to 0.89, and to calibrate the average entry rate reported by Lee and Mukoyama (2008), which is equal to 0.62. We obtain the Gini coefficient from the small establishment distribution and it is given by the expression below:[4]

$$\text{Gini} = \frac{1}{E_{n_1} z} \int_{s_*}^{\infty} F(z) \times (1 - F(z)) dz = \frac{2}{E_{n_1} z} \frac{(\xi - 1)^2}{(\xi - 2)^3} z_1 = \frac{2}{(\xi - 2)}.$$

The Gini and the entry rate λ have a direct implication to the drift μ_1 and the variance σ_z of the Brownian motion. As it show below:

$$\sigma_z^2 = \frac{\lambda(\text{Gini} + 2)^2}{2} \quad \text{and} \quad \mu_1 = \left(\frac{2}{(\text{Gini} + 2)} + \frac{1}{2} \right) \sigma_z^2.$$

These two moments are independent of the firing cost and they map to the fundamentals of the economy. Table 1 summarizes the calibrated parameters.

Table 1: Benchmark Calibration to US Data

Parameter	Value	Target
α	0.85	Literature
ρ	0.05	Literature
n_1	1.00	Normalization
n_2	10.00	Average firm size
μ_1	0.0327	Entry rate
σ_z	0.0831	Gini coefficient
c_e	1.000	Normalization

4 Quantitative Experiments

We quantify the impact of firing cost on aggregate TFP, aggregate output, productivity, and firm size distribution by comparing these statistics in each distorted economy relative to the

[4]The expression for the small establishments cumulative distribution function is given by:

$$F(z) = \int_{s_*}^{s} \frac{(\xi - 1)^2}{z_1} \left(\frac{z_1}{z} \right)^{\xi} \log(z_1/z) ds = 1 - \left(\frac{z_1}{z} \right)^{\xi - 1} \left[(1 - \xi) \ln\left(\frac{z_1}{z} \right) + 1 \right].$$

benchmark economy which is assumed to have no distortions. We highlight the quantitative impact of firing cost in our model by analyzing the effect of firing cost output, aggregate TFP, small establishments' average TFP, large establishments' average TFP, wages, and the relative mass of establishments relative to the benchmark economy.

4.1 Firing Costs

We study the the impact of different firing cost policies by changing c_f on the economy and report statistics relative to the undistorted economy. The firing cost c_f has a direct interpretation with other values in the model, since a period in the model is equal to 1 year, a value of c_f equal to 1/12 corresponds to 1 months' wages, and a value c_f equal to 1 corresponds to 1 years' wage. We report the results in Table 2 for a number of statistics such as aggregate output, aggregate TFP, small establishments' average TFP, large establishments' average TFP, wages, and the relative mass of establishments. All statistics reported are relative to the benchmark economy in percent.

Table 2: Effects of Changes in Firing Cost c_f

	c_f					
	1/12	1/4	1/2	1	2	5
Relative Y	0.8043	0.7046	0.6183	0.5089	0.3743	0.1729
Relative TFP	0.8043	0.7046	0.6183	0.5089	0.3743	0.1729
Relative wages w	0.9365	0.8826	0.8264	0.7436	0.6240	0.3928
Relative small establishments TFP	0.9224	0.8442	0.7716	0.6760	0.5544	0.3535
Relative large establishments TFP	0.7987	0.6979	0.6110	0.5009	0.3657	0.1643

As is apparent in Table 2, firing costs have a substantial impact on aggregate output and TFP. An economy with a 1 month firing cost has aggregate output and TFP that is around 80 percent of the benchmark economy. While an economy with 5 years of firing cost generates an economy that has 17 percent of the output and TFP of the benchmark economy. Overall, differences in output and TFP relative to the benchmark economy are increasing in the amount of firing cost, more firing cost implies lower TFP and output.

Regarding the distribution of firm, an increase in firing cost generates economies where establishments are on average less productive independent of their size. Both small and larger firms have on average lower productivity and larger firms are affected more than smaller ones. Wages also fall with an increase in firing cost, because on average the expected value of incumbents are lower.

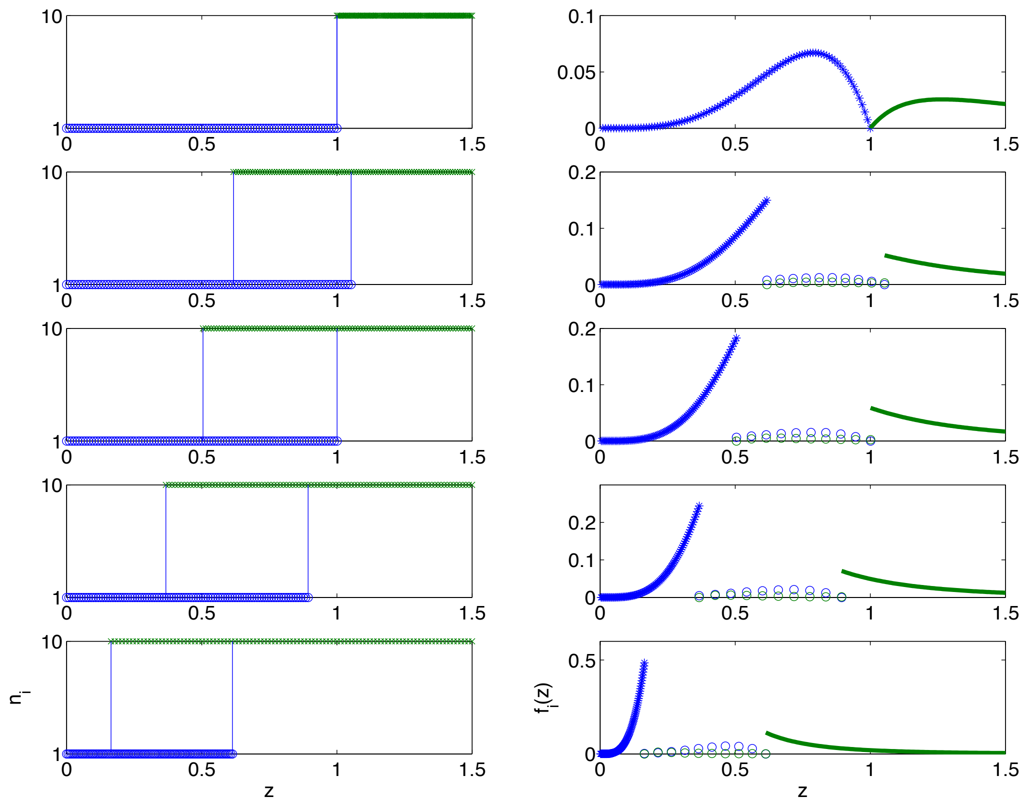
It is also instructive to analyze how firing costs affect the decision rules of small establishments, large establishments, and job turnover of the economy. Table 3 contains information about the inaction zone rate, job turnover, and decision rules of small and large establishments relative to the benchmark economy.

Table 3: Effect of Firing Cost c_f on Decision Rules

	c_f					
	1/12	1/4	1/2	1	2	5
Relative Inaction rate $\theta = z_2/z_1$	0.7494	0.6574	0.5863	0.5044	0.4109	0.2681
Job turnover ϕ	0.0783	0.0721	0.0526	0.0362	0.0201	0.0049
Relative decision rule z_1	1.0758	1.0737	1.0529	1.0006	0.8934	0.6134
Relative decision rule z_2	0.8062	0.7059	0.6174	0.5047	0.3671	0.1644

The first result from Table 3 is that increases in the firing cost increase the inaction zone rate, which decreases job turnover. An increase in firing cost for 1 month to 1 year reduces job turnover by 50 percent. Compare with previous results of the literature **?** 1 year wage of firing cost in **?** reduces TFP by 3 percent and job turnover by 8 percent. In our economy for the same level of firing cost, TFP decreases by 15 percent and job turnover by 50 percent. The main reason for such a difference in magnitude is because in our model the distribution of TFP is endogenous and changes with firing cost. In figure 3, we report both the inaction zone on the left and the endogenous distribution of small establishments in blue and the endogenous distribution of large establishments on green for an economy without firing cost on the top of the figure, an economy with 1 3 months, 6 months, and 1 year of firing costs.

Figure 3: Inaction zones and Distribution of Productivity



The key effect to understand the significant impact of firing cost on TFP is the interaction between inaction zones, the panel on the left, and role of the endogenous distribution, on the right. On the left panel, we can see that increases in firing cost increase the inaction zone, by moving both the decision rules of large establishments to the left and the decision rules of small establishments to the right. On the right panel, we observe how the distribution interact with the inaction zone. As the inaction zone increases, the mass of small establishments move to left to zone where the productivity of TFP lower, reducing the average of TFP of small establishments. The mass of large establishments move to more productivity firms, since small establishments wait longer to become small, however, this effect is not able compensate the overall increase in the inaction zone. As a result, increases in the firing cost have significant impact on TFP, wanes, and job turnover.

4.2 Hiring Costs

We establish an equivalence between firing costs and hiring costs. This equivalence is important because it allows us to relate our results to the broad literature on size-dependent policies in Guner et al. (2008) and more specifically in Gourio and Roys (2014) who study a size-dependent regulation in the form of firing taxes in France that only apply to establishments with 50 or more employees.

5 Conclusions

TBW.

References

Buera, F. J., Kaboski, J. P., and Shin, Y. (2011). Finance and development: A tale of two sectors. *The American Economic Review*, pages 1964–2002.

Da-Rocha, J.-M., Tavares, M. M., and Restuccia, D. (2014). Policy distortions and aggregate productivity with endogenous establishment-level productivity. Working Paper, University of Toronto.

Dixit, A. (1989). Entry and exist decisions under uncertainty. *Journal of Political Economy*, 97(3):620–638.

Gourio, F. and Roys, N. (2014). Size-dependent regulations, firm size distribution, and reallocation. *Quantitative Economics*, 5:377–416.

Greenwood, J., Sanchez, J. M., and Wang, C. (2013). Quantifying the impact of financial development on economic development. *Review of Economic Dynamics*, 16(1):194–215.

Guner, N., Ventura, G., and Xu, Y. (2008). Macroeconomic implications of size-dependent policies. *Review of Economic Dynamics*, 11(4):721–744.

Hopenhayn, H. (1992). Entry, exit, and firm dynamics in long run equilibrium. *Econometrica*, 60:1127–50.

Hopenhayn, H. (2013). Distortions, firms, and aggregate productivity. slides, INEGI/ITAM Workshop on Productivity.

Hopenhayn, H. and Rogerson, R. (1993). Job turnover and policy evaluation: A general equilibrium analysis. *Journal of Political Economy*, 101(5):915–38.

Hopenhayn, H. A. (2014). Firms, misallocation, and aggregate productivity: A review. *Annu. Rev. Econ.*, 6(1):735–770.

Hsieh, C. and Klenow, P. (2014). The life-cycle of plants in india and mexico. Quarterly Journal of Economics, forthcoming.

Lee, Y. and Mukoyama, T. (2008). Entry, exit and plant-level dynamics over the business cycle. *FRB of Cleveland Working Paper*, (07-18R).

Luttmer, E. (2010). Models of growth and firm heterogeneity. *Annual Reviews of Economics*, pages 547–76.

Moscoso-Boedo, H. J. and Mukoyama, T. (2012). Evaluating the effects of entry regulations and firing costs on international income differences. *Journal of Economic Growth*, 17(2):143–170.

Restuccia, D. (2013). Factor misallocation and development. In Edition, O., editor, *The New Palgrave Dictionary of Economics*. Steven N. Durlauf and Lawrence E. Blume, Palgrave Macmillan.

Restuccia, D. and Rogerson, R. (2008). Policy distortions and aggregate productivity with heterogeneous establishments. *Review of Economic Dynamics*, 11(4):707–720.

Restuccia, D. and Rogerson, R. (2013). Misallocation and productivity. *Review of Economic Dynamics*, 16(1):1–10.

A Appendix

A.1 Proof Lemma 1

The problem is standard. The proof is by guessing and verifying, we guess the following functional form for the value function $W_i(z) = a_i + A_i z + B_i z^{\beta_i}$ and solving the Hamilton-Jacobi-Bellman equation we find that $a_i = -\frac{wn_i}{\rho}$, $A_i = \frac{{n_i}^{\alpha}}{\rho - \mu_i}$ and β_i is equal:

$$\beta_i = -\left(\frac{\mu_i}{\sigma_z^2} - \frac{1}{2}\right) \pm \sqrt{\left(\frac{\mu_i}{\sigma_z^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma_z^2}}.$$

Finally from the boundary and smooth pasting conditions we find B_1, B_2, z_1, and z_2 solving the following system of nonlinear equations:

$$\begin{aligned}
(1-\beta_1)B_1 z_1^{\beta_1} &= (a_2 - a_1) + (1-\beta_2)B_2 z_1^{\beta_2} && (7)\\
(1-\beta_1)B_1 z_2^{\beta_1} &= (a_2 - a_1) + (1-\beta_2)B_2 z_2^{\beta_2} + c_f(n_2 - n_1)w, && (8)\\
\beta_1 B_1 z_1^{\beta_1} &= (A_2 - A_1)z_1 + \beta_2 B_2 z_1^{\beta_2}, && (9)\\
\beta_1 B_1 z_2^{\beta_1} &= (A_2 - A_1)z_2 + \beta_2 B_2 z_2^{\beta_2}. && (10)
\end{aligned}$$

And this conclude the proof. ■

A.2 Proof Lemma 2

First we need to show that the positive root β_1 is greater than one. The root β_1 is the solution of the following polynomial:

$$\Omega(\beta) = \frac{\beta^2}{2} + \left(\frac{\mu_1}{\sigma_z^2} - \frac{1}{2}\right)\beta - \frac{\rho}{\sigma_z^2}.$$

First note that this polynomial is convex, since $\Omega''(\xi) = 1$, and at zero $\Omega(0) = -\frac{\rho}{\sigma_z^2}$, which is negative. So, $\Omega(\cdot)$ has a positive and a negative root. At one $\Omega(1) = \frac{\mu_1}{\sigma_z^2} - \frac{\rho}{\sigma_z^2}$, since ρ is greater than μ_1, $\Omega(1)$ is also negative. Consequently, the positive root must be greater than one. Now, we can prove that B_1 and B_2 are positive.

From Lemma 1 and the two equations on the smoothing pasting conditions (10) and (10), we can write B_1 and B_2 as a function of the parameters and z_1 and z_2, the constant B_2 is equal to:

$$B_2 = \frac{(A_2 - A_1)(z_2^{1-\beta_1} - z_1^{1-\beta_1})}{\beta_2(z_1^{\beta_2-\beta_1} - z_2^{\beta_2-\beta_1})} \tag{11}$$

Note that the numerator is positive (negative), because $A_2 - A_1$ is positive and $(z_2^{1-\beta_1} - z_1^{1-\beta_1})$ is positive, since $z_1 > z_2$ and $\beta_1 > 1$, as we can see from ($\beta_1 < 1$). The denominator is also positive, because β_2 is negative and $(z_1^{\beta_2-\beta_1} - z_2^{\beta_2-\beta_1})$ is algo negative. As a result B_2 is positive. Substituting the expression for B_2 into equation 10, we find B_1 equal to:

$$B_1 = \frac{(A_2 - A_1)(z_1 z_2)^{-\beta_1}(z_1^{\beta_2} z_2 - z_1 z_2^{\beta_2})}{\beta_1(z_1^{\beta_2-\beta_1} - z_2^{\beta_2-\beta_1})} \tag{12}$$

Note that the numerator is negative, because $z_1 > z_2$ and $z_1^{\beta_2} < z_2^{\beta_2}$ since $\beta_2 < 0$. In addition, the denominator is also negative, because $(z_1^{\beta_2-\beta_1} - z_1^{\beta_2-\beta_1})$ is negative and β_1 is positive, as result B_1 is positive. ■

A.3 Proof Lemma 3

From the market clearing condition in flows we find that $\xi = \xi_1 = \xi_2$. Therefore $\frac{\mu_1}{\sigma^2} - \frac{1}{2} = -\frac{\mu_2}{\sigma^2} + \frac{1}{2}$, and

$$\beta_1 = \frac{1}{2} - \frac{\mu_1}{\sigma_z^2} + \sqrt{\left(\frac{\mu_1}{\sigma_z^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma_z^2}} = -\xi + \sqrt{\xi^2 + \frac{2\rho}{\sigma_z^2}}$$

$$\beta_2 = \frac{1}{2} - \frac{\mu_2}{\sigma_z^2} - \sqrt{\left(\frac{\mu_2}{\sigma_z^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma_z^2}} = -\frac{1}{2} + \frac{\mu_1}{\sigma_z^2} - \sqrt{\left(-\frac{\mu_1}{\sigma_z^2} + \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma_z^2}} = -\beta_1,$$

Define the positive root as

$$\beta = -\xi + \sqrt{\xi^2 + \frac{2\rho}{\sigma_z^2}}$$

and rewrite the boundary and smooth pasting conditions as:

$$(1-\beta)B^1 z_1^{\beta} - (1+\beta)B^2 z_1^{-\beta} = -\frac{(n_2 - n_1)}{\rho} w, \tag{13}$$

$$(1-\beta)B^1 z_2^{\beta} - (1+\beta)B^2 z_2^{-\beta} = -\frac{(n_2 - n_1)(1 - \rho c_f)}{\rho} w, \tag{14}$$

$$B^1 z_1^{\beta} = \frac{(A^2 - A^1)z_1 - \beta B^2 z_1^{-\beta}}{\beta}, \tag{15}$$

$$B^1 z_2^{\beta} = \frac{(A^2 - A^1)z_2 - \beta B^2 z_2^{-\beta}}{\beta}. \tag{16}$$

From (15) and (16) we have **note that is equal to eq (12)**)

$$B_2 = \frac{(A_2 - A_1)}{-\beta}\left(\frac{z_2^{1-\beta} - z_1^{1-\beta}}{z_1^{-2\beta} - z_2^{-2\beta}}\right)$$

$$B_1 = \frac{(A_2 - A_1)}{\beta} z_1^{1-\beta} - B_2 z_1^{-2\beta}$$

Assume that $z_2 = \theta z_1$, with $\theta \in (0,1]$. Then using

$$B_2 = \frac{(A_2 - A_1)}{\beta}\left(\frac{1-\theta^{1-\beta}}{1-\theta^{-2\beta}}\right) z_1^{1+\beta} = \frac{(A_2 - A_1)}{\beta}\Theta z_1^{1+\beta}$$

we have that

$$\frac{B_1}{B_2} = \frac{(A_2 - A_1)}{\beta B_2} z_1^{1-\beta} - z_1^{-2\beta} = \left[\frac{1}{\Theta} - 1\right] z_1^{-2\beta}$$

Using

$$\frac{(1-\beta)B^1}{(1+\beta)B^2} z_1^{\beta} - z_1^{-\beta} = \frac{(1-\beta)}{(1+\beta)}\left(\frac{1}{\Theta} - 1\right) z_1^{-2\beta} z_1^{\beta} - z_1^{-\beta} \tag{17}$$

$$\frac{(1-\beta)B^1}{(1+\beta)B^2} z_2^{\beta} - z_2^{-\beta} = \frac{(1-\beta)}{(1+\beta)}\left(\frac{1}{\Theta} - 1\right) z_1^{-2\beta} z_2^{\beta} - z_2^{-\beta} \tag{18}$$

Rewrite (13) and (14) as

$$\left[\frac{(1-\beta)}{(1+\beta)}\left(\frac{1}{\Theta} - 1\right) - 1\right] z_1^{-\beta} = -\frac{(n_2 - n_1)}{\rho(1+\beta)B^2} w, \tag{19}$$

$$\left[\frac{(1-\beta)}{(1+\beta)}\left(\frac{1}{\Theta} - 1\right)\theta^{\beta} - \theta^{-\beta}\right] z_1^{-\beta} = -\frac{(n_2 - n_1)(1-\rho c_f)}{\rho(1+\beta)B^2} w, \tag{20}$$

Then

$$\varphi = \frac{\left[\frac{(1-\beta)}{(1+\beta)}\left(\frac{1}{\Theta} - 1\right) - 1\right]}{\left[\frac{(1-\beta)}{(1+\beta)}\left(\frac{1}{\Theta} - 1\right)\theta^{\beta} - \theta^{-\beta}\right]} = \frac{1}{1-\rho c_f} \tag{21}$$

Note that, if $f_c = 0$, then $\theta = 1$ and $z_1 = z_2$. **me must show that $\lim_{\theta \to 0} \varphi = \infty$, and φ is decreasing as in the figure**

Finally, from equation (19) we have

$$\left[\frac{(1-\beta)}{(1+\beta)}\left(\frac{1}{\Theta} - 1\right) - 1\right] z_1^{-\beta}\frac{(A_2 - A_1)}{\beta}\Theta z_1^{1+\beta} = -\frac{(n_2 - n_1)}{\rho(1+\beta)} w \tag{22}$$

$$z_1 = \kappa(\theta) w \tag{23}$$

where

$$\kappa(\theta) = \frac{-\frac{(n_2 - n_1)}{\rho(1+\beta)}}{\left[\frac{(1-\beta)}{(1+\beta)}\left(\frac{1}{\Theta} - 1\right) - 1\right]\frac{(A_2 - A_1)}{\beta}\Theta}$$

■

A.4 Proof Lemma 4

We start the proof by characterizing the distribution of small size establishments $f^1(\cdot)$. This distribution satisfies the two boundary conditions $f^1(0) = 0$ and $f^1(-\infty) = 0$, and outside the entry point x^1 the distribution satisfies the Kolmogorov forward equation (6). The proof is by guess and verify, we first guess the distribution $f^1(\cdot)$ to be equal to $f^1(x) = -xe^{\xi_1 x}$, for x in the interval $(-\infty, x_1^-]$ and $f^1(x) = -\phi_1 x e^{\xi_1 x}$, for x in the interval $(x_1^+, 0)$. $f^1(\cdot)$ satisfies the Kolmogorov Forward Equation below:

$$-\lambda_1 f^1(x) = -\hat{\mu}_1 \left(-e^{\xi_1 x} + \xi_1 f^1(x)\right) + \frac{\hat{\sigma}^2}{2}\left(-2\xi_1 e^{\xi_1 x} + \xi_1^2 f^1(x)\right).$$

After simple algebraic calculation we find that ξ_1 and λ_1 that solve the equation above are $\xi_1 = \frac{\hat{\mu}_1}{\hat{\sigma}^2}$ and $\lambda_1 = \frac{\hat{\mu}_1^2}{2\hat{\sigma}^2}$. In addition, we find the same ξ_1 and λ_1 for $f^1(x) = -\phi_1 x e^{\xi_1 x}$.

For large establishments using the same methodology we find $\xi_2 = -\frac{\hat{\mu}_2}{\sigma^2}$ and $\lambda_2 = \frac{\hat{\mu}_2^2}{2\sigma^2}$. Now, we still need to find the two constants ϕ_1 and ϕ_2, and we use the two conditions at the entry points, equation (5), to characterize these constants. Using the fact that $x_2 = -x_1$, and substituting then in equation equations (5), we find the following equations:

$$\begin{aligned}
\phi_1 &= \frac{x_1 e^{\xi_1 x_1}(1 - x_1 e^{\xi_2 x_1})}{1 - x_1^2 e^{(\xi_1+\xi_2)x_1}} \\
\phi_2 &= \frac{x_1 e^{\xi_2 x_1}(1 - x_1 e^{\xi_1 x_1})}{1 - x_1^2 e^{(\xi_1+\xi_2)x_1}}
\end{aligned}$$

■

A.5 Proof Lemma 5

From the market clearing condition in flows ϕ_1 and ϕ_2 must be the same. Consequently, from the conditions at the switching points, equation 5, we find that ξ_1 must be equal to ξ_2. ■

A.6 Proof Lemma 6

Taking the first order condition with respect to x_1, we find $\phi'(\cdot) = \frac{-(1+\xi x_1)e^{\xi x_1}}{(1-x_1 e^{\xi x_1})^2}$, since ξ is always positive and x_1 is also positive, we find that $\phi'(\cdot) < 0$. ■

A.7 Proof Lemma 7

First, the same entry rate for large and small establishments is a direct consequence that ξ_1 is equal to ξ_2, which implies that λ_1 is equal to λ_2. Second, from Lemma 4 the mass of small and the mass of large establishment are given by:

$$
\begin{aligned}
m_1 &= \int_{-\infty}^{x_1} -xe^{\xi_1 x}dx + \int_{x_1}^{0} -\phi_1 x e^{\xi_1 x}dx = \frac{\phi_1}{\xi_2} + (\phi_1 - 1)\left(\frac{x_1 e^{\xi_1 x_1}}{\xi_1} - \frac{e^{\xi_1 x_1}}{\xi_1^2}\right) \\
m_2 &= \int_{0}^{x_2} \phi_2 x e^{-\xi_2 x}dx + \int_{x_2}^{+\infty} x e^{-\xi_2 x}dx = \frac{\phi_2}{\xi_2^2} + (1 - \phi_2)\left(\frac{x_2 e^{-\xi_2 x_2}}{\xi_2} + \frac{e^{-\xi_2 x_2}}{\xi_2^2}\right)
\end{aligned}
$$

From 5 we know that ϕ_1 is equal to ϕ_2 and ξ_1 is equal to ξ_2. In addition, from the entry point definition we know that $x_1 = -x_2$. Substituting all these terms to the expression above we find that the two masses are equal. ■

A.8 Proof Lemma 8

Output in this economy is given by the total mass of establishments multiplied by the total output produced by small and large establishments. Substituting the total mass of establishments M from the labor's market clearing condition on levels, we find that output does not depend on the mass of small and large establishments and neither on the total mass of establishments.

$$
Y = M\left(m_1 n_1^{\alpha} E_{n_1} z + m_2 n_2^{\alpha} E_{n_2} z\right) = \frac{n_1^{\alpha} E_{n_1} z + n_2^{\alpha} E_{n_2} z}{n_1 + n_2}
$$

TFP also does not depend on the total mass of establishments neither in the mass of small and large establishments. Using again the labor's market clearing condition in levels and some simple analytical manipulation we find that is TFP is given by:

$$
TFP = \frac{Y}{M(m_1 n_1^{\alpha} + m_2 n_2^{\alpha})} = \frac{m_1 n_1^{\alpha} E_{n_1} z + m_2 n_2^{\alpha} E_{n_2} z}{m_1 n_1^{\alpha} + m_2 n_2^{\alpha}} = \frac{n_1^{\alpha} E_{n_1} z + n_2^{\alpha} E_{n_2} z}{n_1^{\alpha} + n_2^{\alpha}}.
$$

■