Market Based, Segregated Exchanges in Securities with Default Risk

Weerachart T. Kilenthong and Robert M. Townsend*

April 1, 2009

Abstract

This paper studies a competitive general equilibrium model with default and endogenous collateral constraints. Even though all collateralized contracts are allowed, the possibility and desirability of trade in spot markets (or the equivalent trade in ex ante asset backed securities) creates externalities, as spot prices (or security prices) and the bindingness of collateral constraints interact. We show that if agents are allowed to contract ex ante on market fundamentals determining the state-contingent spot price, over and above contracting on true underlying states of the world, then competitive equilibria with bundled securities and commodities and with endogenous collateral constraints are equivalent with Pareto optima. Examples show that it is possible to have multiple market fundamentals in equilibrium. Equivalently, it is possible for there to be segregation into distinct competitive securities exchanges with endogenous (positive and negative) entry fees. Fees accrue to borrowers who are otherwise collateral constrained.

Keywords: Default, Walrasian Equilibrium, Limited Commitment, Endogenous Collateral, Externalities, Market Fundamentals, Welfare Theorems

1 Introduction

This paper uses a competitive general equilibrium with directly-collateralized and asset-backed securities to analyze the interaction between the endogenous valuation of collateral and corresponding

*This paper previously circulated under the title "Endogenous Valuation of Collateral and Contingent Trade in Price Fundamentals." Tee Kilenthong is at Department of Economics, University of California at Santa Barbara, Email: tee@econ.ucsb.edu. Robert Townsend is at Department of Economics, MIT, Email: rtownsen@mit.edu. We would like to thank seminar participants at MIT, UCSB, and 2008 SED conference. Robert Townsend is grateful to financial supports from NSF, NICHD, and Templeton Foundation through the University of Chicago where he is research associate. Tee Kilenthong is grateful to financial support from David Marshall-Merrill Lynch Faculty Fellowship.

default decisions. The interaction creates an "externality", which causes a collateral-constrained equilibrium to be inefficient (e.g. Kiyotaki and Moore (1997), Geanakoplos (2003) and Lustig (2007)). The externality exists because the decisions of all agents in the contracting period determine the spot prices and the prices of asset backed securities that in turn influence the feasible consumption sets of all individuals. The impact on the feasible set in turn affects the allocations of all agents whenever the collateral or borrowing constraints of some agents are binding.

With an extended commodity space which allows for endogenous selection into security market exchanges, we prove that competitive equilibria with endogenous collateral constraints are equivalent with (constrained) Pareto optima. Specifically, we show that if agents are allowed to contract ex ante on market fundamentals determining the state-contingent spot price, over and above contracting on true underlying states of the world, and to bundle commodity and security trades accordingly, then competitive equilibria with endogenous collateral constraints are equivalent with Pareto optima.

We take it as a primitive that default is possible or equivalently that collateral is required to make borrowers (or issuers of securities) repay their loans. A borrower may choose to default on a particular loan, or a particular state-contingent promise, and in doing so would lose the value of collateral backing that particular loan or security. A rational borrower will base her default decision security by security on the value of the collateral backing each liability, compared to the original promise to pay. Of course the value of the collateral good at the time of repayment decisions (called the execution period) and in the market for asset backed securities (in the contract period) is an equilibrium phenomenon. Yet this market-clearing price of collateral in spot markets determines whether borrowers default or not, and the overall amount of debt.

A contract or security consists of two items, a state-contingent promise and the collateral backing that promise. Contracts which do not default have to be backed by a sufficient, minimum level of collateral, again depending on the promise and the value of collateral. Likewise asset backed securities which are issued have to be backed in collateral by an equivalent value of asset backed securities acquired. For every set of securities which actually default, handing over collateral, there is another set which is equivalent, with the same overall payoff and no default. Adding up all such promises, over state contingent security promises directed backed by collateral and over state contingent securities backed by the promises of others generates a state contingent collateral constraint on trades in the ex ante contract market. But contracts which do default also require collateral that is to be handed over when the borrower does not repay. That is, partially collateralized securities are still intimately associated with the exactly amount of collateral which serves a backing. Again, by rescaling contracts, these constraints can be shown to be equivalent. We label such constraints collateral constraints, for brevity.

Collateral constraints generate endogenous short-sale constraints, limiting promises issued. These constraints depend on the spot (market-clearing) relative price of the collateral good in the spot market (and/or the relative price of securities in the contract market paying the collateral good.) Here the flow of ideas is from subsequent spot prices to current collateral requirements. On the other hand, the spot prices in execution periods are influenced by the aggregate actions in the trading period, as savings in the form of (endogenous) collateral backing issued promises to pay in the trading period is carried over time and impacts the spot prices (and therefore affects the allocations in execution period). Here the flow of ideas is from ex ante contracting to subsequent spot prices. This simultaneity or externality causes a collateral constrained equilibrium to be inefficient. This externality remains even if trade in asset backed securities replaces all spot trade. Collateral used in backing of promises still distorts relative prices.

The externality problem is thus a missing-market problem; that is, the markets for contracts over the "market fundamentals", those aspects of the environment which determine the spot-market-clearing price, are missing. The market fundamentals are, in general, defined by the distribution of the resources across types of agents. In this paper, with identical homothetic preferences, the market fundamental is simply the aggregate ratio of a pair of physical commodities. Note again that a market fundamental is endogenous; that is, it depends on the aggregate saving, which is a result of the actions in the trading period of all agents as a group.

In this paper, we extend the commodity space so that contracts are contingent on these market fundamentals and contingent contracts can be bought or sold, over and above contracts contingent on true underlying states of the world, though these are bundled together. Allowing agents to contract ex-ante on market fundamentals determining the state-contingent spot-market-clearing price in effect allows them to contract on the price, and internalizes the externality. We thus prove that the competitive equilibria with endogenous collateral constraints and the extended commodity space are equivalent with Pareto optima. One could view these results as normative, indicative of the need for a systematic but market determined way for traders to unwind commitments.

A price island is a way to conceptualize the consistent execution of the contingencies on fundamentals. That is, a price island specifies the spot price, and the set of agents that end up there have to support that price. This is like a club constraint in other literature, e.g. Prescott and Townsend (2006). Agents can carry in goods in such a way that their pre-trade ratio of endowments in a spot market deviates from the market fundamental, but the sum of the deviations must, by the definition of consistency, be zero. Lotteries are then a way to assign agents to price islands jointly with other decisions such as security holdings and end-of-trading-period collateral. In a decentralized equilibrium, in which prices are taken as given, all price islands including out-of-equilibrium price islands are available for agents to purchase. Specifically, we internalize the externality by making household types pay or be paid for their influence on the spot market prices. On another interpretation, ex post spot trades are replaced by ex ante trade in asset backed securities. In this interpretation, a household has to pay or be paid for the rights to trade in particular security exchanges. This is related to the consumption right in Bisin and Gottardi (2006). Here the coupling of asset backed securities with directly backed securities is even more natural.

We do not require that the markets keep track of individual trades and contracts, only that the over all composition of traders be such as to deliver the contracted price. This takes a certain commitment to prevent retrading across the "price islands." With that in place, prices will direct trade and traders efficiently.

More generally, endogenous collateral constraints generate a non-convexity problem, as prices reflecting assignments and collateral decisions interact multiplicatively. As Prescott and Townsend (1984b), Hansen (1985), Rogerson (1988) have shown, lotteries can (weakly) improve on deterministic allocations when feasible sets are not convex. For computational purposes, we allow each variable to take only finite values (finite grids).

Importantly, the amount a household pays (or receives) in unit of account in the contract market does depend on its type, its individual endowment position relative to the market fundamental. Again, the amount that an agent will pay (or receive) depends on the difference between the market fundamental and endowment ratio of good-1 to good-2 (including collateral holding). If her endowment ratio is exactly equal to the market fundamental, then she does not pay (nor is paid). On the other hand, if she comes with a low (high) endowment ratio relative to the market fundamental, i.e., holding little of good-1 and lots of good-2 relative to the market fundamental, so that with good 1 as the numeraire, the spot price of the abundantly held good 2 is high (holding lots of good-1), then she will pay (will be paid) for the right to trade at the specified market fundamental.

The collateralization structure in this model incorporates both "tranching" and "pyramiding". With "tranching", a specific piece of collateral can be used to back up several contracts as long as their promises to pay are in different states. With "pyramiding", agents are allowed to use financial assets, the contracts for promises to receive goods of others, as collateral for their own promises. One could interpret these contracts as the asset-backed securities which are much in the news these days, e.g., the set of securitized mortgage obligation, a promise ultimately but indirectly backed by an underlying collateral asset. What actually gets traded is an equilibrium phenomenon. This is different from the contract-specific collateralization structure as in Geanakoplos (2003), among others, where the collateral of a contract cannot be used as collateral for any other contract. On the other hand, our structure is similar to that of Lustig (2007), where several state-contingent contracts can be backed by the same collateral.

Of course agents are allowed to retrade in spot markets, and that is what delivers the spot-market-clearing prices. However, with pyramiding, agents are indifferent between ex-ante contracting versus retrading in spot markets. This is because anything which can be done in the spot market, trading one good for another, can be done in the ex ante contract market, with promises to receive one good backing promises to pay the other. Agents do not need to retrade in spot markets, but they may well do so. It is worth noting, however, that most of results in this paper would obtain even without the "pyramiding" assumption. We do so to allow for realism and generality. Even without pyramiding, state contingent collateral constraints generate an externality. This externality is what we seek to remedy.

Related Literature

This paper is related to several strands of literature on limited commitment. The first group is a class of general equilibrium models with limited commitment, e.g., Kehoe and Levine (1993), Kocherlakota (1996), Alvarez and Jermann (2000), and Lustig (2007). Similar to our model, they allow ex ante complete contracts. These papers, except Lustig (2007), study dynamic general equilibrium models with financial-market exclusion as a punishment scheme to prevent households from defaulting on their promises. The punishment mechanism is clearly different from ours, as for us defaulting agents will only lose their collateral. More importantly, in Alvarez and Jermann (2000) there is no interaction between spot prices and borrowing constraints, and hence, there are no externalities in those models. Alvarez and Jermann (2000) used a competitive equilibrium with state-contingent portfolio constraints, called solvency constraints, to decentralize the constrained optimal allocations. The solvency constraints in that context are defined using individual utility, while our collateral constraints are based on the spot-market prices or marginal rate of substitutions. Kehoe and Levine (1993) decentralized constrained optimal allocations with limited commitment by imposing participation constraints directly on the consumption possibility sets.

On the other hand, Lustig (2007) studies the asset pricing implications of a competitive equilibrium with collateral constraints similar to ours. He also considers all state-contingent collateralized contracts, and this is implicitly equivalent with "tranching." In addition, with spot markets, the interaction between spot prices and collateral constraints should result in externality as well. As a result the competitive equilibrium in Lustig (2007) is inefficient. Again our competitive equilibrium with those contracts over the market fundamentals is efficient.

The second strand of the literature is a class of general equilibrium models with endogenous collateral constraints, Kiyotaki and Moore (1997), Geanakoplos and Zame (2002), Geanakoplos (2003), and Kilenthong (2006), among others. Our model economy environment is the same as the environment of these papers, but our contractual structure is different. These papers assume a contract-specific collateralization structure while we allow for both "tranching" and "pyramiding". This literature follows the tradition of Bewley (1986) and assumes exogenously incomplete contracts as part of the specification. Our paper allows for all state- contingent contracts, but with with limited collateral, so in this sense our contracts are endogenously incomplete. The closest paper to our is Kilenthong (2006) in making endogenous the contract space and showing that contracts are genuinely incomplete but on the other hand with one good, in Kilenthong (2006) there is no externality.

In addition, this paper is also related to literature on liquidity, e.g., Holmström and Tirole (1998), Caballero and Krishnamurthy (2004), and Lorenzoni (Lorenzoni). Liquidity demand originates in the technology in case of Holmström and Tirole (1998), as used in Caballero and Krishnamurthy (2004). On the contrary, our model generates liquidity demand from endowment profiles. Lorenzoni (Lorenzoni) incorporate spot markets into Holmström and Tirole (1998), and Caballero and Krishnamurthy (2004), and shows again that it is the interaction between spot prices and liquidity constraints that create an externality.

This paper also contributes to security design literature. There are several papers dealing with asset-backed securities (or "pyramiding"), e.g., Demarzo and Duffie (1999), Biais and Mariotti (2005), DeMarzo (2005), and Steinert and Torres-Martínez (2007). Demarzo and Duffie (1999) study a signalling game of asset-backed security design under asymmetric information. Biais and Mariotti (2005) use a mechanism design approach to study a similar problem. Demarzo and Duffie (1999) and DeMarzo (2005) focus on a tranching mechanism in a private information environment. Steinert and Torres-Ma (2007) study the existence of competitive equilibrium with asset-backed securities. The last paper is similar to ours in that both employ competitive equilibrium framework to deal with limited commitment problem. On the other hand, we allow all potential contracts while they assumed exogenous incomplete markets. In addition, we also discuss the welfare properties of a competitive equilibrium with both "tranching" and "pyramiding".

There is also related literature on retrading or anonymous trading in spot markets. Jacklin (1987) showed that demand deposits are inefficient when retrading in spot markets are allowed. Greenwald and Stiglitz (1986) illustrated that a competitive equilibrium with private information and anonymous trading is inefficient. On the contrary, we show that our competitive equilibrium is efficient under retrading. Retrading in spot markets has also been discussed in recent optimal taxation literature, e.g. Golosov and Tsyvinski (2007). Note that we do not need taxes in this paper.

The remaining of the paper proceeds as follows. Section 2 describes the primitive ingredients of the model. The collateralization structure, namely tranching and pyramiding, is articulated in this section. We establish the existence of the externality and introduce the lotteries in section 3. Section 4 defines the extended commodity space. The constrained feasible allocation and the Pareto program are formulated in this section. Section 5 introduces a competitive equilibrium with lotteries and contracts over market fundamentals. In section 6, the first and second welfare theorems, and the existence theorem are proved. Some properties of equilibrium prices are discussed in section 7. Numerical examples are shown in sections 8 and 9. Section 10 discusses some potential extensions. Section 11 concludes the paper. Appendix A contains proofs that are omitted from the main text.

2 The Model Economy

This is a two-period economy, $t = 0, 1$. All contracts are traded in period-0, henceforth called the "contracting period". In addition, in period-0, both of two consumption goods can be traded and consumed, and one of them can be saved. All contracts will be executed in period-1, henceforth called the "execution period". There are finite S possible states of nature in period-1, i.e., $s = 1, 2, ..., S$. This allows $S = 1$ so there is only intertemporal trade. Let $0 < \pi_s < 1$ be the objective and common assessed probability of state s occurring, where $\sum_s \pi_s = 1$. The two goods can be traded and consumed in each state s. We refer to these as spot markets.

Again there are two goods, called good-1 and good-2. Good-1 cannot be stored (perishable) from $t = 0$ to $t = 1$ while good-2 is storable. The good-2 that is stored is collateralizable, i.e., can serve as collateral to back promises. Henceforth, good-2 and collateral good will be used interchangeably. Furthermore, good-1 will be the numeraire good in every date and state.

There is a continuum of agents of measure one. The agents are divided into H types, each of which is indexed by $h = \{1, 2, \cdots, H\}$. Each type h consists of $\alpha^h \in (0, 1)$ fraction of the population such that $\sum_h \alpha^h = 1$. Each agent type h is endowed with good-1 and good-2, $\mathbf{e}_0^h = \left(e_{10}^h, e_{20}^h\right)$ in period-0 and $\mathbf{e}_s^h = \left(e_{1s}^h, e_{2s}^h\right)$, in each state $s = 1, \cdots, S$. Let $\mathbf{e}^h = \left(\mathbf{e}_0^h, \cdots, \mathbf{e}_S^h\right)$ be the endowment profile of agent type h over period-0 and states s in period-1. Heterogeneity of agents originates in part from the endowment profiles $\mathbf{e}^h$. As a notational convention, vectors or matrices will be represented by bold letters.

Let $k^h \in \mathbb{R}_+$ denote the collateral allocation of an agent h at the end of period-0. Note that this collateral allocation does not need to be equal to his initial endowment of good-2. In particular, since it can be exchanged in the contracting period (at date $t = 0$), it will be equal to the net-position of collateral good after trading in period-0. The collateral good is assumed to be kept in escrow, and cannot be taken away either by borrowers or lenders. The storage technology of good-2 whether in collateral or normal saving is linear but potentially with a random return. In some applications, it is natural to treat the returns as a constant, and focus on how collateral interacts with intertemporal trade. In other applications, the risk is in the collateral itself, i.e., what happens if housing values could fall. Each unit of good-2 stored will become R_s units of good-2 in state $s = 1, \cdots, S$. Specifically, storing I units of good-2 will deliver $R_s I$ units of good-2 in state s. It is noteworthy that the results in this paper are valid even if the technology is not random. In most of the exposition, uncertainty originates in the endowment, primarily.

Preferences are identically homothetic. The preferences of agent h are represented by the utility function $U\left(c_1^h, c_2^h\right) : \mathbb{R}_+^2 \to \mathbb{R}$, where $\left(c_1^h, c_2^h\right)$ are the consumption of good-1 and good-2 of agent h, respectively. Let $0 < \beta \leq 1$ be the common discount factor. The discounted von Neumann-Morgenstern expected utility of h is thus

$$\mathscr{U}^h\left(\mathbf{c}^h\right) \equiv U\left(c_{10}^h, c_{20}^h\right) + \beta \sum_{s=1}^{S} \pi_s U\left(c_{1s}^h, c_{2s}^h\right)$$

where $\mathbf{c}^h = \left(\mathbf{c}_0^h, \cdots, \mathbf{c}_S^h\right)$ is the consumption allocation with $\mathbf{c}_0^h \equiv \left(c_{10}^h, c_{20}^h\right) \in \mathbb{R}_+^2$ and $\mathbf{c}_s^h \equiv \left(c_{1s}^h, c_{2s}^h\right) \in \mathbb{R}_+^2$ for $s = 1, \ldots, S$ as the consumption of good-1 and good-2 in period-0, in state s, respectively. The utility function satisfies

Assumption 1. *For each agent h, common utility function $U\left(c_1^h, c_2^h\right)$ is homothetic, continuous, strictly concave, strictly increasing in both arguments, and satisfies the usual Inada conditions.*

Homotheticity will allow us closed form solutions in the determination of spot prices. Risk aversion with random endowments motivates trade in state-contingent securities. Heterogeneous intertemporal endowments motivates trade in bonds. We will on occasion put superscript h on the utility function for clarity, but preference heterogeneity is not an essential part of what we do here.

2.1 Collateralization Structure

A specific piece of collateral can be used to back up several contracts as long as their promises to pay are in different states. So there is no conflict in a given state s. This is known as *tranching.* This is distinct from the contract-specific collateralization structure (in Geanakoplos (2003) among others), in which the collateral of a given security cannot be used as collateral for any other security. For full generality here, we will consider state-contingent contracts as the primitives and otherwise let the security structure be endogenous. Accordingly, we focus on contracts paying in each state s, one at a time.

We will first show that there is no loss of generality in restricting attention to contracts without default, and also in excluding over-collateralized contracts, whose collateral value is strictly larger than the promise. We prove the result for a contract paying in good-1 in state s with good-2 as collateral. Then, we will argue that the same logic applies for all other types of contracts.

A (contingent) contract paying one unit of good-1 in state s with $\widehat{C}$ units of good-2 as collateral is a promise to pay a unit of good-1 if the state of nature is s and nothing otherwise. For notational convenience, we use $\widehat{}$ to distinguish contracts paying in good-1, the numeraire, from contracts paying in good-2. With limited commitment, the payoff of this contract is given by

$$\widehat{D} = \begin{cases} \min\left(1, \widehat{C} R_s P_{2s}\right) & \text{if state is } s \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where this payoff is in units of good-1 in period-1, and P_{2s} is the price of good-2 (in units of good-1) in state s. Note that this defaulting condition depends on the spot price P_{2s}.

Equation (1) is the central of limited commitment problem in this model. The issuer or "borrower" in period-0 may not wish to honor the state-contingent obligation. This creates the limited commitment problem; that is, she will keep the promise if that promise is no larger than the value of the collateral, i.e., $1 \leq \widehat{C} R_s P_{2s}$, and will "default" otherwise, $\widehat{C} R_s P_{2s} < 1$. In case of default, the payoff of the contract in state s is equal to the value of its collateral in that state, $\widehat{C} R_s P_{2s}$ units of good-1.

Now consider a contingent contract *that will be in default* in state s, with collateral $\widehat{C} < \frac{1}{R_s P_{2s}}$. That is, an issuer of this contract will "default" in state s. Hence, according to condition (1), the payoff of this contract (in units of good-1) in state s is

$$\min\left(1, \widehat{C} R_s P_{2s}\right) = \widehat{C} R_s P_{2s} < 1 \tag{2}$$

We now argue that there is an alternative contract that generates exactly the same total payoffs using an identical amount of collateral overall. Consider a state-s contingent contract with collateral amount $\widetilde{C} = \frac{1}{R_s P_{2s}}$. The payoff of this contract (in units of good-1) in state s is equal to

$$\min\left(1, \widetilde{C} R_s P_{2s}\right) = \min\left(1, \left(\frac{1}{R_s P_{2s}}\right) R_s P_{2s}\right) = 1 \tag{3}$$

This also implies that this contract will not default. Now consider $\widehat{C} R_s P_{2s}$ *units* of the *alternative* contract. That collection of securities pay in state s one per unit or $\widehat{C} R_s P_{2s}$ in total. This is exactly the same as the payoff of the original contract with default: see (2). In addition, the total collateral for $\widehat{C} R_s P_{2s}$ units of the alternative contract with $\frac{1}{R_s P_{2s}}$ collateral per unit is equal to

$$\left(\widehat{C} R_s P_{2s}\right)\left(\frac{1}{R_s P_{2s}}\right) = \widehat{C}$$

which is exactly the same as the collateral level of the original contract. Therefore, the alternative contract can generate the same payoffs using the same total amount of collateral but without default.

A similar argument also applies to other types of contracts. The following lemma implies that there is no loss of generality in restricting attention to no-default contracts only. In particular, issuing contracts that do default requires no less collateral than contracts that do not. In other words, contracts with default, i.e., with little collateral, do not economize on collateral. In addition, we will show in the next section that default cannot make collateral constraints, formally defined later, less binding. The result is summarized in the following lemma.

Lemma 1. *For any state-contingent contract, there exists a contract with* ***no default*** *that can generate the* ***same total payoffs*** *using the* ***same amount of collateral****.*

In addition, with perfectly divisible collateral, there is no loss of generality in excluding over-collateralized contracts, whose collateral value is strictly larger than the promise. More precisely, an over-collateralized contract paying in good-1 in state s is a contract with a collateral $\widehat{C}$ such that $\widehat{C} R_s P_{2s} > 1$. The payoff of this contract in state s is 1. This contract is equivalent to a no-default contract with $\frac{1}{R_s P_{2s}} < \widehat{C}$ units of good-2 as collateral, whose payoff in state s is also 1. A similar result applies to other types of contracts as well.

It is worthy of emphasis, however, that own saving should not be interpreted as over-collateralization, as no securities are acquired from others; that is, each agent can save. This saving will result in the slackness of the collateral constraint (4) defined below. In particular, an agent may hold at the end of period-0 more collateral good than the (minimum) amount needed to collateralized all contracts issued.

2.2 Collateral Constraints on Directly Collateralized Contracts

To generalize a bit, let $\hat{\psi}^h \equiv \left(\hat{\psi}_s^h\right)_{s=1}^S \in \mathbb{R}^S$ and $\psi^h \equiv \left(\psi_s^h\right)_{s=1}^S \in \mathbb{R}^S$ denote agent h's portfolios of contracts demanded, held at the end of period-0 paying in good-1 and also now in good-2, both with good-2 as collateral, respectively. We adopt the convention that positive means demand and negative means sale. So, holding a positive amount of a contract, $max\left(0, \psi_s^h\right) = \psi_s^h$, a positive number, is equivalent to buying the contract (or lending) while holding a negative amount of a contract, $min\left(0, \psi_s^h\right) = \psi_s^h$, a negative number, is equivalent to selling the contract (or borrowing). In short, the max and min operators pick off demand and supply, respectively. A wedge is created by the need to back the supply by collateral but not the demand.

More generally, a contract paying a unit of good-1 in state s backed by good-2 pays the minimum of 1 unit of good-1 or the value of its collateral in state s. By an argument similar to the one given earlier, the minimum no-default collateral is $\frac{1}{P_{2s}R_s}$ per unit. Similarly, with no-default and no-over-collateralization, a contract paying in good-2 in state s requires $\frac{1}{R_s}$ units of good-2 as collateral. The results so far are summarized in the first two rows of the Table 1 with collateral requirement in the last column.

	payment unit	collateral unit	issued liabilities	purchased assets available as collateral	total collateral requirement for no default securities
$\hat{\psi}_s^h$	good-1	good-2	$-\min\left(0, \hat{\psi}_s^h\right)$	$\max\left(0, \hat{\psi}_s^h\right)$	$-\left(\frac{1}{R_s P_{2s}}\right)\min\left(0, \hat{\psi}_s^h\right)$
ψ_s^h	good-2	good-2	$-\min\left(0, \psi_s^h\right)$	$\max\left(0, \psi_s^h\right)$	$-\left(\frac{1}{R_s}\right)\min\left(0, \psi_s^h\right)$
$\hat{\sigma}_s^h$	good-1	contracts paying in good-2	$-\min\left(0, \hat{\sigma}_s^h\right)$	$\max\left(0, \hat{\sigma}_s^h\right)$	$-\left(\frac{1}{P_{2s}}\right)\min\left(0, \hat{\sigma}_s^h\right)$
σ_s^h	good-2	contracts paying in good-1	$-\min\left(0, \sigma_s^h\right)$	$\max\left(0, \sigma_s^h\right)$	$-P_{2s}\min\left(0, \sigma_s^h\right)$
$\hat{\nu}_s^h$	good-1	contracts paying in good-1	$-\min\left(0, \hat{\nu}_s^h\right)$	$\max\left(0, \hat{\nu}_s^h\right)$	$-\min\left(0, \hat{\nu}_s^h\right)$
ν_s^h	good-2	contracts paying in good-2	$-\min\left(0, \nu_s^h\right)$	$\max\left(0, \nu_s^h\right)$	$-\min\left(0, \nu_s^h\right)$

Table 1: Collateral requirements for each type of contracts.

For contracts $\left(\hat{\psi}_s^h, \psi_s^h\right)$ with good-2 as collateral, paying in good-1 and good-2, respectively, agent h must hold good-2 at the end of period-0 no less than the collateral requirement in any state (shown in Table 1):

$$k^h \;\; \geq \;\; -\min\left(0, \hat{\psi}_s^h\right)\left(\frac{1}{R_s P_{2s}}\right) - \min\left(0, \psi_s^h\right)\left(\frac{1}{R_s}\right), \;\forall s \tag{4}$$

which can be rewritten as

$$P_{2s} R_s k^h + \min\left(0, \hat{\psi}_s^h\right) + P_{2s}\min\left(0, \psi_s^h\right) \;\; \geq \;\; 0, \;\forall s \tag{5}$$

These are *state-contingent collateral requirement constraints* with directly collateralized contracts. We incorporate asset-backed securities in the next section.

Note that when an agent h's collateral requirement constraints (5) are not binding for every state s (i.e., the LHS of (4) exceeds its RHS or (5) holds with strict inequality for every state s), then the agent h holds collateral k^h more than needed to back issued securities. The extra part of collateral is normal saving.

2.3 Pyramiding: Asset-Backed Securities

In real world economies, agents are allowed to use the *promises to receive* goods of others as collateral to back their own promises. This is termed *pyramiding*. In other words, there are two types of collateral, good-2 itself (described in the preceding section) and "assets" backed by such collateral. The prototypical example of an asset-backed promise in this paper is an ex-ante agreement for an agent to give up good-1 in the spot market in state s backed by someone else's promise, a receipt of good-2, or vice versa. The promise of receipt is the asset and this backs the promise to pay. Indeed, if the planned spot-market trade is at equilibrium price of P_{2s}, then one is moving along a budget line and so the value of collateral, the good to be recovered, exactly equals the promise and there is no need for additional underlying collateral.

With two physical commodities, there are four possible types of asset-backed securities; (1) securities paying in good-1 with assets paying in good-2 as collateral, $\hat{\sigma}^h \equiv \left(\hat{\sigma}_s^h\right)_{s=1}^S \in \mathbb{R}^S$, (2) securities paying in good-2 with assets paying in good-1 as collateral, $\sigma^h \equiv \left(\sigma_s^h\right)_{s=1}^S \in \mathbb{R}^S$, (3) securities paying in good-1 with assets paying in good-1 as collateral, $\hat{\nu}^h \equiv \left(\hat{\nu}_s^h\right)_{s=1}^S \in \mathbb{R}^S$, and (4) securities paying in good-2 with assets paying in good-2 as collateral, $\nu^h \equiv \left(\nu_s^h\right)_{s=1}^S \in \mathbb{R}^S$. The last four rows of Table 1 summarize their characteristics including their collateral requirements. For example, a unit of an asset-backed security $\hat{\sigma}_s$ paying in good-1 in state s needs $\frac{1}{P_{2s}}$ units of assets paying in good-2 as collateral. The value of the payoff of $\frac{1}{P_{2s}}$ units of securities paying in good-2 in state s equals $P_{2s} \times \frac{1}{P_{2s}} = 1$ unit of good-1, which is exactly the face-value promise to pay. These collateral requirements are minimum no-default levels.

As shown in the third row of Table 1 (see the column titled total collateral requirement), an asset-backed security paying a unit of good-1 in state s, $\hat{\sigma}_s^h$, requires that the total amount of purchased assets paying in good-2 in state s is no less than

$$-\left(\frac{1}{P_{2s}}\right)\min\left(0, \hat{\sigma}_s^h\right) \tag{6}$$

Similarly, an asset-backed security ν_s^h requires that the total amount of purchased assets paying in good-2 in state s is no less than (see the last row of Table 1)

$$-\min\left(0, \nu_s^h\right) \tag{7}$$

On the other hand, the total amount of purchased assets paying in good-2 is, as shown in the second, fourth and last rows of Table 1 (see the next-to-last column titled purchased assets):

$$\max\left(0, \psi_s^h\right) + \max\left(0, \sigma_s^h\right) + \max\left(0, \nu_s^h\right) \tag{8}$$

Hence, the collateral requirement condition regarding issued securities $\hat{\sigma}_s^h$ and ν_s^h that require financial assets paying in good 2 as collateral can be written as, for any state s,

$$\max\left(0, \psi_s^h\right) + \max\left(0, \sigma_s^h\right) + \max\left(0, \nu_s^h\right) \quad \geq \quad -\left(\frac{1}{P_{2s}}\right)\min\left(0, \hat{\sigma}_s^h\right) - \min\left(0, \nu_s^h\right) \tag{9}$$

This states that the agent purchases enough assets or promises paying in good-2, $\theta_s^h, \sigma_s^h, \nu_s^h$, to back up her own asset-backed securities or issued promises $\hat{\sigma}_s^h, \nu_s^h$. The above condition can be rearranged as

$$P_{2s}\max\left(0, \psi_s^h\right) + P_{2s}\max\left(0, \sigma_s^h\right) + P_{2s}\nu_s^h \quad \geq \quad -\min\left(0, \hat{\sigma}_s^h\right) \tag{10}$$

where we applies the fact that $\max\left(0, \nu_s^h\right) + \min\left(0, \nu_s^h\right) = \nu_s^h$.

Similarly, the collateral requirement condition for issued securities that require financial assets paying in good 1 as collateral is given by

$$\max\left(0, \hat{\psi}_s^h\right) + \max\left(0, \hat{\sigma}_s^h\right) + \hat{\nu}_s^h \quad \geq \quad -P_{2s}\min\left(0, \sigma_s^h\right), \ \forall s \tag{11}$$

where the right-hand-side comes from the fourth and fifth rows of Table 1.

2.4 Collateral Requirement Constraints and Default

We will show in this section that contracts that do actually default do not relax the collateral requirement constraints (5), (10), and (11); that is, contracts that do default are not necessary.

They may exist and get traded, but we can support an equivalent allocation without them. In particular, we now derive collateral requirement constraints with contracts that do default, and then show that the same net-payoff and same collateral requirement constraints can be reached using no-default contracts. This is, in fact, a result of Lemma 1 but it is nice to be explicit, as the result seems counterintuitive.

For brevity, we will discuss only constraint (5) here. Similar arguments apply to the others, (10),(11). Let $\widehat{C}$, C be the collateral levels of defaulting contracts promising to pay a unit of good-1 with good-2 as collateral and promising to pay a unit of good-2 with good-2 as collateral, respectively. Note that, for expositional reasons, we assume that all contracts are contracts that do default. Accordingly, the payoffs of those contracts, which by construction with default, in state s are

$$\widehat{D}_s = \min\left(P_{2s}R_s\widehat{C}, 1\right) = P_{2s}R_s\widehat{C} \tag{12}$$

$$D_s = \min\left(R_sC, 1\right) = R_sC \tag{13}$$

The collateral requirement condition for contracts using physical good-2 as collateral is simply the nominal collateral that backs the contracts, i.e.,

$$k^h \geq -\widehat{C}\min\left(0, \hat{\psi}_s^h\right) - C\min\left(0, \psi_s^h\right)$$

Multiplying both sides by $P_{2s}R_s$ gives

$$\begin{aligned} P_{2s}R_sk^h &\geq -P_{2s}R_s\widehat{C}\min\left(0, \hat{\psi}_s^h\right) - P_{2s}R_sC\min\left(0, \psi_s^h\right) \\ &= -\min\left(0, \widehat{D}_s\hat{\psi}_s^h\right) - P_{2s}\min\left(0, D_s\psi_s^h\right) \end{aligned} \tag{14}$$

where the last equality follows from (12)-(13), and the fact that $\widehat{D}_s, D_s \geq 0$.

We will now find equivalent contracts with no-default that satisfy the same collateral constraint, by re-normalizing the original contracts. In particular, consider contracts paying in good-1 and good-2, respectively, with collateral $\widehat{C}' = \frac{1}{P_{2s}R_s}$, $C' = \frac{1}{R_s}$. Hence, their payoffs $\widehat{D}'_s$ and D'_s in state s are, by construction,

$$\widehat{D}'_s = \min\left(P_{2s}R_s\widehat{C}', 1\right) = 1 \tag{15}$$

$$D'_s = \min\left(R_sC', 1\right) = 1 \tag{16}$$

Thus $\widehat{D}'_s$ and D'_s are no-default contracts.

In order to reach the same total payoff as originally, let the agent hold securities with "$'$", $\hat{\psi}'^h_s = \widehat{D}_s\hat{\psi}^h_s$, and $\psi'^h_s = D_s\psi^h_s$. With these substitutions, the collateral constraint (14) becomes

$$P_{2s}R_sk^h + \min\left(0, \hat{\psi}'^h_s\right) + P_{2s}\min\left(0, \psi'^h_s\right) \geq 0 \tag{17}$$

which is identical to the collateral constraint (5), derived from no-default securities $\hat{\psi}_s^h$ and ψ_s^h only.

3 Collateral Constraints and Externality

This section shows that the interaction between the bindingness of collateral constraints and spot prices generates an externality. With potentially active spot markets, the constrained planner can influence the resource allocation in period-1 through period-0 assignments, aggregate saving. On the other hand, an infinitesimal agent has no influence on aggregate saving. The asymmetry between the options available to the planner and to agents generates the inefficiency when a collateral constraint is binding.

3.1 Collateral Constrained Attainable Allocations

Consider an economy at the beginning of period-1 (but before ex-post spot trade), a spot economy, represented by the distribution of pre-trade endowment $\tilde{\mathbf{e}}_s^h \equiv \left(\tilde{e}_{1s}^h, \tilde{e}_{2s}^h\right)$, $s = 1, \ldots, S$; $h = 1, \ldots, H$. For example, the pre-trade endowment in a state s, $\tilde{\mathbf{e}}_s^h$, is represented by

$$\tilde{e}_{1s}^h = e_{1s}^h + \hat{\psi}_s^h + \hat{\sigma}_s^h + \hat{\nu}_s^h, \ \forall h \tag{18}$$

$$\tilde{e}_{2s}^h = e_{2s}^h + R_s k^h + \psi_s^h + \sigma_s^h + \nu_s^h, \ \forall h \tag{19}$$

A Walrasian equilibrium or spot-market equilibrium is defined:

Definition 1. *A **spot-market equilibrium** in state s, with $\tilde{\mathbf{e}}_s = \left(\tilde{e}_{1s}^h, \tilde{e}_{2s}^h\right)_h$, is a specification of price of good-2, p_s^*, called the spot price, and spot trade transfers of good-1 and good-2, respectively, $\left(\hat{\tau}_s^h, \tau_s^h\right)_h$ such that*

(i) taking p_s^* as given, for each h, $\left(\hat{\tau}_s^h, \tau_s^h\right)$ solves

$$\max_{\left(\hat{\tau}_s^h, \tau_s^h\right)} U\left(\tilde{e}_{1s}^h + \hat{\tau}_s^h, \tilde{e}_{2s}^h + \tau_s^h\right) \tag{20}$$

subject to budget constraint:

$$\hat{\tau}_s^h + p_s^* \tau_s^h = 0$$

(ii) all markets clear:

$$\sum_h \alpha^h \hat{\tau}_s^h = 0 \tag{21}$$

$$\sum_h \alpha^h \tau_s^h = 0 \tag{22}$$

The set of all spot prices is denoted by $\mathbf{p}^*(\tilde{\mathbf{e}}_s) = p^*_s$.

Note that different pre-trade endowments could result in the same spot price. For example, with an identical homothetic utility function, any pre-trade endowment with the same aggregate ratio of good-1 to good-2 will yield the same spot price (see Lemma 2 below). As a result, we will characterize the spot price using the aggregate ratio of good-1 to good-2, which is a lower-dimensional object.

To be precise, let $z_s = z(\tilde{\mathbf{e}}_s)$ be a *market fundamental* that determines the spot price in state s. Accordingly, the *spot-price function* is defined by $p(z_s) = \mathbf{p}^*(\tilde{\mathbf{e}}_s)$. $z_s = z(\tilde{\mathbf{e}}_s)$ is thus said to be the market fundamental of a spot economy with endowment $\tilde{\mathbf{e}}_s$.

With identical homothetic preferences, the aggregate ratio of good-1 to good-2 in state s is the market fundamental in state s; that is, $z_s = z(\tilde{\mathbf{e}}_s) = \frac{\sum_h \alpha^h \tilde{e}^h_{1s}}{\sum_h \alpha^h \tilde{e}^h_{2s}}$. Here then the spot price function can be represented by a single-valued function $p(z_s)$ such that $p(z_s) = p(z'_s)$ implies that $z_s = z'_s$. In other words, the market fundamental is necessary and sufficient to pin down the spot price. This ensures that working with spot prices is equivalent to working with market fundamentals. We summarize:

Lemma 2. *With identical homothetic preferences, the market fundamental in state s is given by*

$$z_s = \frac{\sum_h \alpha^h e^h_{1s}}{R_s K + \sum_h \alpha^h e^h_{2s}} \tag{23}$$

where $K = \sum_h \alpha^h k^h$ is the aggregate (endogenous) saving including collateral. Further, $p(z_s)$ is a one-to-one function, i.e. $p(z_s)$ is a single-valued, and $p(z_s) = p(z'_s)$ implies that $z_s = z'_s$.

Condition (23), $z_s = z(\tilde{\mathbf{e}}_s)$, will be called a "consistency constraint". It ensures that the market fundamental is consistently well-defined. That is, $p(z_s)$ is exactly the spot price that constitutes the spot equilibrium. Note that we use the market clearing conditions for contracts $(\hat{\psi}^h_s, \psi^h_s, \hat{\sigma}^h_s, \sigma^h_s, \hat{\nu}^h_s, \nu^h_s)$ (formally defined below) to show that the aggregate ratio (23) does not depend further on the details of contract allocations. This is an implication of the homotheticity assumption.

Attainable allocations are those that can be achieved by exchanges of securities and collateral in date 0 and exchanges of consumption goods in date 1 at state s, respecting spot prices $p(z_s)$. Note in particular with potentially active spot markets, there is a connection between consumption allocations and transfers in period-1 at state s. Accordingly, attainable allocations are defined using the spot-price function $p(z_s)$. For notational convenience, let $\hat{\tau}^h = \left(\hat{\tau}^h_s\right)_s$, and $\tau^h = \left(\tau^h_s\right)_s$.

Definition 2. *An allocation* $\mathbf{x} \equiv \left(\mathbf{c}^h_0, k^h, \hat{\psi}^h_s, \psi^h_s, \hat{\sigma}^h_s, \sigma^h_s, \hat{\nu}^h_s, \nu^h_s, \hat{\tau}^h, \tau^h\right)_h$ *is* ***attainable*** *if* *(i)* $\left(\mathbf{c}_0^h, k^h, \hat{\psi}_s^h, \psi_s^h, \hat{\sigma}_s^h, \sigma_s^h, \hat{\nu}_s^h, \nu_s^h\right)_h$ *satisfies standard feasibility (or market-clearing) conditions:*

$$\sum_h \alpha^h c_{10}^h \leq \sum_h \alpha^h e_{10}^h \tag{24}$$

$$\sum_h \alpha^h \left[c_{20}^h + k^h\right] \leq \sum_h \alpha^h e_{20}^h \tag{25}$$

$$\sum_h \alpha^h \hat{\psi}_s^h = 0, \ \forall s \tag{26}$$

$$\sum_h \alpha^h \psi_s^h = 0, \ \forall s \tag{27}$$

$$\sum_h \alpha^h \hat{\sigma}_s^h = 0, \ \forall s \tag{28}$$

$$\sum_h \alpha^h \sigma_s^h = 0, \ \forall s \tag{29}$$

$$\sum_h \alpha^h \hat{\nu}_s^h = 0, \ \forall s \tag{30}$$

$$\sum_h \alpha^h \nu_s^h = 0, \ \forall s \tag{31}$$

(ii) *for each h,* $\left(k^h, \hat{\psi}_s^h, \psi_s^h, \hat{\sigma}_s^h, \sigma_s^h, \hat{\nu}_s^h, \nu_s^h\right)$ *satisfies collateral requirement constraints (5), (10), and (11) with* $P_{2s} = p(z_s)$*,*

(iii) *for each s and h,*

$$c_{1s}^h = e_{1s}^h + \hat{\psi}_s^h + \hat{\sigma}_s^h + \hat{\nu}_s^h + \hat{\tau}_s^h \tag{32}$$

$$c_{2s}^h = e_{2s}^h + \psi_s^h + \sigma_s^h + \nu_s^h + \tau_s^h \tag{33}$$

where the pre-trade-endowment are defined in (18) and (19) and the ex-post spot trade allocation $\left(\hat{\tau}_s^h, \tau_s^h\right)_h$ *satisfies the following spot-trade conditions*

$$\hat{\tau}_s^h + p(z_s)\tau_s^h = 0, \quad \forall s, h \tag{34}$$

(iv) *the feasibility (or market-clearing) conditions for the spot trade transfers:*

$$\sum_h \alpha^h \hat{\tau}_s^h = 0, \ \forall s \tag{35}$$

and

(v) *the market fundamental in state s:*

$$z_s = z\left(\tilde{\mathbf{e}}_s\right) = \frac{\sum_h \alpha^h e_{1s}^h}{R_s K + \sum_h \alpha^h e_{2s}^h} \tag{36}$$

where $K = \sum_h \alpha^h k^h$*.*

The set of all attainable allocations is denoted by $\mathbf{A} = \{\mathbf{x} : \mathbf{x}$ *is attainable*$\}$, *and is called an* ***attainable set***.

In addition, with Walras' law, the market-clearing condition for $\hat{\tau}_s^h$ (35) implies that the market-clearing condition for τ_s^h:

$$\sum_h \alpha^h \tau_s^h = 0, \ \forall s \tag{37}$$

Notice that we do not impose utility maximization (20) in the definition of attainable allocations here.

3.2 The Collateral Constraints

The attainability definition (Definition 2) imposes three types of collateral requirement conditions (with three types of collateral), (5), (10), and (11). We will now argue that there is no loss of generality to replace these conditions by the unified collateral constraint:

$$p(z_s)R_s k^h + \hat{\theta}_s^h + p(z_s)\theta_s^h \geq 0, \ \forall s \tag{38}$$

where $\hat{\theta}_s^h = \hat{\psi}_s^h + \hat{\sigma}_s^h + \hat{\nu}_s^h$ and $\theta_s^h = \psi_s^h + \sigma_s^h + \nu_s^h$ be state-s contingent contracts paying in good-1 and in good-2, respectively, which can be backed either by good-2 or purchased assets (other people's promises). Note that $\hat{\theta}_s^h$ and θ_s^h include both directly collateralized and asset-backed securities. The collateral constraint (38) results from summing (5), (10), and (11) altogether, and then applying $\max(0, x) + \min(0, x) = x$ to get rid of max and min operators.

The collateral constraint (38) states that, for each state s, the net-value of all assets, including collateral-good and securities, must be non-negative. The net-value of securities depends on the total amount of securities paying in each good regardless of collateral units, i.e., physical or someone else promises. In other words, collateral types do not matter.

In addition, the market-clearing conditions for all contracts (26)-(31) can be replaced by the market-clearing conditions of $\left(\hat{\theta}_s^h, \theta_s^h\right)$:

$$\sum_h \alpha^h \hat{\theta}_s^h = 0, \ \forall s \tag{39}$$

$$\sum_h \alpha^h \theta_s^h = 0, \ \forall s \tag{40}$$

The result is formalized in the following lemma. Note that $\hat{\theta}_s^h = \hat{\psi}_s^h + \hat{\sigma}_s^h + \hat{\nu}_s^h$ and $\theta_s^h = \psi_s^h + \sigma_s^h + \nu_s^h$.

Lemma 3. *The following statements are true:*

(i) if $\left(\mathbf{c}_0^h, k^h, \hat{\psi}_s^h, \psi_s^h, \hat{\sigma}_s^h, \sigma_s^h, \hat{\nu}_s^h, \nu_s^h, \hat{\tau}^h, \tau^h\right)_h$ *is attainable, then the collateral constraint (38) and the market-clearing conditions (39)-(40) hold, and*

(ii) if $\left(k^h, \hat{\theta}_s^h, \theta_s^h\right)_h$ *satisfies the collateral constraint (38) for all h and s, and the market-clearing conditions (39)-(40), then there exists a collateral and contract allocation* $\left(k^h, \hat{\psi}_s^h, \psi_s^h, \hat{\sigma}_s^h, \sigma_s^h, \hat{\nu}_s^h, \nu_s^h\right)_h$ *that satisfies collateral requirement conditions (5), (10), (11) and the market-clearing conditions (26)-(31).*

Proof. See Appendix A. *Q.E.D.*

Following this lemma, we will use $\left(\hat{\theta}_s^h, \theta_s^h\right)$ as a typical contract allocation, henceforth, unless stated otherwise. Accordingly, the pre-trade endowment in state s becomes

$$\tilde{e}_{1s}^h = e_{1s}^h + \hat{\theta}_s^h, \ \forall h \tag{41}$$

$$\tilde{e}_{2s}^h = e_{2s}^h + R_s k^h + \theta_s^h, \ \forall h \tag{42}$$

In addition, the proof of this lemma also shows how to recover contract allocation $\left(\hat{\psi}_s^h, \psi_s^h, \hat{\sigma}_s^h, \sigma_s^h, \hat{\nu}_s^h, \nu_s^h\right)_h$ from $\left(\hat{\theta}_s^h, \theta_s^h\right)$.

Ex-ante Contracting versus Ex-post Spot Trading

This section shows that the spot markets are redundant when all types of contracts are available (see Lemma 4 below). In other words, agents do not need to trade in spot markets, but they may well do so. Importantly, the spot markets are open and deliver the spot price $p(z_s)$. In addition, we also show that the asset-backed securities are not necessary[1] when the spot markets are open and active (see Lemma 5 below). Put differently, agents simply are indifferent between trading in spot markets or ex-ante contracts.

To be more precise, an allocation is said to be *equivalent* to an attainable allocation if it is attainable and generates the same consumption allocation and market fundamental in each state s as the original attainable allocation.

Lemma 4. *For any attainable allocation* $\left(\mathbf{c}_0^h, k^h, \hat{\theta}^h, \theta^h, \hat{\tau}^h, \tau^h\right)_h$*, there exists an **equivalent** allocation* $\left(\mathbf{c}_0'^h, k'^h, \hat{\theta}'^h, \theta'^h, \hat{\tau}'^h, \tau'^h\right)_h$ *such that*

$$\hat{\tau}_s'^h = \tau_s'^h = 0, \forall s, h \tag{43}$$

[1]Kilenthong (2008) shows that asset-backed securities (pyramiding) are potentially welfare improving when the collateralization is contract-specific. In fact, it is proved that pyramiding with contract-specific collateralization is equivalent to tranching.

Proof. See Appendix A. *Q.E.D.*

Condition (43) in Lemma 4 implies that the spot markets in period-1 are redundant when all contracts are allowed; that is, anything that can be done through the spot markets and a type of contracts is feasible under both types of contracts without spot markets. Henceforth, the ex-post spot trade transfers will be set to zero, ($\hat{\tau}^h = 0, \tau^h = 0$ as in (43)) and the spot-trade constraints (34) will be neglected, unless stated otherwise. Accordingly, the consumption allocation of an agent type h in state s is given by

$$c_{1s}^h = e_{1s}^h + \hat{\theta}_s^h \quad (44)$$

$$c_{2s}^h = e_{2s}^h + R_s k^h + \theta_s^h \quad (45)$$

Lemma 5. *For any attainable allocation* $\left(\mathbf{c}_0^h, k^h, \hat{\psi}_s^h, \psi_s^h, \hat{\sigma}_s^h, \sigma_s^h, \hat{\nu}_s^h, \nu_s^h, \hat{\tau}^h, \tau^h\right)_h$*, there exists an* ***equivalent*** *allocation* $\left(\mathbf{c}_0'^h, k'^h, \hat{\psi}_s'^h, \psi_s'^h, \hat{\sigma}_s'^h, \sigma_s'^h, \hat{\nu}_s'^h, \nu_s'^h, \hat{\tau}'^h, \tau'^h\right)_h$ *such that*

$$\hat{\sigma}_s'^h = \sigma_s'^h = \hat{\nu}_s'^h = \nu_s'^h = 0, \forall s, h \quad (46)$$

Proof. See Appendix A. *Q.E.D.*

It is worthy of emphasis that Lemma 4 and Lemma 5 imply that the asset-backed securities that we need in this model are the ones that replicate spot markets. In other words, the asset-backed securities in this model (with tranching) are simply substitutes for spot markets. Henceforth, we let asset-backed securities paly this role and shut down active trade in spot markets. The result is summarized in the following corollary.

Corollary 1. *Asset-backed securities and the spot markets are perfect substitute in this model.*

Collateral Constraints and Non-convex Attainable Set

With non-constant spot-price function, the attainable set is non-convex. The non-constant price condition is typical. For instance, this is the case with identical homothetic preferences. The main source of the non-convexity is the product of spot-price function and the sum of collateral and contract allocations, $p(z_s)\left(R_s k^h + \theta_s^h\right)$, in the collateral constraints (38).

Lemma 6. *With identical homothetic and concave preferences, the attainable set is non-convex.*

Proof. See Appendix A. *Q.E.D.*

3.3 Collateral Constrained Optimality

A constrained optimal allocation is an attainable allocation such that there is no other attainable allocation that can make at least one agent strictly better off without making any other agent worse off.

Definition 3. *An attainable allocation* $\mathbf{x}^* \in \mathbf{A}$ *is said to be a* ***constrained optimal*** *allocation if there is no other attainable allocation* $\mathbf{x} \in \mathbf{A}$ *such that*

$$\mathscr{U}^h\left(\mathbf{c}^h\right) \geq \mathscr{U}^h\left(\mathbf{c}^{*h}\right) \text{ for every } h, \text{ and } \mathscr{U}^{\bar{h}}\left(\mathbf{c}^{\bar{h}}\right) > \mathscr{U}^{\bar{h}}\left(\mathbf{c}^{*\bar{h}}\right) \text{ for some } \bar{h}$$

We characterize the constrained optimal allocations using the following planner's problem. Let $\overline{\mathscr{U}}^h$ be the reservation utility level for an agent type h.

Program 1

$$\max_{\left(\mathbf{c}_0^h, k^h, \hat{\theta}^h, \theta^h\right)_h} \quad U\left(c_{10}^1, c_{20}^1\right) + \beta \sum_s \pi_s U\left(e_{1s}^1 + \hat{\theta}_s^1, e_{2s}^1 + R_s k^1 + \theta_s^1\right) \tag{47}$$

subject to

$$U\left(c_{10}^h, c_{20}^h\right) + \beta \sum_s \pi_s U\left(e_{1s}^h + \hat{\theta}_s^h, e_{2s}^h + R_s k^h + \theta_s^h\right) \quad \geq \quad \overline{\mathscr{U}}^h, \text{ for } h = 2, \cdots, H \tag{48}$$

and (24), (25), (36), (38), (39), and (40), and non-negativity constraints for consumption and collateral allocations.

From the planner's perspective, $p(z_s)$ is not a price, but rather a function of market fundamental. The planner can manipulate the consumption allocation in period-1 through period-0 assignments of collateral and saving , which affect the market fundamental.

For expositional reasons, we focus only on interior solutions; that is, the non-negativity constraint for k^h is neglected without loss of generality here. Let μ_{cc-s}^h, and $\mu_{\bar{u}}^h$ denote the Lagrange multipliers for the collateral constraint (38) for agent h in state s, and for the participation constraint (48) for agent h. For notational convenience, let $\mu_{\bar{u}}^1 = 1$. A necessary condition for constrained optimality related to collateral allocation k^h is given by, for any h,

$$\frac{U_{20}^h}{U_{10}^h} = \sum_s \pi_s \beta \frac{U_{2s}^h}{U_{10}^h} R_s \quad + \quad \sum_s \frac{\mu_{cc-s}^h}{\mu_{\bar{u}}^h U_{10}^h} p(z_s) R_s - \sum_s \frac{\alpha^h}{\mu_{\bar{u}}^h U_{10}^h} \frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \sum_{\tilde{h}} \mu_{cc-s}^{\tilde{h}} \hat{\theta}_s^{\tilde{h}} \tag{49}$$

where $U_{i0}^h = \frac{\partial U^h\left(c_{10}^h, c_{20}^h\right)}{\partial c_{i0}}$, $U_{is}^h = \frac{\partial U^h\left(c_{1s}^h, c_{2s}^h\right)}{\partial c_{is}}$ for $i = 1, 2$, $p'(z_s) = \frac{\partial p(z_s)}{\partial z_s}$, and $K = \sum_h \alpha^h k^h$. See the derivation in the proof of Lemma 7 stated below in Appendix A. Of special interest, the last term depends not only on the bindingness of her collateral constraints but also the bindingness of other agents' collateral constraints. This implies that if an agent's collateral constraint is binding, it will impact everyone. This is the source of the externality.

Given that the constraint set is not convex (Lemma 6), this optimality condition is necessary but may not be sufficient. Nevertheless, this does not cause any problem to our externality argument, as we simply need to show that a collateral equilibrium cannot be constrained optimal, i.e. does not satisfy the necessary optimal condition (49).

3.4 Collateral Constrained Equilibrium and The Externality

We will show in this section that a "collateral equilibrium" is constrained inefficient unless it is first-best optimal. In particular, there is an externality resulting from the interaction between binding collateral constraints and spot prices. Let $\hat{P}_{as}$ and P_{as} be the prices of securities paying in good-1 and in good-2 in state s respectively. For notational convenience, we denote the vectors of security prices by $\hat{P}_a = \left(\hat{P}_{as}\right)_s$ and $P_a = (P_{as})_s$.

A collateral equilibrium is defined:

Definition 4. *A collateral equilibrium is a specification of prices of good-2 in period-0, P_{20}, the prices of contracts paying in good-1, $\hat{P}_a$, and the prices of contracts paying in good-2, P_a, the spot price of good-2 in each state s, $p(z_s)$, and an allocation $\left(\mathbf{c}_0^h, k^h, \hat{\theta}^h, \theta^h\right)_h$ such that*

(i) taking prices $P_{20}, \hat{P}_a, P_a, (p(z_s))_{s=1}^S$ as given, for any h, $\left(\mathbf{c}_0^h, k^h, \hat{\theta}^h, \theta^h\right)$ solves

$$\max_{\left(\mathbf{c}_0^h, k^h, \hat{\theta}^h, \theta^h\right)} U\left(c_{10}^h, c_{20}^h\right) + \beta \sum_s \pi_s U\left(e_{1s}^h + \hat{\theta}_s^h, e_{2s}^h + R_s k^h + \theta_s^h\right) \tag{50}$$

subject to ex-ante budget in period $t = 0$, and collateral constraint in each state s :

$$c_{10}^h - e_{10}^h + P_{20}\left[c_{20}^h + k^h - e_{20}^h\right] + \hat{P}_a \cdot \hat{\theta}^h + P_a \cdot \theta^h \leq 0 \tag{51}$$

$$p(z_s) R_s k^h + \hat{\theta}_s^h + p(z_s)\theta_s^h \geq 0 \tag{52}$$

(ii) all markets clear:

$$\sum_h \alpha^h c_{10}^h \leq \sum_h \alpha^h e_{10}^h \tag{53}$$

$$\sum_h \alpha^h \left[c_{20}^h + k^h\right] \leq \sum_h \alpha^h e_{20}^h \tag{54}$$

$$\sum_h \alpha^h \hat{\theta}_s^h = 0, \ \forall s \tag{55}$$

$$\sum_h \alpha^h \theta_s^h = 0, \ \forall s \tag{56}$$

(iii) the market fundamental in state s is consistent with (36).

The necessary optimal condition for a collateral equilibrium that is comparable to the optimal condition for a constrained optimality (49) is given by, for any h,

$$P_{20} = \frac{U_{20}^h}{U_{10}^h} = \sum_s \pi_s \frac{\beta U_{2s}^h}{U_{10}^h} R_s + \sum_s \frac{\gamma_{cc-s}^h}{U_{10}^h} p(z_s) R_s \tag{57}$$

where γ_{cc-s}^h is the Lagrange multiplier for the collateral constraint of contracts paying in state s for agent h. This condition follows the first-order conditions with respect to c_{20}^h and k^h. See the derivation in the proof of Lemma 7 in the appendix. Note that if the last term in (49) is zero and we set $\gamma_{cc-s}^h = \frac{\mu_{cc-s}^h}{\mu_{\bar{u}}^h}$, then condition (57) is exactly the same as (49). On the other hand, if the last term in (49), $\sum_s \frac{\alpha^h}{\mu_{\bar{u}}^h U_{10}^h} \frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \sum_{\tilde{h}} \mu_{cc-s}^{\tilde{h}} \hat{\theta}_s^{\tilde{h}}$, is positive, as in all examples in this paper, the equilibrium price of good-2 in period-0 will be too high relative to the optimal one. Therefore, the planner can do better by lowering the aggregate saving or collateral (see Example 1).

Note that an infinitesimal agent takes a spot price, $p(z_s)$, as a constant. On the contrary, the constrained planner can influence the spot prices $p(z_s)$ through collateral assignments, k^h, for the agents of type h in period-0, which affect the market fundamentals z_s. This key influence is the term in $\frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K}$. The difference between the impact of the planner and that of the agents creates the externality and causes an inefficiency.

The last term in (49) could be zero if $\mu_{cc-s}^{\tilde{h}} = 0$ for all $\tilde{h}$ or $\frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} = 0$. With strictly concave utility function, the spot price varies with the market fundamental[2] (not constant), i.e. $\frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \neq 0$. As a result, when the collateral constraints are binding, i.e., $\mu_{cc-s}^{\tilde{h}} > 0$ for some $\tilde{h}$, the last term in (49) will be non-zero in general. With this non-zero term, a collateral equilibrium will not be constrained efficient. It is the interaction between the bindingness of collateral constraints and spot prices that is the key.

Lemma 7. *A collateral equilibrium is constrained optimal if and only if all collateral constraints are not binding, i.e.* $\gamma_{cc-s}^h = \mu_{cc-s}^h = 0$ *for all* h *and all* s.

Proof. See Appendix A. *Q.E.D.*

As an exceptional case a collateral equilibrium could be a full first-best optimum. But otherwise the collateral equilibrium is constrained *sub*optimal. In particular, we argue that a collateral

[2]If the utility function is linear in both goods, then the spot price is constant, i.e. $\frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} = 0$. This clearly results in constrained efficiency. Similarly, if the amount of aggregate saving is fixed exogenously, then the market fundamental (the ratio of good-1 to good-2) is fixed. This also implies that the last term is zero, and so constrained efficiency.

equilibrium is first-best optimal when all collateral constraints are slack. Using lemma 7, we then prove that a collateral equilibrium will be constrained optimal *only if* it is first-best optimal.

Theorem 1. *If a collateral equilibrium is not first-best optimal, then it is constrained* suboptimal.

Proof. The proof is a contrapositive. Suppose a collateral equilibrium is constrained optimal. Lemma 7 implies that a necessary condition for a collateral equilibrium to be constrained optimal is that all collateral constraints are not binding. No binding collateral constraints implies first-best optimality. In short, we have shown that first-best optimality is a necessary condition for constrained optimality. Thus we can conclude that if a collateral equilibrium is not first-best optimal, then it is constrained suboptimal. *Q.E.D.*

3.5 A Graphical Illustration of The Externality

The externality exists because the (state-contingent) feasible set depends on the spot price $p(z_s)$. This dependency on spot price matters when the collateral constraints (or borrowing constraints in general) of some agents are binding.

To be more precise, consider an economy with two representative agents. If there were no collateral constraint, the feasible set would be the set of consumption allocation in state s that satisfies the resource constraints

$$\sum_h \alpha^h c_{1s}^h = \sum_h e_{1s}^h \Longrightarrow c_{1s}^1 + c_{1s}^2 = e_{1s}^1 + e_{1s}^2,\ \forall s \tag{58}$$

$$\sum_h c_{2s}^h = \sum_h e_{2s}^h + R_s \sum_h k^h \Longrightarrow c_{2s}^1 + c_{2s}^2 = e_{2s}^1 + e_{2s}^2 + KR_s,\ \forall s \tag{59}$$

where $K = k^1 + k^2$ is the total saving. The first condition is the resource constraint for good-1. Similarly, the second one is the resource constraint for good-2, which depends on the total saving K. The feasible set of this frictionless economy is the whole Edgeworth box, and the feasible set does not depend on the spot price.

On the contrary, with the collateral constraints, only a subset of the Edgeworth box is feasible. More importantly, the feasible set depends on the spot price. Specifically, a feasible allocation must satisfy resource constraints (58)-(59), plus the collateral constraint for each agent type h in each state s (38). Recall that the consumption for an agent type h in state s is given by (see (44) and (45))

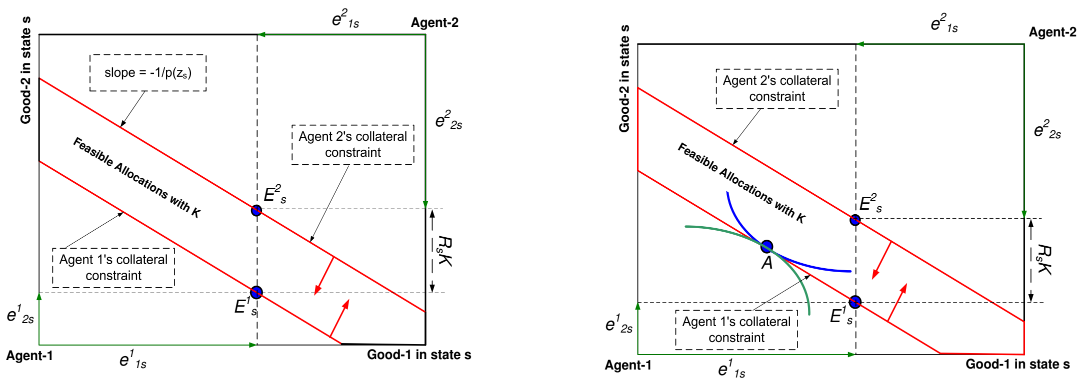
$$c_{1s}^h = e_{1s}^h + \hat{\theta}_s^h \Longrightarrow \hat{\theta}_s^h = c_{1s}^h - e_{1s}^h \tag{60}$$

$$c_{2s}^h = e_{2s}^h + R_s k^h + \theta_s^h \Longrightarrow R_s k^h + \theta_s^h = c_{2s}^h - e_{2s}^h \tag{61}$$

Substituting these two equations into the collateral constraint (38), it becomes

$$p(z_s)\left[c_{2s}^h - e_{2s}^h\right] + c_{1s}^h - e_{1s}^h \geq 0 \Longrightarrow c_{1s}^h + p(z_s)c_{2s}^h \quad \geq \quad e_{1s}^h + p(z_s)e_{2s}^h \tag{62}$$

This condition follows from collateral constraint (38), and conditions (60) and (61). It implies that the value of period-1 consumption of an agent, $c_{1s}^h + p(z_s)c_{2s}^h$, cannot have a lower value than the value of her endowment (without collateral k^h), $e_{1s}^h + p(z_s)e_{2s}^h$. This constraint causes the feasible set to depend on the spot price.

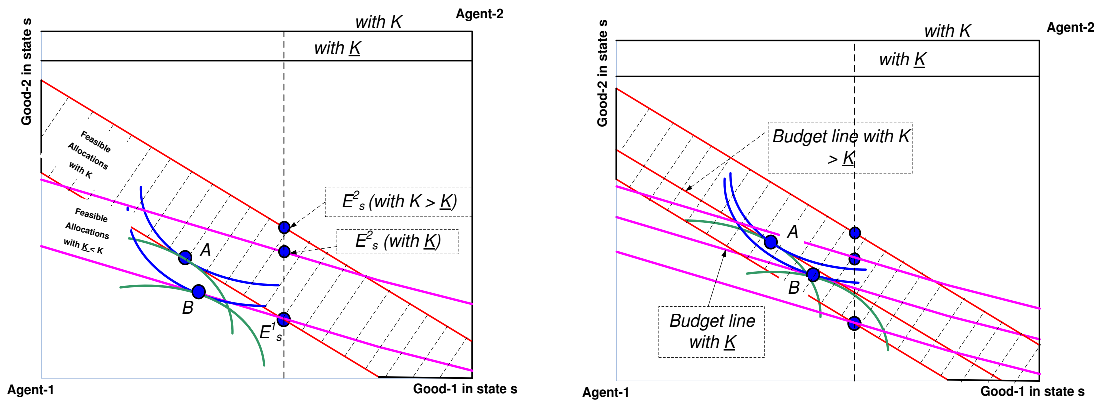


(a) (b)

Figure 1: Collateral-constrained feasible sets. (a) Feasible allocations (the shaded area) depends on the spot price $p(z_s)$ which depends on total saving K. (b) The resulting consumption allocation will be on the lower border of the feasible set when agent 1's collateral constraint is binding.

See figure 1a as an example of a feasible set. The size of the Edgeworth box is determined by the resource constraints (58)-(59). $E_s^h = \left(e_{1s}^h, e_{2s}^h\right)$ is the endowment of agent h in state s. Note that when $K > 0$, the endowments points of the two agents will not coincide. The collateral-constrained feasible set is represented by the shaded area or band. The lower (upper) border of the feasible set corresponds to the bindingness of agent 1's (agent 2's) collateral constraint. As a result, when agent 1's collateral constraint is binding, the resulting consumption allocation will be on the lower border (point A in figure 1b), and vice versa.

Figure 2a illustrates that different spot prices lead to different feasible sets. Note that lower total saving ($\underline{K} < K$) implies lower slope of the boundary of the feasible set. This is because lower total saving implies higher spot price, which in turn implies that the slope, $-\frac{1}{p(z_s)}$, is lower in absolute value. As shown in the figure, two feasible sets with different spot prices are overlapped, and importantly not contained in one another. In the figure, it is assumed that $k^1 = 0, k^2 = \underline{K}$.



(a) (b)

Figure 2: Collateral-constrained feasible sets. (a) Different levels of (endogenous) aggregate saving imply different (overlapped) feasible sets. When agent 1's collateral constraint is binding, changing total saving from K to $\underline{K}$ moves the resulting consumption from A to B which is not feasible with K. (b) When none of the collateral constraints is binding, changing total saving from K to $\underline{K}$ moves the resulting consumption from A to B, which is feasible with K.

Hence, when agent 1's collateral constraint is binding, the marginal change in total saving (say, from K to $\underline{K}$) could lead to the resulting consumption allocation that is not feasible under the original level of total saving (moving from point A to point B which is not feasible with total saving K). This marginal effect, therefore, is not priced in collateral equilibrium without lotteries. That is, there is an externality.

On the other hand, if none of the collateral constraints is binding, this dependency on the spot price will not generate the externality. See figure 2b. With no binding collateral constraints, the resulting consumption allocations will be strictly inside (interior of) the feasible set. This implies that its neighborhood is also in the feasible set. As a result, the marginal change in total saving will lead to the resulting consumption allocation that is still feasible under the original level of total saving (A and B are both feasible with total saving K). This implies that the collateral equilibrium prices this marginal effect. Hence, there is no externality in this case.

4 Internalizing The Externality: The Economy with Lotteries

Let $\mathbf{z} = (z_s)_{s=1}^{S}$ denotes a vector of the market fundamental in all states. The composition of agents determines the market fundamental. We can go further and interpret each market fundamental z_s

as an *price-island* in state s, where the composition of agents forms into market fundamental z_s. In other words, the market fundamental in an price-island-z_s is exactly z_s.

Now suppose it is possible to assign agents to different islands even in state s; island assignments are still state-contingent. We assume that an agent who resides in an island-z_s can trade within the island-z_s at spot price $p(z_s)$ only. Realistically, one can imagine of having segregated security exchanges trading bundles of securities, collateral and current consumption.

In addition, being in island-z_s in state s also means that an agent h can trade in spot markets at spot price $p(z_s)$, which is determined by the market fundamental z_s. Equivalently, even if the spot markets are shut down[3], an agent with a bundle contingent on an island z_s can trade in ex-ante securities which are traded in the segregated exchanges and whose prices depend on z_s. Let $\Delta_s \in \mathbb{R}$ define "individual h's *deviation from the market fundamental*[4]" in state s.

$$\Delta_s = z_s \left(e^h_{2s} + R_s k^h \right) - e^h_{1s}, \ \forall s \tag{63}$$

Note that if $\Delta_s = 0$, then $\frac{e^h_{1s}}{e^h_{2s} + R_s k^h} = z_s$ and individual h's pre-trade endowment is exactly equal to the market fundamental. If $\Delta_s > 0$, then individual h holds a relative low amount of good-1 and abundant amount of good-2. Adding one unit of good-2 (via collateral k) adds to the individual deviation Δ_s by exactly $z_s R_s$ (see Eq.(63)). Note also that there is a part of Δ_s over which h has no control, namely her endowments. The "individual deviation from the fundamental" will be priced in a competitive equilibrium. In addition, the price of the right to trade in each island will be proportional to the "individual deviation from the fundamental".

For each agent type h, let $x^h(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta) \geq 0$ denote a probability measure on $\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$, where Δ_s satisfies (63) for all s. In other words, $x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ is the probability of receiving period-0 consumption, $\mathbf{c}_0 \equiv (c_{10}, c_{20})$, collateral, k, contracts paying in good-1, $\hat{\theta}_s$, contracts paying in good-2, θ_s, and being in island-z_s in state s where all contracts are executed and all spot-trade takes place. Recall that a positive (negative) amount of trade means receiving (transferring out) the specified good. Importantly, a resident of an island-z_s can trade contracts with other residents in the same island only. In other words, contracts are executed at spot prices within each island only.

[3]As proved in Lemma 4, the spot markets are redundant in that agents are indifferent between trading in ex-ante contracts or in spot markets. Importantly, the spot markets are opened.

[4]If we were in the underlying spaces of k and z_s, they would enter multiplicatively, hence and so we would have a non convexity problem. This is not a problem with lotteries, however.

As a probability measure, a lottery of an agent type h satisfies

$$\sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)} x^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)=1 \tag{64}$$

With continuum of agents, $x^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)$ can be interpreted as the fraction of agent type h assigned to a bundle $\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)$.

More formally, the commodity space L is assumed to be a finite n-dimensional linear space[5]. Let $C,K,\widehat{\Theta},\Theta,Z,D$ be the sets of period-0 consumption allocations, collateral allocation, portfolios of contracts paying in good-1, portfolios of contracts paying in good-2, vectors of market fundamentals, and vectors of "individual deviations from the fundamental", respectively. The commodity space is a set of lotteries or probability measures on $C\times K\times\widehat{\Theta}\times\Theta\times Z\times D\equiv\mathscr{B}$, the set of all available bundles. For notational purposes, let $b=\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)$ be a typical element of $\mathscr{B}$, called a bundle. We will use b and $\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)$ interchangeably. Accordingly, we can write $\mathbf{x}^h\equiv\left[x^h(b)\right]_{b\in\mathscr{B}}\in\mathbb{R}^n_+$ as a typical lottery for an agent type h.

4.1 Consumption Possibility Set

A holder of a bundle $\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)$ will receive k units of collateral, and hold portfolio of contracts $\left(\hat{\theta},\theta\right)$. With limited commitment, each bundle will be feasible only if the collateral and contract assignments satisfy the following collateral constraint:

$$p\left(z_s\right)R_sk+\hat{\theta}_s+p\left(z_s\right)\theta_s\geq 0,\ \forall s \tag{65}$$

Accordingly, we impose the following condition on a probability measure $x^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)$.

$$\begin{aligned} x^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) &\geq 0 \text{ if } \left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) \text{ satisfies (63) and (65)} \\ &= 0 \text{ if otherwise} \end{aligned} \tag{66}$$

In words, a positive measure can be defined only on feasible bundles, which have to satisfy condition (65). More formally, the consumption possibility set of an agent type h is defined by

$$X^h=\left\{\mathbf{x}^h\in\mathbb{R}^n_+:\sum_b x^h\left(b\right)=1,\text{ and for any } b\in\mathscr{B}, x^h\left(b\right)\text{ satisfies (66)}\right\} \tag{67}$$

Let $\mathbf{x}^h$ be a typical element of X^h. Note that $X^h\subset L$ is compact and convex. In addition, the non-emptiness of X^h is guaranteed by assigning mass one on each agent's endowment. In particular, we assume that the endowment of each agent h is on the grids; that is, $\mathbf{c}_0=\mathbf{e}^h_0\in C$ for every h, $k=0\in K$, $\hat{\theta}=0\in\widehat{\Theta}$, $\theta=0\in\Theta$, $\Delta=0\in D$, and for each h, $\mathbf{z}\in Z$ where $z_s=\frac{e^h_{1s}}{e^h_{2s}}$.

[5] The limiting arguments under weak-topology used in Prescott and Townsend (1984a) can be applied to establish the results if L is not finite.

4.2 Attainable Allocations

An attainable allocation with lotteries will be defined in an analogous manner to the ones without lotteries in Section 3.3. In particular, an allocation $\mathbf{x} \equiv \left(\mathbf{x}^h\right)_h$ is attainable if $\mathbf{x}^h \in X^h$ for all h, and it satisfies the following feasibility constraints.

Recall that good-1 cannot be stored; only good-2 is storable. The aggregate endowment of good-1 in period-0 is $\sum_h \alpha^h e_{10}^h$. Therefore, the resource constraint for good-1 in period-0 is given by

$$\sum_h \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)} \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) c_{10} \leq \sum_h \alpha^h e_{10}^h \tag{68}$$

Similarly, the feasibility constraint for good-2 in period-0 is given by

$$\sum_h \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)} \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) [c_{20} + k] \leq \sum_h \alpha^h e_{20}^h \tag{69}$$

Note that the nonnegativity constraint on k guarantees that the aggregate saving is nonnegative.

Recall that all contracts are executed within each island only. In particular, for an island-z_s in state s, the net supply of contract paying in good-1 in state s, $\hat{\theta}_s$ must be zero

$$\sum_h \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta\right)} \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s, \Delta\right) \hat{\theta}_s = 0 \ , \forall s, z_s \tag{70}$$

where $\mathbf{z}_{-s} = (z_1, \ldots, z_{s-1}, z_{s+1}, \ldots, z_S)$ is a vector of market fundamentals in all states but state s. This feasibility condition holds for every state s and every island z_s. Similarly, the feasibility or market-clearing constraints for a contracts paying in good-2 are given by

$$\sum_h \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta\right)} \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s, \Delta\right) \theta_s = 0 \ , \forall s, z_s \tag{71}$$

Similar to the economy without lotteries in Section 3.3, the market fundamental in each island must be consistent. In other words, the planner will choose the composition of agents to set the market fundamental for each island to its specified level. With identical homothetic preferences, the consistency constraint for an island-z_s is that the aggregate ratio of good-1 to good-2 within the island-z_s is exactly z_s:

$$z_s = \frac{\sum_h \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta\right)} \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s, \Delta\right) \left(e_{1s}^h + \hat{\theta}_s\right)}{\sum_h \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta\right)} \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s, \Delta\right) \left(e_{2s}^h + R_s k + \theta_s\right)} \tag{72}$$

Using the feasibility conditions for contracts within each island, (70)-(71), the consistency constraints can be rewritten as

$$z_s = \frac{\sum_h \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s}\right)} \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s\right) e_{1s}^h}{\sum_h \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s}\right)} \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s\right) \left(e_{2s}^h + R_s k\right)}, \ \forall s, z_s \tag{73}$$

They can be further simplified as

$$
\begin{aligned}
z_s \sum_h \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta\right)} \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s, \Delta\right) \left(e^h_{2s} + R_s k\right) &= \\
\sum_h \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta\right)} \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s, \Delta\right) e^h_{1s} & \\
\sum_h \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta\right)} \alpha^h x^h \left(c, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s, \Delta\right) \left[z_s \left(e^h_{2s} + R_s k\right) - e^h_{1s}\right] &= 0 \\
\sum_h \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta\right)} \sum_h \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s, \Delta\right) \Delta_s &= 0, \ \forall s, z_s \qquad (74)
\end{aligned}
$$

where the last equation follows the definition of the "individual deviation from the fundamental" in (63).

Definition 5. An allocation $\mathbf{x} \equiv \left(\mathbf{x}^h\right)_{h=1}^H \in X^1 \times \ldots \times X^H$ is said to be *attainable* if $\mathbf{x}^h \in X^h$ for every h, and it satisfies (68)-(71) and (74). Accordingly, the attainable set $X = \{\mathbf{x} : \mathbf{x} \text{ is attainable}\}$ is the set of all attainable allocations

With finite linear weak-inequality constraints, the feasible set X is compact and convex. In addition, the assumption that the endowment is on the grids also ensures that X is nonempty.

4.3 Constrained Optimal Allocations

A constrained optimal allocation is an attainable allocation such that there is no other attainable allocation that can make at least one agent strictly better off without making any other agent worse off. To be precise, the expected utility of an agent type h, holding a lottery $\mathbf{x}^h$, is given by

$$
\mathscr{U}^h\left(\mathbf{x}^h\right) = \sum_{\left(c,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)} x^h\left(c, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \left\{U\left(c_{10}, c_{20}\right) + \beta V^h\left(k, \hat{\theta}, \theta, \mathbf{z}\right)\right\}
$$

where $V^h\left(k, \hat{\theta}, \theta, \mathbf{z}\right)$ is the "indirect" utility of agent h that is derived from a bundle $\left(c, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$;

$$
V^h\left(k, \hat{\theta}, \theta, \mathbf{z}\right) = \sum_s \pi_s U\left(e^h_{1s} + \hat{\theta}_s, e^h_{2s} + R_s k + \theta_s\right) \qquad (75)
$$

This object is the result of the assignments of a bundle $\left(c, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ that is executed in period-1 over state s. In other words, it summarizes all actions happening to the holder of the bundle in period-1. In each state s, a holder type h receives $\hat{\theta}_s$ units of good-1 as the net-payment from portfolio $\hat{\theta}$, θ_s units of good-2 as the net-payment from portfolio θ, $R_s k$ units of good-2 from the collateral good, and also e^h_{1s} units of good-1 and e^h_{2s} units of good-2 as endowments.

Definition 6. *An attainable allocation* $\mathbf{x}^* \in X$ *is said to be a* ***constrained optimal*** *allocation if there is no another attainable allocation* $\mathbf{x} \in X$ *such that*

$$\mathscr{U}^h\left(\mathbf{x}^h\right) \geq \mathscr{U}^h\left(\mathbf{x}^{*h}\right) \text{ for every } h, \text{ and } \mathscr{U}^{\bar{h}}\left(\mathbf{x}^{\bar{h}}\right) > \mathscr{U}^{\bar{h}}\left(\mathbf{x}^{*\bar{h}}\right) \text{ for some } \bar{h}$$

We characterize the constrained optimality using the following Pareto program. Let $\lambda^h \geq 0$ be the Pareto weight of agent type h. There is no loss of generality to normalize the weights such that $\sum_h \lambda^h = 1$. A constrained Pareto optimal allocation $\mathbf{x}^*$ solves the following Pareto program.

Program 2: The Pareto Program with Lotteries

$$\max_{\mathbf{x}} \sum_h \lambda^h \alpha^h \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)} x^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)\left\{U^h(c_{10},c_{20}) + \beta V^h\left(k,\hat{\theta},\theta,\mathbf{z}\right)\right\} \tag{76}$$

subject to

$$\mathbf{x}^h \in X^h, \ \forall h \tag{77}$$

$$\sum_h \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)} \alpha^h x^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta\right) c_{10} \leq \sum_h \alpha^h e_{10}^h \tag{78}$$

$$\sum_h \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)} \alpha^h x^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)\left(c_{20}+k\right) \leq \sum_h \alpha^h e_{20}^h \tag{79}$$

$$\sum_h \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta\right)} \alpha^h x^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},z_s,\Delta\right)\hat{\theta}_s = 0 \ ,\forall s, z_s \tag{80}$$

$$\sum_h \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta\right)} \alpha^h x^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},z_s,\Delta\right)\theta_s = 0 \ ,\forall s, z_s \tag{81}$$

$$\sum_h \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta\right)} \alpha^h x^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},z_s,\Delta\right)\Delta_s = 0 \ ,\forall s, z_s \tag{82}$$

Note again that we already embedded the collateral constraints (65) and the "individual deviations from the fundamental" (63) into the consumption possibility sets X^h.

It is clear that the objective function now is linear in x^h, and thereby it is continuous and weakly concave. As discussed earlier, the feasible set X is non-empty, compact, and convex. Therefore, a solution to the Pareto program for given positive Pareto weights exists and is a global maximum. The proof of the equivalence between Pareto optimal allocations and the solutions to the program is omitted for brevity (see Prescott and Townsend (1984b) for a similar proof).

5 Decentralized Equilibrium

Let P_{20} be the price of good-2 in period-0, and $P\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)$ be the price of a bundle $\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)$. Note that the price of good-1 in period-0 is $P_{10}=1$ as good-1 is the numeraire good. Each agent is infinitesimally small relative to the entire economy and will take all prices as given. The market-makers introduced below will also act competitively. Now as well that Δ is also priced.

Consumers: Each agent h, taking prices $\left(P_{20}, P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)\right)$ as given, chooses $\mathbf{x}^h$ in period $t = 0$ to maximize its expected utility:

$$\max_{\mathbf{x}^h} \sum_{\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)} x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)\left\{U\left(c_{10}, c_{20}\right) + \beta V^h\left(k, \hat{\theta}, \theta, \mathbf{z}\right)\right\} \tag{83}$$

subject to $\mathbf{x}^h \in X^h$, and period-0 budget constraint

$$e_{10}^h + P_{20} e_{20}^h - \sum_{\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)} P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \geq 0 \tag{84}$$

where

$$V^h\left(k, \hat{\theta}, \theta, \mathbf{z}\right) = \sum_s \pi_s U\left(e_{1s}^h + \hat{\theta}_s, e_{2s}^h + R_s k + \theta_s\right) \tag{85}$$

The period-0 budget constraint (84) states that the agent sells all her endowments[6] including good-2 at price P_{20} and uses this income to buy lotteries $\mathbf{x}^h$, which includes consumption in period-0, $\left(c_{10}^h, c_{20}^h\right)$.

In state-s, a type-h holder of bundle $\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ receives in addition of her endowments of good-1 and good-2, $\left(e_{1s}^h, e_{2s}^h\right)$, $\hat{\theta}_s$ units of good-1 as the net-payment of portfolio $\hat{\theta}$, $R_s k$ units of good-2 from the collateral good, θ_s units of good-2 as the net-payment of portfolio θ. Of course, if $\hat{\theta}_s$ and θ_s are negative, promises to pay. The result of the actions in period-1 is summarized in $V^h\left(k, \hat{\theta}, \theta, \mathbf{z}\right)$. It is worthy of emphasis that the agent will reside in island z_s, where she can in principle trade good-1 and good-2 at price $p(z_s)$ in spot markets.

Market-Makers: The primary role of a market-maker is to put together deals, i.e., buying consumption goods and collateral and selling the bundle, including securities backed by collateral. In order to do so, the market maker issues (sells) $y\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \in \mathbb{R}_+$ units of each bundle $\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$, at the unit price $P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$. Note that the market-maker can issue any non-negative number of a bundle; that is, the number of a bundle issued does not have to be between zero and one and is not a lottery. It is the number of bundles. Let $\mathbf{y} \in L$ be the vector of the number of bundles issued on $C \times K \times \widehat{\Theta} \times \Theta \times Z \times D$. With constant returns to scale, the profit

[6] It is worthy of emphasis that we can write an equivalent problem specifying consumption transfers in period-0, instead of consumption allocation. By doing so, agents do not need to sell their entire endowments but simply buy and sell consumption transfers. In other words, it is not restrictive to make agents sell their entire endowments and buy consumption allocation through lotteries.

of a market-maker must be zero and the number of market-makers becomes irrelevant. Therefore, without loss of generality we assume there is one representative market-maker, which takes prices as given.

By issuing or selling $\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$, the market-maker promises to deliver k units of collateral good to a holder. In order to do so, the market-maker needs to hold a sufficient amount of collateral to back the promises. In particular, it buys I units of good-2 at price P_{20} (in terms of good-1) in period-0, and distributes it according to $\mathbf{y}$.

$$\sum_{\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)} y\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) k = I \tag{86}$$

This constraint states that the market-maker uses I as collateral for the promises to deliver the collateral good.

Similarly, the market-maker will also deliver (c_{10}, c_{20}) units of good-1 and good-2 to a holder of a bundle $\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$. In order to do so, the market-maker buys C_1 units of good-1 and C_2 units of good-2, and distributes it according to $\mathbf{y}$:

$$\sum_{\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)} y\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) c_{10} = C_1 \tag{87}$$

$$\sum_{\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)} y\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) c_{20} = C_2 \tag{88}$$

In conclusion, the total (market) cost to the market-maker from buying good-1 and good-2 is equal to $C_1 + P_{20}C_2 + P_{20}I$.

Furthermore, it also delivers portfolio of contracts $\left(\hat{\theta}, \theta\right)$ to the holder. We do not require the market-maker to back securities delivered to consumers, $\hat{\theta}, \theta < 0$, by collateral k as that is up to the consumers themselves. The market-maker itself does not hold any contracts, but simply distributes them according to $\mathbf{y}$. Hence, the net-supply of each contract from the market-maker must be zero:

$$\sum_{\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, \Delta\right)} y\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s, \Delta\right) \hat{\theta}_s = 0, \ \forall s, z_s \tag{89}$$

$$\sum_{\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, \Delta\right)} y\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s, \Delta\right) \theta_s = 0, \ \forall s, z_s \tag{90}$$

These constraints will be equivalent to the market-clearing constraints for contracts in the competitive equilibrium with $y\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s, \Delta\right) = \sum_h \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s, \Delta\right)$, which is the market-clearing condition for lotteries.

The market-maker's technology also requires that the sum of all "individual fundamental" must be zero in each island-z_s for every state s:

$$\sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta\right)} y\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},z_s,\Delta\right)\Delta_s \quad = \quad 0\ ,\forall s,z_s \tag{91}$$

This constraint is the counter part of the consistency constraint (74) in the Pareto program. In particular, using the market-clearing condition for lotteries, we can show that this consistency constraint is identical to (74). Hence, it is also called the consistency constraint for an island-z_s for every state s.

The objective of an intermediary is to maximize its profit by choosing $(\mathbf{y},C_1,C_2,I)$, taking prices $(P_{20},P(b))$ as given:

$$\begin{array}{ll}\max\limits_{(\mathbf{y},C_1,C_2,I)} & \sum\limits_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)} y\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)P\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)-\left[C_1+P_{20}C_2+P_{20}I\right] \\ \text{s.t.} & (86)-(91)\end{array}$$

where the first term is the total revenue of the market-maker and the bracketed term denotes its total (market) cost. Using conditions (86)-(88), we can rewrite the problem as

$$\begin{array}{ll}\max\limits_{\mathbf{y}} & \sum\limits_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)} y\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)\left[P\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)-c_{10}-P_{20}c_{20}-P_{20}k\right] \\ \text{s.t.} & (89)-(91)\end{array} \tag{92}$$

This profit-maximization problem is a well-defined linear program whose (linear) constraints satisfy Slater's condition (Uzawa (1958)). Hence, the problem can be written as a saddle-point problem

$$\max_{\mathbf{y}} \min_{\left(\widehat{P}_a(\mathbf{z}),P_a(\mathbf{z}),P_\Delta(\mathbf{z})\right)} \sum_{b\in\mathscr{B}} y(b)\left[P(b)-c_{10}-P_{20}c_{20}-P_{20}k-\widehat{P}_a(\mathbf{z})\cdot\hat{\theta}-P_a(\mathbf{z})\cdot\theta-P_\Delta(\mathbf{z})\cdot\Delta\right] \tag{93}$$

where $\widehat{P}_a(\mathbf{z})\equiv\left(\widehat{P}_a(z_s,s)\right)_{s=1}^{S}$, $P_a(\mathbf{z})\equiv(P_a(z_s,s))_{s=1}^{S}$, and $P_\Delta(\mathbf{z})\equiv(P_\Delta(z_s,s))_{s=1}^{S}$ are the vectors of Lagrange multipliers for the market-clearing constraints for contracts paying in good-1 (89), for the market-clearing constraints for contracts paying in good-2 (90), and for consistency constraints (91), respectively. In particular, for an island z_s in state s, $\widehat{P}_a(z_s,s)$, $P_a(z_s,s)$, and $P_\Delta(z_s,s)$ are the shadow prices of a contract paying in good-1 and good-2, respectively, and the shadow price of "individual fundamental" in the island-z_s.

The existence of an optimum to the market-maker's problem requires that for any bundle $\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)$,

$$P\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)\le c_{10}+P_{20}c_{20}+P_{20}k+\widehat{P}_a(\mathbf{z})\cdot\hat{\theta}+P_a(\mathbf{z})\cdot\theta+P_\Delta(\mathbf{z})\cdot\Delta \tag{94}$$

where it holds with equality if $y\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) > 0$. Here $P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ is the revenue from the sale of one unit of bundle $\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$. This condition is in fact the necessary and sufficient condition for the saddle-point problem (93).

Define $c_{10} + P_{20}c_{20} + P_{20}k$ as the market cost of the bundle, and $\widehat{P}_a(\mathbf{z}) \cdot \hat{\theta} + P_a(\mathbf{z}) \cdot \theta + P_\Delta(\mathbf{z}) \cdot \Delta$ as its shadow cost. The market-maker considers the sum of both market cost and shadow cost as its total cost for issuing a bundle. The optimal condition states that the market-maker will issue a bundle only if it does not cause a total loss (the revenue is strictly less than the sum of market cost and shadow cost). On the other hand, the revenue of a bundle cannot be strictly larger than the total cost. Otherwise, the market-maker will issue an unbounded amount of such bundle, which cannot be in equilibrium. In addition, the total market profit of the market-maker in equilibrium is zero.

Market Clearing: In period-0, the market-clearing condition for good-1 is

$$\sum_{\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)} y\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) c_{10} = \sum_h \alpha^h e_{10}^h \tag{95}$$

Similarly, the market-clearing condition for good-2 in period-0 is

$$\sum_{\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)} y\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) [c_{20} + k] = \sum_h \alpha^h e_{20}^h \tag{96}$$

The market-clearing conditions for lotteries in period-0 are

$$\sum_h \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) = y\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right), \ \forall \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \tag{97}$$

Definition 7. A competitive equilibrium is a specification of allocation $(\mathbf{x}, \mathbf{y})$, and the price of good-2 in period-0, P_{20}, and the price of a bundle $b \in \mathscr{B}$, $P(b)$, such that

(i) for each h, $\mathbf{x}^h \in X^h$ solves (83) subject to (84), taking prices $(P_{20}, P(b))$ as given,

(ii) for the market-maker, $\left\{\mathbf{y}, \widehat{P}_a(\mathbf{z}), P_a(\mathbf{z}), P_\Delta(\mathbf{z})\right\}$ solves (93), taking prices $(P_{20}, P(b))$ as given,

(iii) in period-0, markets for good-1, good-2 and lotteries clear, i.e. (95)-(97) hold,

6 Existence and Welfare Theorems

As in the classical general equilibrium model, the economy is a well-defined convex economy, i.e., the commodity space is Euclidean, the consumption set is compact and convex, the utility function is linear. As a result, the first and second welfare theorems hold, and a competitive equilibrium exists. In particular, this section proves that the competitive equilibrium is constrained optimal and any constrained optimal allocation can be supported by a competitive equilibrium with transfers. Then, we will use Negishi's method to prove the existence of a competitive equilibrium.

The standard contradiction argument will be used to prove the following first welfare theorem. We also assume that there is no local satiation point in the consumption set.

Assumption 2. *For any* $\mathbf{x}^h \in X^h$*, there exists* $\tilde{\mathbf{x}}^h \in X^h$ *such that*

$$\mathscr{U}^h\left(\tilde{\mathbf{x}}^h\right) > \mathscr{U}^h\left(\mathbf{x}^h\right) \tag{98}$$

where $\mathscr{U}^h\left(\mathbf{x}^h\right)$ *is the expected utility of agent* h *derived from allocation* $\mathbf{x}^h$*.*

This assumption is easily satisfied using reasonable specifications of the grid of consumption allocation in period-0. For example, with a strictly increasing utility function, if we include a very large consumption allocation in period-0 into the grid (larger than what can be attained with endowments and storage), then the local nonsatiation assumption will be satisfied.

Theorem 2. *With local nonsatiation of preferences (Assumption 2), a competitive equilibrium allocation is (constrained) Pareto optimal.*

Proof. See Appendix A *Q.E.D.*

The Second Welfare theorem states that any Pareto optimal allocation, corresponding to strictly positive Pareto weights, can be supported as a competitive equilibrium with transfers, precisely defined later. The standard approach applies here. In particular, we will first prove that any constrained optimal allocation can be decentralized as a compensated equilibrium. Then, we will use a standard cheaper-point argument (see Debreu (1954)) to show that any compensated equilibrium is a competitive equilibrium with transfers. The compensated equilibrium is defined as follows.

Definition 8. A compensated equilibrium is specification of allocation $(\mathbf{x}, \mathbf{y})$, and prices P_{20}, $P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ such that

(i) for each h, $\mathbf{x}^h \in X^h$ solves

$$\min_{\hat{\mathbf{x}}^h} \sum_{\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)} P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \hat{x}^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \tag{99}$$

subject to

$$\mathscr{U}\left(\hat{\mathbf{x}}^h\right) \geq \mathscr{U}\left(\mathbf{x}^h\right) \tag{100}$$

taking prices $P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ as given,

(ii) for the market-maker, $\left\{\mathbf{y}, \widehat{P}_a(\mathbf{z}), P_a(\mathbf{z})\right\}$ solves (93), taking prices $\left\{P_{20}, P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)\right\}$ as given,

(iii) in period-0, markets for good-1, good-2 and lotteries clear, i.e. (95)-(97) hold,

Note that the sole difference between the compensated equilibrium and the competitive equilibrium is the specification of consumer's problem (99).

The following theorem states that any Pareto optimal allocation can be supported as a compensated equilibrium. To prove this theorem, we will show that a solution to the Pareto program is also a solution to expenditure minimization and profit maximization problems in equilibrium. Since all the optimization problems are well-defined concave problems, the Kuhn-Tucker conditions are necessary and sufficient. Hence, we will show that the Kuhn-Tucker conditions of Pareto program are equivalent to the ones of consumers' and intermediary's problems in equilibrium. In addition, the resource constraints in the Pareto programs are equivalent to the market-clearing conditions in equilibrium.

Theorem 3. *Any solution to the Pareto program with Pareto weight $\lambda^h \geq 0, \forall h$ can be supported as a compensated equilibrium.*

Proof. See Appendix A *Q.E.D.*

We will now use a cheaper-point argument to prove that any compensated equilibrium is a competitive equilibrium with transfers, when all Pareto weights are strictly positive. The competitive equilibrium with transfers is defined analogously to the competitive equilibrium but the consumer's problem is modified. Specifically, every agent h will receive feasible redistributed wealth $w^h \in \mathbb{R}$. In addition, a feasible redistributed-wealth allocation $\left[w^h\right]_h$ satisfies

$$\sum_h \alpha^h w^h = \sum_h \alpha^h \left[e_{10}^h + P_{20} e_{20}^h\right] \tag{101}$$

Given redistributed wealth w^h, each agent h solves

$$\max_{\mathbf{x}^h} \sum_{\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)} x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)\left\{U\left(c_{10}, c_{20}\right) + \beta V^h\left(k, \hat{\theta}, \theta, \mathbf{z}\right)\right\} \tag{102}$$

subject to $\mathbf{x}^h \in X^h$, and budget constraint (with transfers)

$$\sum_{\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)} P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \leq w^h \tag{103}$$

Definition 9. A competitive equilibrium with transfers is specification of allocation $(\mathbf{x}, \mathbf{y})$, and prices $\left(P_{20}, P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)\right)$ and is a feasible redistributed-wealth allocation in period-0, w^h for every h satisfying (101), such that

(i) for each h, $\mathbf{x}^h \in X^h$ solves (102) subject to (103), taking prices $\left(P_{20}, P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)\right)$ as given,

(ii) for the market-maker, $\left\{\mathbf{y}, \widehat{P}_a(\mathbf{z}), P_a(\mathbf{z}), P_\Delta(\mathbf{z})\right\}$ solves (93), taking prices $\left(P_{20}, P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)\right)$ as given,

(iii) in period-0, markets for good-1, good-2 and lotteries clear, i.e., (95)-(97) hold,

Theorem 4. *With Assumption 1, any Pareto optimal allocation corresponding with strictly positive Pareto weights* $\lambda^h > 0, \forall h$ *can be supported as a competitive equilibrium with transfers.*

Proof. See Appendix A *Q.E.D.*

We will use Negishi's mapping method (Negishi (1960)) to prove the existence of competitive equilibrium. The proof benefits from the second welfare theorem. Specifically, a part of the mapping applies the theorem in that the solution to the Pareto program is a competitive equilibrium with transfers. We will show that a fixed-point of the mapping exists and it represents a competitive equilibrium without transfers.

Theorem 5. *For any positive endowments, with Assumption 1, a competitive equilibrium exists.*

Proof. See Appendix A *Q.E.D.*

7 Analysis of Prices

This section characterizes systematic relationships between equilibrium prices.

Spot Markets and Contract Prices

First, the pyramiding mechanism puts a restriction on the prices of contracts traded within each price-island. In particular, the ratio of the equilibrium price of the contract paying in good-2 to the equilibrium price of the contract paying in good-1 in island-z_s in state s, $\frac{P_a(z_s,s)}{\widehat{P}_a(z_s,s)}$, must be equal to the spot price in the island, $p(z_s)$. Intuitively, if the ratio of the prices is different from the spot price, agents will be able to arbitrage in period-0 by trading an unbounded amount of contracts one for another (by keeping the collateral constraints satisfied with pyramiding), and then will trade in spot markets in the opposite direction to undo their positions in period-1. The result is summarized in the following lemma.

Lemma 8. *In a competitive equilibrium,*

$$P_a(z_s, s) = p(z_s)\widehat{P}_a(z_s, s), \ \forall s, z_s \tag{104}$$

Collateral, Contract and Price-Island Prices

Second, trading in price islands also imposes a restriction on collateral, contract and price-island prices, $P_{20}, P_a(z_s, s), P_\Delta(z_s, s)$. Even though holding collateral and a portfolio of contracts paying in good-2 can lead to the same consumption allocation in period-1, holding collateral additionally impacts the spot price $p(z_s)$. Therefore, the equilibrium price of collateral must reflect the role of collateral on the spot price in each price island.

Similar to Lemma 4, the collateral and contract (paying in good-2) allocations are indeterminate; that is, neither k or θ_s can be pinned down (but the net-claim of good-2, $R_s k + \theta_s$, will be uniquely determined). Roughly speaking, agents are indifferent between buying contracts ($\theta_s > 0$), and holding collateral ($k > 0$) and selling contracts against it ($\theta_s < 0$) as long as they lead to the same consumption allocation in period-1 over state s. Note that storage technology is linear and there is no direct utility per se from holding collateral. This result will be further discussed in examples in Section 8. The formal result is summarized in the following lemma.

Lemma 9. *For a feasible bundle of an agent type h* $\left(c_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$*, any bundle* $\left(c_0', k', \hat{\theta}', \theta', \mathbf{z}', \Delta'^h\right)$ *such that (i)* $c_0' = c_0$*, (ii)* $\mathbf{z}' = \mathbf{z}$*, (iii)* $\hat{\theta}' = \hat{\theta}$*, and (iv)* $R_s k' + \theta_s' = R_s k + \theta_s, \ \forall s$*, is also be feasible for the agent h, and lead to the same consumption allocation as the original bundle.*

Proof. The proof is similar to the one of Lemma 4. *Q.E.D.*

From condition (iv) of the lemma, there is some indeterminacy between k and θ_s. In particular, if we set $k' = 0$, then we can reach the same consumption allocation by setting the security position to be $\theta_s' = R_s k + \theta_s$. This implies that there is no loss of generality to assume that a constrained agent holds no collateral, $k = 0$, and therefore, we will do so unless stated otherwise.

Since both bundles lead to the same agent h's consumption allocation, an agent type h must be indifferent between them. As a result, they must have the same prices, i.e.,

$$P\left(\mathbf{c}_0', k', \hat{\theta}', \theta', \mathbf{z}', \Delta'^h\right) - P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) = 0 \tag{105}$$

Recall that $P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ can be rewritten, using the profit maximization condition of a market-maker (94), as

$$P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) = c_{10} + P_{20}c_{20} + P_{20}k + \widehat{P}_a(\mathbf{z}) \cdot \hat{\theta} + P_a(\mathbf{z}) \cdot \theta + P_\Delta(\mathbf{z}) \cdot \Delta \tag{106}$$

Substituting (106) into (105) gives

$$\begin{aligned}
0 &= P\left(\mathbf{c}_0', k', \hat{\theta}', \theta', \mathbf{z}', \Delta'^h\right) - P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \\
&= c_{10}' + P_{20}c_{20}' + P_{20}k' + \widehat{P}_a(\mathbf{z}') \cdot \hat{\theta}' + P_a(\mathbf{z}') \cdot \theta' + P_\Delta(\mathbf{z}') \cdot \Delta'^h \\
&\quad - \left[c_{10} + P_{20}c_{20} + P_{20}k + \widehat{P}_a(\mathbf{z}) \cdot \hat{\theta} + P_a(\mathbf{z}) \cdot \theta + P_\Delta(\mathbf{z}) \cdot \Delta\right] \\
&= P_{20}\left(k' - k\right) + P_a(\mathbf{z}) \cdot \left(\theta' - \theta\right) + \sum_s P_\Delta(z_s, s)\left[\left(z_s e_{2s}^h + z_s R_s k' - e_{1s}^h\right) - \left(z_s e_{2s}^h + z_s R_s k - e_{1s}^h\right)\right] \\
&= \left(k' - k\right)P_{20} - \left(k' - k\right)\sum_s P_a(z_s, s)R_s + \left(k' - k\right)\sum_s P_\Delta(z_s, s)R_s z_s \\
&= \left(k' - k\right)\left[P_{20} - \sum_s P_a(z_s, s)R_s + \sum_s P_\Delta(z_s, s)R_s z_s\right]
\end{aligned}$$

where the third equality follows from conditions (i)-(iii) in Lemma 9 and the definition of Δ in (63), and the fourth one follows from $\theta_s' - \theta_s = -R_s\left(k' - k\right)$ (condition (iv) in Lemma 9). This must be true even if $k \neq k'$. As a result, the term in the bracket must be zero. This result is summarized in the following lemma.

Lemma 10. *In a competitive equilibrium,*

$$P_{20} + \sum_{s=1}^{S} P_\Delta\left(z_s, s\right) z_s R_s = \sum_{s=1}^{S} P_a\left(z_s, s\right) R_s \tag{107}$$

The RHS is the price of contracts paying R_s units of good-2 in every state s. On the other hand, the LHS is the total cost of the same return, received by buying and holding a unit of collateral. The first term on the LHS is the price of the collateral good. The second term on the LHS comes from the fact that holding more a unit of good-2 increases Δ in every state s by the amount $z_s R_s$. In particular, an agent holding an additional unit of collateral must pay for the marginal impact $z_s R_s$ at price $P_\Delta\left(z_s, s\right)$. This term prices the impact of collateral on the market fundamental. In equilibrium, these two values must be the same.

Trading in Price-Islands and Intertemporal Transfers

Trading in price-islands generates additional intertemporal transfers. Recall that a constrained agent would like to smooth consumption by issuing securities or borrowing to transfer future wealth back to the first period but cannot do so much because of the limited commitment. In order to facilitate more consumption smoothing, a constrained agent will trade "future wealth" by trading in price-islands for period-0 transfer, and vice versa.

Specifically, using market-maker's profit maximization (94), an agent type h's budget constraint (84) can be rewritten as

$$\begin{aligned} e_{10}^h + P_{20}e_{20}^h &\geq \sum_b x^h(b)\, P(b) \\ &= \sum_b x^h(b)\,[c_{10} + P_{20}c_{20}] + \sum_b x^h(b)\, P_{20}k + \sum_b x^h(b) \sum_s \widehat{P}_a(z_s, s)\,\hat{\theta}_s \\ &+ \sum_b x^h(b) \sum_s P_a(z_s, s)\,\theta_s + \sum_b x^h(b) \sum_s P_\Delta(z_s, s)\,\Delta_s \end{aligned}$$

where again for brevity $b = \left(c, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$. Using (104) and the definition of Δ_s (63), the budget constraint becomes,

$$\begin{aligned} \sum_b x^h(b)\,[c_{10} + P_{20}c_{20}] &+ \sum_b x^h(b)\, P_{20}k \leq e_{10}^h + P_{20}e_{20}^h + \sum_b x^h(b) \sum_s \widehat{P}_a(z_s, s)\left[-\hat{\theta}_s - p(z_s)\theta_s\right] \\ &+ \sum_b x^h(b) \sum_s P_\Delta(z_s, s)\left[\frac{e_{1s}^h}{e_{2s}^h + R_s k} - z_s\right]\left(e_{2s}^h + R_s k\right) \qquad (108) \end{aligned}$$

For the sake of discussion, we will consider a case with two types of agents one of which is constrained, and without uncertainty (i.e., $S = 1$). In addition, as shown in the previous section, we assume that a constrained agent holds no collateral, $k = 0$. Therefore, the budget constraint for a constrained agent becomes

$$\begin{aligned} \sum_b x^h(b)\,[c_{10} + P_{20}c_{20}] \leq e_{10}^h + P_{20}e_{20}^h + \sum_b x^h(b)\, \widehat{P}_a(z_1, 1)\left[-\hat{\theta}_1 - p(z_1)\theta_1\right] \\ + \sum_b x^h(b)\, P_\Delta(z_1, 1)\left[\frac{e_{11}^h}{e_{21}^h} - z_1\right] e_{21}^h \qquad (109) \end{aligned}$$

The third term on the RHS is the revenue from borrowing via $\left(\hat{\theta}_1, \theta_1\right)$. Using the collateral constraint (65), $p(z_1)R_1k + \hat{\theta}_1 + p(z_1)\theta_1 \geq 0$. Since the constrained agent holds no collateral, $k = 0$, her collateral constraint becomes $\hat{\theta}_1 + p(z_1)\theta_1 = 0$. Of course, this constrained agent would like to go short on the contracts (i.e., having $\hat{\theta}_1 + p(z_1)\theta_1 < 0$) but cannot do so because she holds no collateral. In other words, with zero collateral, the agent cannot borrow from trading in contracts.

Of special interest, last term on the RHS shows that the constrained agent could potentially receive positive period-0 wealth by trading in price-islands. In particular, a constrained agent could smooth consumption intertemporally by trading in price-islands in such a way that this term is positive, giving her more resources to purchase date zero consumption. For example, if

$P_\Delta(z_s, s) > 0$, then the constrained agent will buy a price island-z_s whose market fundamental is lower than her own endowment, i.e., $z_s < \frac{e^h_{1s}}{e^h_{2s}}$, and vice versa.

On the other hand, an unconstrained agent will potentially hold strictly positive amount of collateral, $k > 0$. She will in fact transfer out period-0 wealth from trading in price-islands. For example, if a constrained agent has $z_s < \frac{e^h_{1s}}{e^h_{2s}}$, then the consistency constraint (74) implies that an unconstrained agent must have $z_s > \frac{e^h_{1s}}{R_s k + e^h_{2s}}$. Hence, if $P_\Delta(z_s, s) > 0$, the last term on the RHS will be negative for an unconstrained agent.

8 Numerical Examples without Uncertainty

Two simple economies without uncertainty are discussed in this section. An example with uncertainty will be discussed in the next section. As mentioned earlier (see Lemma 9), all equilibria presented here have constrained agents holding zero collateral, $k = 0$.

The first economy consists of two types of agents. A collateral equilibrium with an externality and a competitive equilibrium with lotteries (without externality) are both presented. We find that the externality leads to a larger amount of aggregate saving relative to the one in the competitive equilibrium with lotteries. Interestingly, the competitive equilibrium with lotteries has a *unique active price-island* even though all price-islands are available. This does not have to be true in general, however. In particular, the second environment with three types of agents illustrates a lottery equilibrium with *multiple active price-islands*.

The effects of internalizing the externality on prices and allocations are discussed. We find that internalizing the externality could make (i) the price of good-2 fluctuates less over time relative to the equilibrium with externalities, and (ii) someone is strictly better off and someone is strictly worse off than being in the collateral equilibrium. The first outcome suggests naturally enough that the externality causes collateral price to fluctuate too much. It is the good in short supply when there are borrowing constrained agents. The second outcome implies that trading in the price islands generate a redistribution of wealth across agents. This is a general equilibrium effect of the model. See figure 3. Note that, as shown in the example in the section, the second statement may not be true in general. The redistribution of wealth could be very small or vanished when the agents are quite similar. As a result, internalizing the externality will lead to a Pareto improving.

This example also shows that a constrained agent may hold no collateral and therefore issue no directly-collateralized contacts, yet her collateral constraint is binding still. The fact that a constrained agent is effectively not borrowing at all in equilibrium seems counterintuitive at first.

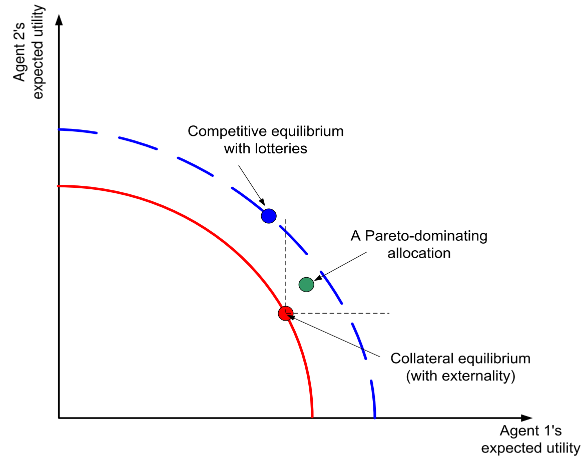


Figure 3: Bold curve: the Pareto frontier and collateral equilibrium without lotteries; Dash curve: the Pareto frontier and competitive equilibrium with lotteries.

In a partial equilibrium setting when the price of collateral good is fixed exogenously, one would imagine that the agent will try to buy more of the collateral good and then borrow against to increase current consumption. In this *general equilibrium* setting where collateral price is determined endogenously, however, the price of the collateral good rises so in effect those transactions will offset each other and lead to a zero net transfer.

Environment 1. There is a single state, $S = 1$, and two types of agents, $H = 2$, both of which have an identical constant relative risk aversion (CRRA) utility function

$$U(c_1, c_2) = \frac{c_1^{1-\gamma}}{1-\gamma} + \frac{c_2^{1-\gamma}}{1-\gamma}, \quad \forall h \tag{110}$$

where $\gamma = 2$. Each type consists of $\frac{1}{2}$ fraction of the population, i.e. $\alpha^h = \frac{1}{2}$. In addition, the discount factor $\beta = 1$. The storage technology is given by $R = 1$. The endowment profiles of the agents are shown in Table 2 below. Recall that e_{it}^h is an agent h's endowment of good-i in period t.

Type of Agent	e_{10}^h	e_{20}^h	e_{11}^h	e_{21}^h
$h = 1$	3	3	1	1
$h = 2$	1	1	3	3

Table 2: Endowment profiles of the agents.

Note that endowments for both agents are symmetric. In particular, an agent type 1 is well endowed with both goods in period-0 and vice versa for type 2. The symmetry of endowments with $\beta = 1$, and of the utility function across goods implies that both types of agents are ex-ante identical in a frictionless economy. The first-best aggregate saving will be zero, and each agent gets the average 2 units of each good in each periods (see Table 3). The first-best price of good-2, the collateral good, in period-0 is $P_{20}^{fb} = 1$. Furthermore, zero per-capita saving implies that the market fundamental, the aggregate ratio of good-1 to good-2, is $z = 1$, and the first-best spot price of good-2 in period $t = 1$ is $p(z) = z^{\gamma} = 1$. Unfortunately, the first-best allocation is not attainable; that is, it violates the collateral constraints.

Type of Agent	k^h	c_{10}^h	c_{20}^h	c_{11}^h	c_{21}^h
$h = 1$	0	2	2	2	2
$h = 2$	0	2	2	2	2

Table 3: First-best consumption and collateral allocations.

We now consider the economy with default and collateral (with an externality). The endowment profile and the first-best allocation suggest that agent 2 would like to move resources forward from $t = 1$ to $t = 0$, and therefore will be constrained. Hence, we will assume that agents type 2 hold no collateral, i.e. $k^1 = k$ and $k^2 = 0$. We will then solve for an equilibrium k. From the market clearing conditions of contracts, we can set $\hat{\theta}_1^1 = \hat{\theta} = -\hat{\theta}_1^2$ and $\theta_1^1 = \theta = -\theta_1^2$. Note that this does not mean agent 1 is demanding both securities. In addition, using the specified collateral allocation, the market fundamental in period-1 is now $z = \frac{4}{4+k}$ (the ratio of endowment of good 1 to the sum of endowment of good 2 and saving), and consequently the spot price of good-2 in period 1 is $p(z) = \left(\frac{4}{4+k}\right)^2$.

With homothetic preferences, the first-order conditions of the problem (50) for both types imply that in spot markets at date $t = 0$

$$P_{20} = \left(\frac{c_{10}^1}{c_{20}^1}\right)^2 = \left(\frac{c_{10}^2}{c_{20}^2}\right)^2 = \left(\frac{4}{4-k}\right)^2 \tag{111}$$

Since agent 1's collateral constraint is not binding, the first-order conditions of her utility-maximization problem (50) with respect to θ_1^1 and c_{10}^1 lead to

$$P_1 = \frac{U_{21}^1}{U_{10}^1} = \left(\frac{c_{10}^1}{c_{21}^1}\right)^2 \tag{112}$$

where $U_{it}^h = \frac{\partial U^h}{\partial c_{it}}$ is the marginal utility with respect to c_{it}, and P_1 is the price of a security paying in good 2 in period $t = 1$, θ_1^h. Note that we put superscript h on the utility function for clarity. Further, the first-order conditions of the consumer's problem (50) with respect to θ_1^1 and k^1 (interior solutions) lead to

$$P_{20} = P_1 \tag{113}$$

Intuitively, this is the case because their payoffs are identical and both are collateralizable. Using (111) and (112), condition (113) implies that

$$\frac{c_{10}^1}{c_{20}^1} = \frac{c_{10}^1}{c_{21}^1} \Longrightarrow c_{20}^1 = c_{21}^1 \tag{114}$$

That is, an unconstrained agent consumes the same amount of good 2 in both periods.

Substituting (111) and (112) into (113) gives

$$\begin{aligned}\left(\frac{4}{4-k}\right)^2 &= \left(\frac{c_{10}^1}{c_{21}^1}\right)^2 \\ \frac{4}{4-k} &= \frac{c_{10}^1}{1+k+\theta} \Longrightarrow (4-k)\, c_{10}^1 = 4+4k+4\theta \end{aligned} \tag{115}$$

where we use $c_{21}^1 = 1+k+\theta$.

On the other hand, an agent type 2's collateral constraint is binding; with $k^2 = 0$,

$$\hat{\theta}^2 + p(z)\theta^2 = 0 \Longrightarrow -\hat{\theta} - p(z)\theta = 0 \Longrightarrow \hat{\theta} = -\left(\frac{4}{4+k}\right)^2 \theta \tag{116}$$

where the second and the last equations use $\hat{\theta}^2 = -\hat{\theta}$ and $\theta^2 = -\theta$, and $p(z) = \left(\frac{4}{4+k}\right)^2$, respectively.

The budget constraint of an agent 1 (51) can be written as

$$c_{10}^1 - 3 + P_{20}\left[c_{20}^1 + k - 3\right] + \hat{P}_1\hat{\theta} + P_1\theta \quad = \quad 0 \tag{117}$$

A standard no-arbitrage argument (similar to the one used in Lemma 8) implies that

$$P_1 = p(z)\widehat{P}_1 \tag{118}$$

It thus true from (118) that

$$\hat{P}_1\hat{\theta} + P_1\theta = \widehat{P}_1\hat{\theta} + \widehat{P}_1 p(z)\theta = \widehat{P}_1\left[\hat{\theta} + p(z)\theta\right] p(z) = 0 \tag{119}$$

where the last equation follows the fact that the term in the bracket is zero, from (116). Now the LHS of the budget constraint (117) can be rewritten as

$$c_{10}^1 + P_{20}\left[c_{20}^1 + k - 3\right] = 3 \tag{120}$$

Using (111), we can replace c_{20}^1 by $\left(\frac{4-k}{4}\right) c_{10}^1$. Then using $P_{20} = \left(\frac{4}{4-k}\right)^2$ gives

$$\begin{aligned} c_{10}^1 + \left(\frac{4}{4-k}\right)^2 \left[\left(\frac{4-k}{4}\right) c_{10}^1 + k - 3\right] &= 3 \\ \Longrightarrow (4-k)\, c_{10}^1 &= \frac{3k^2 - 40k + 96}{8-k} \end{aligned} \tag{121}$$

Substituting (115)into (121) gives

$$\frac{3k^2 - 40k + 96}{8 - k} = 4 + 4\theta + 4k \Longrightarrow 4\theta + 4k = \frac{3k^2 - 36k + 64}{8 - k} \tag{122}$$

With the identical homothetic preferences, the period-1 consumption allocations must satisfy

$$z = \frac{4}{4+k} = \frac{c_{11}^1}{c_{21}^1} \Longrightarrow \frac{4}{4+k} = \frac{1+\hat{\theta}}{1+k+\theta} \tag{123}$$

Substitute (116) into (123) gives

$$4\theta + 4k = -3k\left(\frac{4+k}{8+k}\right) + 4k \tag{124}$$

Using (122) and (124), we have

$$\frac{3k^2 - 36k + 64}{8 - k} = -3k\left(\frac{4+k}{8+k}\right) + 4k \Longrightarrow 4k^3 - 384k + 512 = 0 \tag{125}$$

There are three roots for equation (125). Using the condition that $0 \leq k \leq 4$, there is only one feasible solution, i.e. $k \approx 1.3595$. To sum up, the equilibrium collateral allocation is $k^1 = k = 1.3595$ and $k^2 = 0$.

As a result, the price of good-2 in period 0 is $P_{20} = \left(\frac{4}{4-k}\right)^2 = 2.2948$, the price of a security paying in good-1 in period-1 is $\widehat{P}_1 = \left(\frac{c_{10}^1}{c_{11}^1}\right)^2 = 4.1198$, and the price of a security paying in good-2 in period-1 is $P_1 = \left(\frac{c_{10}^1}{c_{21}^1}\right)^2 = 2.2948$. In addition, the market fundamental and spot price in period-1 are $z = \frac{4}{4+k}0.7463$ and $p(z) = z^\gamma = 0.5570$, respectively. These prices are summarized in Table 4 below.

The collateral price is higher in the equilibrium with an externality, i.e., $P_{20}^{fb} = 1 < P_{20} = 2.2948$. On the other hand, the spot price of good-2 in period-1 is lower in the equilibrium with an externality, i.e., $p^{fb}(z) = 1 > p(z) = 0.5570$. In words, the collateral distortion makes the price of good-2 high in the first period and low in the second period. In addition, the price of good 2 is significantly higher in period 0 than in period 1, i.e., $P_{20} > p(z)$. Nevertheless, an agent 1 is saving. This is because saving is the only way she can transfer resources to $t = 1$, given that a constrained agent holds zero collateral. Note also that the security paying in good 2 is cheaper than the security paying in good 1, i.e., $P(z) < \widehat{P}(z)$. This is because, good 2 is more abundant than good 1 at the time when the securities are executed ($t = 1$).

The equilibrium allocation with externality is summarized in Table 5. An agent type 1 consumes more of both goods in period-0 than an agent type 2, and vice versa. Since the price of good-2 is higher, both types consume less of good-2 in period-0 relative to the first-best (compare c_{20}^h in Table 3 and Table 5). On the other hand, with the storage, both types consume more of good-2

Market fundamentals	P_{20}	$p(z)$	$\widehat{P}(z)$	$P(z)$
$\mathbf{z = 0.7463}$	**2.2948**	**0.5570**	**4.1198**	**2.2948**

Table 4: Equilibrium prices. The bold numbers are (actively traded) equilibrium prices.

in period-1 than their endowments. In addition, using the consumption allocations reported in the table, the expected utility of an agent type 1 and type 2 are $\mathscr{U}^1 = -2.2527$ and $\mathscr{U}^2 = -2.5724$, respectively. Note that, for comparison, we also report the deviation from the fundamental Δ even though it is not traded in this equilibrium.

Agent	c_{10}^h	c_{20}^h	k^h	$\hat{\theta}^h$	θ^h	z	Δ	c_{11}^h	c_{21}^h	CC
$h = 1$	2.6899	1.7756	1.3595	0.3252	-0.5839	0.7463	0.7610	1.3252	1.7756	NB
$h = 2$	1.3101	0.8649	0	-0.3252	0.5839	0.7463	-0.7610	2.6748	3.5839	B

Table 5: A collateral equilibrium allocation. CC, B, and NB stand for collateral constraint, binding, and non-binding, respectively.

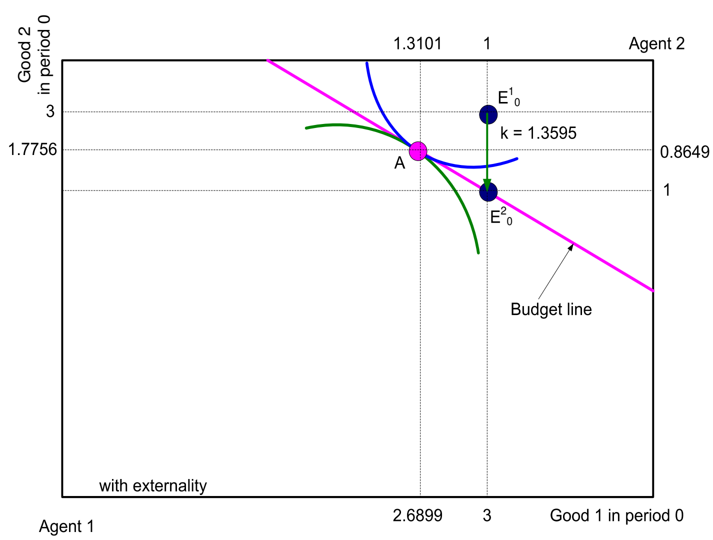


Figure 4: Period-0 equilibrium allocation. Point A is period-0 equilibrium allocation with externality, where each agent consumption is $c_0^1 = (2.6899, 1.7756)$, $c_0^2 = (1.3101, 0.8649)$.

Figures 4 illustrates period-0 equilibrium allocation with the externality[7]. It shows that agent 1 sells good 1 and buys good 2, and vice versa for agent 2. In addition, the allocation is on the

[7]The key equation for this figure is the budget constraint of a collateral-constrained agent $h = 2$ (51), which can

budget line of a constrained agent, which is the line passing through E_0^2. This implies that an unconstrained agent will effectively do all the saving.

Figures 5 illustrates period-1 equilibrium allocation with the externality, including security trades. Agent 1 buys $\hat{\theta}^1 = 0.3252$ and sells $\theta^1 = -0.5839$, and vice versa for agent 2. Equivalently, one can imagine that agent 1 buys 0.3252 units of good 1 and sells 0.5839 units of good 2 at price $p(z) = 0.5570$ in spot markets, and vice versa.

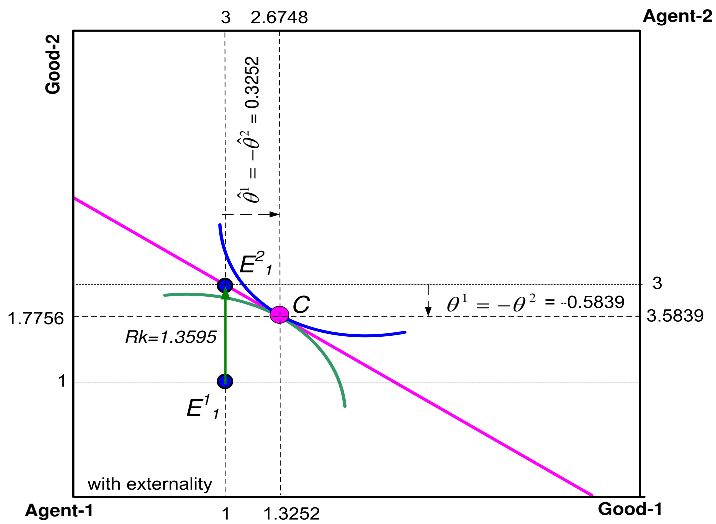


Figure 5: Period-1 equilibrium allocation. Point C is the period-1 equilibrium allocation with externality, where each agent consumption is $c_1^1 = (1.3252, 1.7756)$, $c_1^2 = (2.6748, 3.5839)$. Agent 1 buys good-1 and sells good-2, and vice versa.

---

be rewritten as

$$c_{10}^2 + P_{20}c_{20}^2 + \left[P_{20}k^2 + \hat{P}_1\hat{\theta}^2 + P_1\theta^2\right] \quad \leq \quad e_{10}^2 + P_{20}e_{20}^2 \tag{126}$$

Using the fact that $P_{20} = P_1$ and $P_1 = p(z)\hat{P}_1$ (the proof is based on a no-arbitrage argument similar to the proof of Lemma 8), the budget constraint becomes

$$c_{10}^2 + P_{20}c_{20}^2 + \hat{P}_1\left[p(z)k^2 + \hat{\theta}^2 + p(z)\theta^2\right] \quad \leq \quad e_{10}^2 + P_{20}e_{20}^2 \tag{127}$$

Since agent-2's collateral constraints are binding, the last term on the LHS is vanished. Hence, her budget constraint is

$$c_{10}^2 + P_{20}c_{20}^2 \quad \leq \quad e_{10}^2 + P_{20}e_{20}^2 \tag{128}$$

The slope of the budget line is equal to $-\frac{1}{P_{20}}$. Since P_{20} is increasing in average saving K, the budget line will be steeper when average saving decreases (see figure 6).

We will now turn to a competitive equilibrium with lotteries. The equilibrium allocation with lotteries reported in this paper is a numerical solution[8] to the Pareto program (76) that corresponds to a competitive equilibrium with lotteries (without transfers[9]).

An equilibrium allocation is reported in Table 6. Interestingly, there is *only one active price-island* $z = 0.7729$ even though all price-islands are available for trade. We will now compare this equilibrium allocation without any externality to the one with an externality. The equilibrium average or per capita saving (without externality) is $\sum_h \sum_b \alpha^h x^h(b)k = \frac{1.1753}{2} = 0.5877$, which clearly smaller than the aggregate saving in the equilibrium with externality. With lower aggregate saving, both agents consume more of good 2 in period 0 but less in period 1, relative to the equilibrium allocation with externality. An agent type 1 buys and sells securities $\hat{\theta}^1 = 0.2970$ and $\theta^1 = -0.4972$, and vice versa for agent 2. The volume of trade here is less than the volume of trade in the equilibrium with an externality (in Table 5). Equivalently, all securities trade in this equilibrium can be replicated by trading in spot markets. In other words, only asset-backed securities are traded here. In addition, the consumption allocation also implies that both types are better off in period 0 and worse off in period 1, relative to the equilibrium allocation with externality.

Agent	c^h_{10}	c^h_{20}	k	$\hat{\theta}^h$	θ^h	z	Δ	c^h_{11}	c^h_{21}	mass
$h = 1$	2.6073	1.8410	1.1753	0.2970	-0.4972	0.7729	0.6813	1.2970	1.6781	1.0000
$h = 2$	1.3927	0.9837	0	-0.2970	0.4972	0.7729	-0.6813	2.7030	3.4972	1.0000

Table 6: Equilibrium allocation of (non-zero-mass) lotteries.

Equilibrium prices including the fees of inactive (out-of equilibrium) price-islands are summarized in Table 7 below. With lower aggregate saving, the price of good-2 in period 0 is lower ($P_{20} = 2.0073 < 2.2948$) but the spot price of good 2 is higher ($p(z) = 0.5974 > 0.5570$), relative to the ones in the equilibrium with externality (compare Table 4 and 7). This also implies that the price of good-2 varies less over time when the externality is internalized.

Notice that the fees of price islands are increasing with the market fundamentals; that is, the larger the specified market fundamental of a price island is, the higher the fee of the price island will be. Intuitively, the larger market fundamental, the larger the price of good 2 relative to good 1,

[8]It is computed using Matlab program on a personal computer with AMD Athlon 64X2 Dual Core Processor 3800+ 2.01 GHz, 3.87 GB RAM.

[9]We search for such equilibrium using Negishi's mapping method. The corresponding Pareto weight is $\lambda^1 = 0.7780, \lambda^2 = 0.2220$.

Market fundamentals	P_{20}	$p(z)$	$\widehat{P}(z)$	$P(z)$	$P_\Delta(z)$
$z = 0.7479$	N/A	0.5594	4.2308	2.3668	0.4639
$\mathbf{z = 0.7729}$	**2.0073**	**0.5974**	**4.0621**	**2.4269**	**0.5375**
$z = 0.7979$	N/A	0.6366	3.9257	2.4996	0.6118

Table 7: Equilibrium prices. The bold numbers are (actively traded) equilibrium prices.

means the lower the aggregate amount of good 2 relative to good 1 in the island. Hence, an agent with a larger amount of good 2 relative to good 1 will benefit more from being in a higher price island. Hence, it is optimal to require the benefiting agents to pay more for trading in a higher price island, as in the last column $P_\Delta(z)$.

In addition, since the equilibrium fee $P_\Delta(z) = 0.5375$ is positive in this example, an agent with positive deviation from the fundamental Δ (see (63) for its definition) will pay for trading in such price island. As shown in Table 6, agent 1, who holds all collateral, has positive $\Delta = 0.6813$, and therefore she is paying (receiving a negative transfer) $-P_\Delta(z)\Delta = -0.5375 \times 0.6813 = -0.3662$ units of good 1 at $t = 0$ to be in the equilibrium price-island. On the other hand, agent 2 whose $\Delta = -0.6813$, is receiving transfer $-P_\Delta(z)\Delta = 0.3662$ in period $t = 0$ for being in the equilibrium price-island. This result is illustrated in figure 6 below.

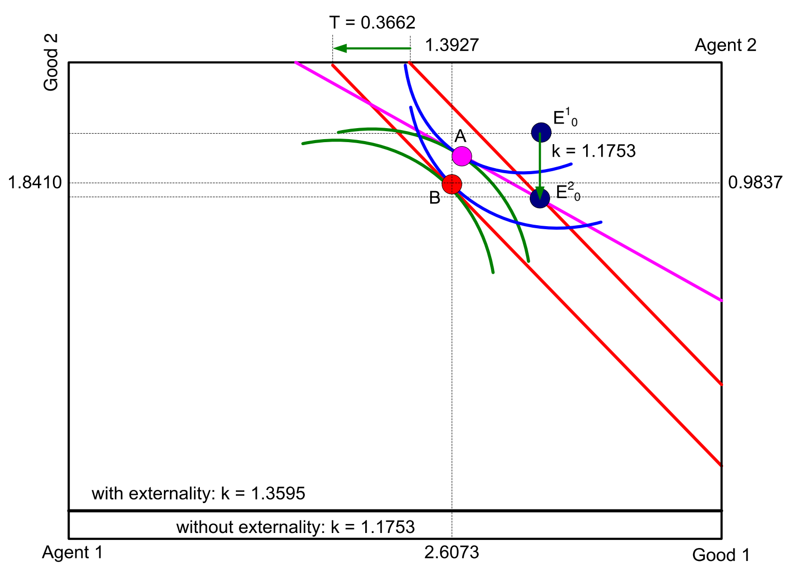


Figure 6: Period-0 equilibrium allocations with and without externality. Point B is the period-0 equilibrium allocation without externality, where each agent consumption is $c_0^1 = (2.6073, 1.8410)$, $c_0^2 = (1.3927, 0.9837)$. Agent 1 receives wealth transfer, $T = 0.366$, from trading in price-islands.

Consider the budget constraint of agent 2 (a constrained agent) in competitive equilibrium with lotteries,(109). For convenience, we write it again here:

$$\sum_b x^h(b)\left[c_{10} + P_{20}c_{20}\right] \le e^h_{10} + P_{20}e^h_{20} + \sum_b x^h(b)\left[-P_\Delta(z)\,\Delta\right] \tag{129}$$

First of all, the slope of this budget line is also equal to $-\frac{1}{P_{20}}$. As discussed earlier, the collateral price P_{20} in competitive equilibrium with lotteries is lower than the one in externality equilibrium. As a result, this budget line will be steeper than the one for the collateral equilibrium in figure 6. In addition, the last term on the RHS is the transfer that the agent receives in period 0 from trading in price-islands, denoted by $T = 0.3662$ units of good 1 which is the numeraire good. With this transfer, agent 2's budget line will shift outward by T, as shown in the figure. As a result, this new allocation could makes agent 2 better-off relative to period-0 equilibrium allocation with externality. Put differently, trading in price-islands enables a constrained agent to intertemporally smooth consumption better. Note that agent 2 uses this extra wealth to buy more consumption in period 0, as shown in the figure.

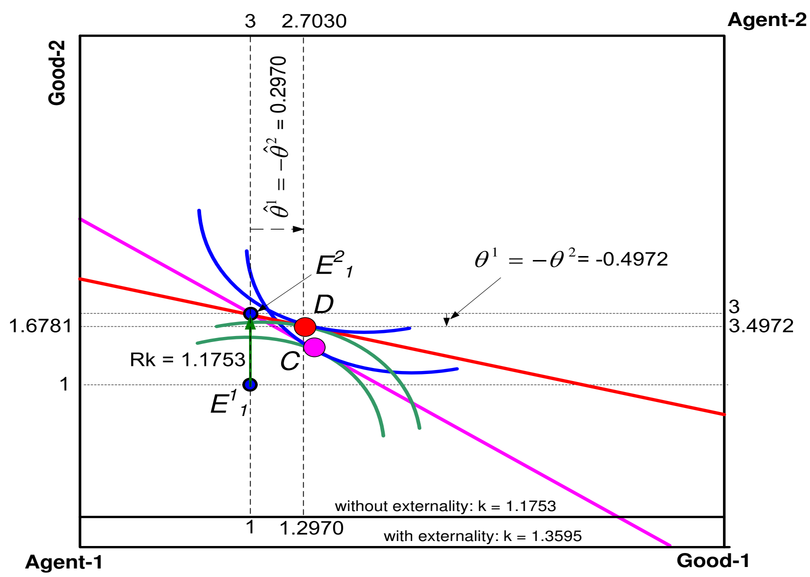


Figure 7: Period-1 equilibrium allocations with and without externality. Point D is the period-1 equilibrium allocation without externality, where each agent consumption is $c^1_1 = (1.2970, 1.6781)$, $c^2_1 = (2.7030, 3.4972)$.

Figure 7 shows period-1 competitive equilibrium allocation with lotteries. As discussed in Section 3.5, the budget line passes through agent 2's period-1 endowment, E^2_1. With higher spot price $p(z)$, the budget line will be flatter when there is no externality. In addition, with a higher spot price agent 2 buys less of good 2. In fact, agents trade securities less relative to the equilibrium allocation with externality.

Using the consumption allocations reported in the table, the expected utility of an agent type 1 and type 2 are $\mathscr{U}^1 = -2.2936, \mathscr{U}^2 = -2.3905$, respectively. Recall that the expected utility in the externality equilibrium of an agent type 1 and type 2 are $\mathscr{U}^1 = -2.2527$ and $\mathscr{U}^2 = -2.5724$, respectively. This shows that internalizing the externality is beneficial to an agent type 2 (constrained agent) but harmful for an agent type 1. This is a (distributional) general equilibrium effect. Internalizing the externality not only improves efficiency of the economy, but also redistributes wealth. All agents benefit from the efficiency effect, which shifts the Pareto frontier outward as shown in figure 3. Some agents may be harmed by the distributional effect, however. In particular, agents with positive deviation from the fundamental (agent 1 in this case) must pay for the access to the price island. This implies that trading in price islands generates the redistribution of wealth in general equilibrium. Alternatively, to induce welfare gain for all of agents, there must be lump sum transfers.

The distribution of welfare gains from internalizing the externality can be defined by the percent of consumption in good-1 in period 0 that is needed to give to an agent in a collateral equilibrium to make her as well off being in the competitive equilibrium. More precisely, let g^h be the welfare gain of an agent type h which is define by

$$U\left((1+g^h)c_{10}^{h*}, c_{20}^{h*}\right) + \beta \sum_s \pi_s U\left(c_{1s}^{h*}, c_{2s}^{h*}\right) = \mathscr{U}^h \tag{130}$$

where c_{is}^{h*} and $\mathscr{U}^h$ are agent h's consumption of good-i in state s in collateral equilibrium, and agent h's expected utility in competitive equilibrium with lotteries (without transfers), respectively. Using the result presented below, the welfare gains of each agent are given by $g^1 = -0.0990, g^2 = 0.3128$. So, agent 2 must get transfer to reach utility up to where it ends up with competitive equilibrium, and agent 1 is taxed.

The next example demonstrates that it is possible to have multiple active price-islands. In particular, with three types of agents, two of which are constrained, there are two distinct active price islands in equilibrium, each of which consists of different composition of agents. Each island, in fact, consists of one constrained type and one unconstrained type.

Environment 2. There are three types of agents, and a single state, i.e. $S = 1$. Each agent is given the same utility function as in (110) with $\gamma = 2$. Each type consists of $\frac{1}{3}$ fraction of the population, i.e. $\alpha^h = \frac{1}{3}$. Similar to previous examples, $\beta = 1$, and $R = 1$. The endowment profile is given in Table 8 below.

Type of Agents	period $t = 0$		period $t = 1$	
	e^h_{10}	e^h_{20}	e^h_{11}	e^h_{21}
$h = 1$	$\frac{12.5}{3}$	11.5	0.5	0.5
$h = 2$	$\frac{12.5}{3}$	0.5	7	5
$h = 3$	$\frac{12.5}{3}$	0.5	5	7

Table 8: Endowment profiles of the agents.

Note that in the first-best world, all agents are again ex-ante identical, and therefore they should end up with identical consumption allocation, $c^h_{it} = \frac{12.5}{3}$ for all $h = 1, 2, 3$, $i = 1, 2$ and $t = 0, 1$, and there is zero aggregate saving, $k^h = 0$ for all h. Hence, the collateral price in period 0 will be $P^{fb}_{20} = 1$. However, this is not the case in this collateral economy with default. Note that we do not present an equilibrium with externality of this economy.

With large endowments in $t = 1$, agent 2 and agent 3 want to move resources forward to $t = 0$. The scarcity of collateral then implies that both of them will be collateral constrained. In addition, type 2 has the larger ratio of period-1 endowment $\frac{e^h_{11}}{e^h_{21}}$ than type 3. As shown below, this difference suggests that agent 2 will be in a higher price-island than agent 3 in equilibrium. In addition, in a competitive equilibrium with lotteries, an agent type 2 is better off than an agent type 3. We will discuss this result below.

A competitive equilibrium allocation with lotteries is presented in Table 9 below. Note that the equilibrium reported here is a competitive equilibrium without lump sum transfers[10]. Interestingly, there are *two active price islands*, $z = 0.6028$ and $z = 0.8167$. The price island $z = 0.6028$ consists of some fraction of agents type-1, and all of agents type-3 (a constrained type). On the other hand, the price island $z = 0.8167$ consists of some fraction of agents type-1, and all of agents type-2 (a constrained type).

The multiplicity of price-islands has to do with nontrivial lotteries. The lotteries are optimal because the collateral constraints create a non-convexity problem (see Lemma 6). AS before, the externality distorts the aggregate saving. Hence, internalizing the externality will correct the aggregate saving level. Then given the optimal aggregate saving, it is beneficial to use lotteries because of a non-convexity problem. In particular, a convex combination over both islands of

[10]We search for such equilibrium using Negishi's mapping method. In particular, the Pareto weight for this particular equilibrium is given by $\lambda^1 = 0.5571, \lambda^2 = 0.3511, \lambda^3 = 0.0918$

the equilibrium allocations of Table 9 is not attainable. At a convex-combination allocation, the market fundamental would be between the two of those islands. For example, suppose $0.6028 < z = 0.7133 < 0.8167$. Then the spot price in island $z = 0.7133$ ($p(z) = 0.5088$) will be lower than in island $z = 0.8167$. This will violate the collateral constraint of a type-2 agent (a constrained agent in island $z = 0.8167$); $p(z)Rk + \hat{\theta}^2 + p(z)\theta^2 = 0 - 1.3111 + 0.5088 \times 1.9657 = -0.3110 < 0$.

Allocation	Agent Type			
	$h=1$		$h=2$	$h=3$
c_{10}^h	5.6823	5.6823	4.5110	2.3066
c_{20}^h	3.4000	3.4000	2.7014	1.3750
k^h	6.4878	4.6339	0.0000	0.0000
$\hat{\theta}^h$	1.5563	1.6601	-1.3111	-0.2935
θ^h	-3.8422	-2.4890	1.9657	0.8077
z	0.6028	0.8167	0.8167	0.6028
c_{11}^h	2.0563	2.1601	5.6889	4.7065
c_{21}^h	3.1456	2.6449	6.9657	7.8077
Δ	3.7122	3.6929	-2.9165	-0.7804
mass: $x^h(b)$	0.2102	0.7898	1.0000	1.0000

Table 9: Equilibrium allocation of (non-zero-mass) lotteries. There are multiple active price islands; $z = 0.6028$ and $z = 0.8167$.

Similar to the previous example, all security trades are equivalent to spot trades; that is, all of them can be replicated by trading in spot markets. In particular, agent 2 and agent 3 buy good 2 and sell good 1 in the spot markets, and vice versa for agent 1. In addition, agent 1's security trading varies across price-islands. This is because different islands have different spot prices. Notice also that there are more of agents type 1 in island $z = 0.8167$ than in island $z = 0.6028$. This follows from the consistency constraints; that is, the market fundamental in each island has to be equal to the specified level. In particular, the market fundamental in island $z = 0.8167$ is given by the aggregate ratio of good 1 to good 2: $\frac{0.7898 \times 0.5 + 1.000 \times 7}{0.7898 \times (4.6339 + 0.5) + 1.000 \times 5} = 0.8167$.

Interestingly, agent 2 and agent 3 have different consumption allocations in period 0 even though they have the same endowment in period 0. This is because they trade in different price-islands, and thereby receive different wealth transfers. In particular, using the allocation in Table 9 and the prices in Table 10, agent 2 receives $-P_{\Delta}(z = 0.8167)\Delta = -2.2316 \times (-2.9165) = 6.5085$ units of good 1 at $t = 0$ while agent 3 receives $-P_\Delta(z = 0.6028)\Delta = -0.7916 \times (-0.7804) = 0.6178$ units of good 1 at $t = 0$. In addition, all agents consume less of good 2 in period 0 relative to the first-best level. This is because the collateral price in period 0 is now $P_{20} = 2.1716 > P_{20}^{fb} = 1$.

Prices	$z = 0.4500$	$\mathbf{z = 0.6028}$	$z = 0.7028$	$z = 0.7667$	$\mathbf{z = 0.8167}$	$z = 0.8667$
$p(z)$	0.2025	**0.3634**	0.4939	0.5878	**0.6670**	0.7512
$\widehat{P}(z)$	14.1883	**8.9797**	8.1863	7.3753	**6.9091**	6.5655
$P(z)$	2.8731	**3.2629**	4.0434	4.3354	**4.6084**	4.9318
$P_\Delta(z)$	-0.1029	**0.7916**	1.7855	2.0212	**2.2316**	2.4761
Members	none	$h = 1, 3$	none	none	$h = 1, 2$	none

Table 10: Equilibrium prices. The bold numbers are (actively traded) equilibrium prices and price-islands.

Table 10 summarizes equilibrium prices, including the prices of non-active price-islands. The price of the security paying in good 1 in island $z = 0.6028$ is larger than the one in island $z = 0.8167$, and vice versa for the one paying in good 2. This is because the lower market fundamental implies that good 1 in island $z = 0.6028$ is scarcer, which implies that good 1 is more valuable. Therefore, the contract paying in good 1 in such island will have a larger price. The similar argument applies to the fact that the price of security paying in good 2 in island $z = 0.8167$ is larger than the one in island $z = 0.6028$. Notice also that the price of island $z = 0.4500$ is negative.

Using the consumption allocations in Table 9, the expected utility of an agent type 1, type 2, and type 3 are $\mathscr{U}^1 = -1.3084, \mathscr{U}^2 = -0.9112, \mathscr{U}^3 = -1.5014$, respectively. Agent 2 is significantly better off than agent 3. Note that they have the same period 0 endowment. The difference thus comes from the difference in their period 1 endowments. Agent 2 holds lots of good 1 in period 1 when is valuable while agent 3 holds lots of good 2 in period 1 when it is not so valuable. This difference leads to the large difference in receiving transfers in period 0 from trading in price island ($T_2 = 6.5085 > T_3 = 0.6178$). In addition, agent 2 benefits from the availability of the lotteries in that it allows her to be in a higher market fundamental island ($z = 0.8167$), which means a larger transfer, than otherwise ($z = 0.7133$). Notice also that a constrained agent type 2 is better off than being in the first-best world, where she will receive expected utility $\mathscr{U}^2 = -0.9600$. This implies that constraining does not necessary mean worse-off.

9 Numerical Examples with Uncertainty

This section presents an example with uncertainty where directly collateralized securities, $\hat{\psi}$, are actively traded.

Environment 3. The economy in this example is similar to the one in example 1 but there are two states, $S = 2$. There are two types of agents, $H = 2$, both of which have an identical constant relative risk aversion (CRRA) utility function (110) with $\gamma = 2$. Each type consists of $\frac{1}{2}$ fraction of the population, i.e. $\alpha^h = \frac{1}{2}$. In addition, the discount factor $\beta = 1$. The storage technology is constant and given by $R_s = 1$ for $s = 1, 2$. The endowment profile is presented in Table 11. Note that the agents are ex-ante identical.

Type of Agents	period $t = 0$		period $t = 1$			
			state $s = 1$		state $s = 2$	
	e^h_{10}	e^h_{20}	e^h_{11}	e^h_{21}	e^h_{12}	e^h_{22}
$h = 1$	2	2	3	3	1	1
$h = 2$	2	2	1	1	3	3

Table 11: Endowment profiles of the agents.

First, the symmetry of the endowments and preferences implies that period-0 allocation should be the same for all agents; that is, $c^h_{10} = c_{10}$ and $c^h_{20} = c_{20}$, for all h. Further, the indeterminacy between k^h and $\theta^h(s)$ implies that there is no loss of generality to consider the case with symmetric collateral allocation, i.e. $k^h = k$, for all h. Moreover, we can set $\hat{\theta}^1_1 = -\hat{\theta}^2_1 = -\hat{\theta} = -\hat{\theta}^1_2 = \hat{\theta}^2_2$ and $\theta^1_1 = -\theta^2_1 = -\theta = -\theta^1_2 = \theta^2_2$.

We will now solve for a competitive equilibrium with the externality. Since it is very similar to the one in example 1, the detailed derivation is omitted and presented in Appendix B.

The unique competitive equilibrium with the externality of this economy has $k^h = k \approx 0.4603$, for all h. Accordingly, the market fundamental and spot price are $z_s = 0.8129$ and $p(z_s) = 0.6608$, respectively, for all s. The price of good-2 in period 0, prices of contracts are $P^{ext}_{20} = 1.6872, \widehat{P}^{ext}_s = 1.2766, P^{ext}_s = 0.8436$, respectively ("ext" stands for externality). The equilibrium allocation is summarized in Table 12 below.

Due to the symmetry of the problem, both agents' consumption allocations are symmetric; that is, (i) both types consume the same amount of good 1 and good 2 in period $t = 0$, (ii) an agent $h = 1$ consumes more of both goods in state $s = 1$ while an agent $h = 2$ consumes more of both goods in state $s = 2$. In addition, the collateral constraints in states $s = 1$ and $s = 2$ are binding

Agent	c^h_{10}	c^h_{20}	c^h_{11}	c^h_{21}	c^h_{12}	c^h_{22}	$\mathscr{U}^h$	k^h
$h=1$	2.000	1.5397	2.7483	3.3808	1.2517	1.5397	-2.2035	0.4603
$h=2$	2.000	1.5397	1.2517	1.5397	2.7483	3.3808	-2.2035	0.4603

Table 12: Collateral equilibrium consumption allocation.

for agent $h=1$ and $h=2$, respectively. In words, an agent will be binding in a state where her endowment is large. This is because she would like to transfer a part of such a large amount of wealth foreword to $t=0$ but cannot do so because of the collateral constraints.

Agent	$\hat{\theta}^h_1$	θ^h_1	$\hat{\theta}^h_2$	θ^h_2	$\hat{\psi}^h_1$	$\hat{\psi}^h_2$	$\hat{\sigma}^h_1$	$\hat{\sigma}^h_2$	σ^h_1	σ^h_2
$h=1$	-0.2517	-0.0794	0.2517	0.0794	-0.3042	0.3042	0.0525	-0.0525	-0.0794	0.0794
$h=2$	0.2517	0.0794	-0.2517	-0.0794	0.3042	-0.3042	-0.0525	0.0525	0.0794	-0.0794

Table 13: Securities traded in competitive equilibrium with the externality.

Table 13 summarizes securities traded in equilibrium. Recall that $\hat{\theta}^h_s$ and θ^h_s include directly-collateralized and asset-backed securities. Nevertheless, using equations (136)-(139) in the proof of Lemma 3, we now can recover the positions of each securities.

$$\begin{aligned}
\hat{\psi}^1_1 &= \hat{\theta}^1_1 + p(z_1)\theta^1_1 = -0.2517 + 0.6608 \times (-0.0794) = -0.3042 = -\hat{\psi}^2_1 \\
\hat{\psi}^1_2 &= 0.3042 = -\hat{\psi}^2_2 \\
\psi^h_s &= \hat{\nu}^h_s = \nu^h_s = 0 \\
\hat{\sigma}^1_1 &= -p(z_s)\theta^1_1 = -0.6608 \times (-0.0794) = 0.0525 = -\hat{\sigma}^2_1 \\
\hat{\sigma}^1_2 &= -p(z_s)\theta^1_2 = -0.6608 \times (0.0794) = -0.0525 = \hat{\sigma}^2_2 \\
\sigma^1_1 &= \theta^1_1 = -0.0794 = -\sigma^2_1 \\
\sigma^1_2 &= \theta^1_2 = 0.0794 = -\sigma^2_2
\end{aligned}$$

Note also that as discussed earlier all asset-back securities are equivalent to spot trades.

Figure 8 illustrates an equilibrium allocation in state $s=1$, including security trades. Point $E^1_1 = (3,3)$ is the endowment of agent $h=1$ in state $s=1$. She holds $k^1 = 0.4603$ units of collateral good. Hence, before all securities are executed at $s=1$, she owns $3+0.4603 = 3.4603$ units of good 2 (she is at point F now). She also issues $\hat{\psi}^1_1 = -0.3042$ units of directly-collateralized security paying good 1 at $s=1$. After executing the security, she ends up with $3-0.3042 = 2.6958$ units of good 1 (point G). Then, she trade in spot markets at spot price $p(z) = 0.6608$. In particular, she will buy 0.0525 units of good 1 and sell 0.0794 units of good 2. She is now moving along the

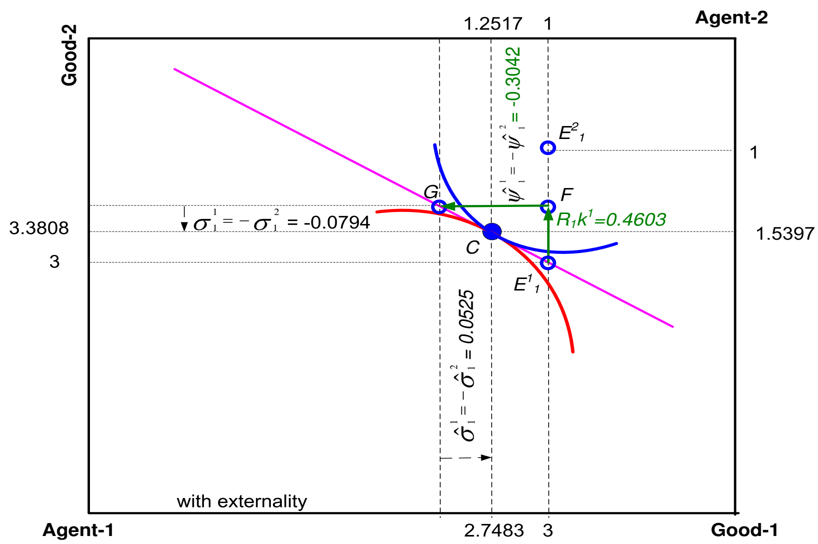


Figure 8: Period-1 equilibrium allocation with the externality. Point C is the period-1 equilibrium allocation with the externality.

budget line from point G to point C which is the equilibrium allocation. Note that we interpret asset-backed securities as spot trades.

We now turn to the competitive equilibrium with lotteries. As mentioned earlier, the equilibrium allocation reported here is a numerical solution to the Pareto program (76) corresponding to a competitive equilibrium without transfers. The equilibrium consumption allocation and traded securities are reported in Table 14 and Table 15, respectively.

Each type of agent holds the same amount of collateral good $k^h = 0.4200 < 0.4603$ which is less than the one in competitive equilibrium with the externality (compare Table 12 and Table 14). As a result, the spot price of good 2 in each state s is higher here, i.e., $p(z_s) = 0.6864 > 0.6608$. In addition, the price of good-2 in period 0 is $P_{20} = 1.5903 < P_{20}^{ext} = 1.6872$, which is again lower than the one in the competitive equilibrium with the externality.

Agent	c_{10}^h	c_{20}^h	c_{11}^h	c_{21}^h	c_{12}^h	c_{22}^h	$\mathscr{U}^h$	k^h
$h=1$	2.000	1.5800	2.7644	3.3449	1.2356	1.4951	-2.2024	0.4200
$h=2$	2.000	1.5800	1.2356	1.4951	2.7644	3.3449	-2.2024	0.4200

Table 14: Competitive equilibrium consumption allocation.

We can recover the positions of each securities using the same approach as in the competitive equilibrium with the externality. First of all, $\psi_s^h = \hat{\nu}_s^h = \nu_s^h = 0$ for all h and s. The rest is reported in Table 15. Note that agents trade less securities relative to the equilibrium with the externality (compare Table 13 and Table 15). This is because the agents save less, and hence can issue less securities. This implies that the externality generates too much borrowing here. This result is also illustrated in figure 9 below. Note that with a higher spot price $p(z)$, the budget line is flatter than the one with the externality.

Agent	$\hat{\theta}_1^h$	θ_1^h	$\hat{\theta}_2^h$	θ_2^h	$\hat{\psi}_1^h$	$\hat{\psi}_2^h$	$\hat{\sigma}_1^h$	$\hat{\sigma}_2^h$	σ_1^h	σ_2^h
$h=1$	-0.2356	-0.0751	0.2356	0.0751	-0.2872	0.2872	0.0515	-0.0515	-0.0751	0.0751
$h=2$	0.2356	0.0751	-0.2356	-0.0751	0.2872	-0.2872	-0.0515	0.0515	0.0751	-0.0751

Table 15: Securities traded in competitive equilibrium without the externality.

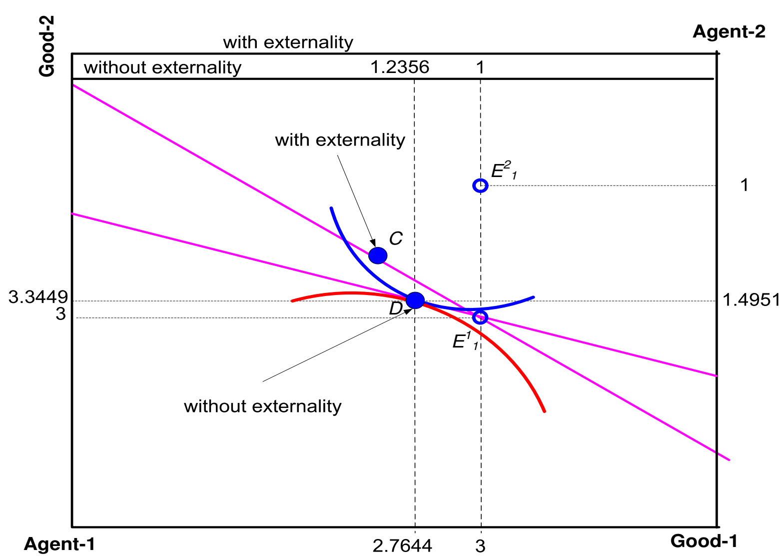


Figure 9: Period-1 equilibrium allocation without the externality. Point D is the period-1 equilibrium allocation without the externality.

10 Extensions

10.1 Contract-Specific Collateralization

This model can be readily extended to incorporate the contract-specific collateralization without pyramiding and tranching as in Geanakoplos (2003) among others. We will consider only contracts paying in good-2; that is, we will set $\hat{\theta} = 0$. A contract j is a bundle of vector of promises $(A_{sj})_s$ in units of good-2 and collateral C_j. The key difference from contracts defined in Section 2 is that a piece of collateral good can be used as collateral for at most one contract. Nevertheless, as shown in Kilenthong (2006), we can focus our attention on no-default contracts. Let Γ^* be a minimal spanning set of finite no-default contracts which spans the space of all contracts of this type. Each of spanning contract in Γ^* has 1 unit of collateral and its promise and also payoff (since there is no default) A_{sj} is either 0 or R_s units of good-2 in state s. See Kilenthong (2006) for more details.

This modification causes two main departures from the original models. First, the collateral constraints now become non-state-contingent borrowing constraints. The collateral constraint of an agent h is then given by

$$k^h + \sum_{j \in \Gamma^*} \min\left(0, \theta_j^h\right) \geq 0 \tag{131}$$

Note that negative (positive) θ_j^h means that an agent h sells (buys) θ_j^h units of contract j. Accordingly, for the lottery, condition (65) now will be replaced by (131).

Secondly, without tranching and pyramiding, the spot markets are certainly active. In other words, we cannot replicate the spot trade by ex-ante contracts anymore. Hence, the ex-post indirect utility must be modified. In state s, an agent h holding $\left(\mathbf{c}_0, k, \hat{\theta} = 0, \theta, \mathbf{z}, \Delta\right)$ will own pre-trade endowment of e_{1s}^h and $e_{2s}^h + R_s k + \sum_j \theta_j A_{sj}$ units of good-1 and good-2, respectively. Then, by being in price-island-z_s, the agent can trade in spot markets at spot price $p(z_s)$ to maximize her own utility:

$$V^h(k, \theta, \mathbf{z}) = \max_{(\hat{\tau}, \tau)} U\left(e_{1s}^h + \hat{\tau}, e_{2s}^h + R_s k + \sum_{j \in \Gamma^*} \theta_j A_{sj} + \tau\right) \tag{132}$$

subject to

$$\hat{\tau} + p(z_s)\tau \leq 0$$

We then replace the indirect utility defined in (75) by the one defined in (132). It is worthy of emphasis that the contract-specific collateralization does not affect the consistency constraint (74) as long as the markets for contracts are cleared within each price-island.

10.2 Generalized Borrowing Constraints

In addition, our model can be extended to include a broader class of borrowing constraints. In particular, the same approach can deal with a borrowing constraint that can be written in the following form

$$B\left(k,\hat{\theta},\theta,\mathbf{z}\right)\geq 0 \tag{133}$$

where $B\left(k,\hat{\theta},\theta,\mathbf{z}\right)$ is potentially a vector-valued function, and $\left(\hat{\theta},\theta\right)$ are contracts paying in units of good-1 and good-2, respectively. Those contracts do not have to follow the same structure as the ones defined in Section 2. In addition, it is not required to have complete contracts; that is, this class includes a borrowing constraint with exogenous incomplete markets as well. The key requirement here is that B is an one-to-one function of an individual's allocation.

The collateral constraints (65) are also in this class. Specifically, $B\left(k,\hat{\theta},\theta,\mathbf{z}\right)$ is a S-dimensional vector, and its s^{th} element is given by $B_s\left(k,\hat{\theta},\theta,\mathbf{z}\right)=p(z_s)R_sk+\hat{\theta}_s+p(z_s)\theta_s$. On the other hand, with the contract-specific collateralization, it is a single-valued function given by (131).

Similarly to the above discussion, the spot markets will be potentially active, and the ex-post indirect utility $V^h\left(k,\hat{\theta},\theta,\mathbf{z}\right)$ is defined as the maximum level of utility that can be achieved from trading in spot markets. Note that different borrowing constraints may lead to different pre-trade endowments. Nevertheless, the same approach applies.

11 Concluding Remarks

As we write this the world financial markets are in much turmoil. Needless to say we do not attempt here to model all possible problems and corresponding solutions. Rather we use theory to try to pinpoint one important aspect of what is going on: default, the consequent use of collateral which moves intertemporal endowments, and endogenous spot prices at the time of repayment decisions all interact to create an externality. It is in this sense that in our model markets do not function efficiently. Essentially all traders take spot prices as given when deciding what claims to buy and issue, and those that issue in the initial securities market need to back their promises with collateral, which determines subsequent spot market prices. This simultaneity happens in complete market set ups without frictions as well, but in our model the set of feasible trades for each agents depends on equilibrium spot market prices, so agents are imposing an externality on one another when they each make their own decisions.

Our solution to this problem is equally intuitive: create a market in the spot market price itself, that is allow agents to contract on what price they will unwind their contract commitments, over and above contracting on intertemporal or state contingent exchange. Of course that price is still endogenous and the contract price must equal the spot market price at which supply equals demand, taking into account exogenous endowments, saving, and contract positions and who is in the market. So when agents contract on the spot price they essentially are counting on having the requisite number and types of traders around to support that contract spot price. No agent cares specifically about the identity or name of other traders, but they do care about the composition of traders (or in our set up with homotheticity, the ratio of pre-trade endowments). So the new market mechanism needs to track who people are and what commitments they have made, in a certain well defined sense. Importantly, the formation of exchanges is determined by the market, so the government or a planner is not directing traffic. Specifically, to support the new constrained efficient allocation, there must be an ex ante market for ex post spot markets, with prices (fees and receipts) paid ex ante at the time of contracting for participation in these ex post exchanges, depending on each agents type (his/her pre trade endowment inclusive of savings to support securities). Agents are in effecting buying and selling their rights to trade in clubs, the set of agents with whom they will execute their promises and unwind positions (but we do not use the word club in a pejorative sense, as each club has a continuum of price taking agents).

Asset-backed securities are allowed in our set up and do not cause a problem. Neither are they essential in that various combinations of securities and markets are equivalent. Essentially asset back security trades mimic spot market trade, and are an essentially part of the set up if and only if spot market exchange is for some reason more limited. As a result, all arguments stated in terms of spot markets can be restated using asset back securities. In particular, we can solve the externality problem by creating segregated exchanges where agents can trade ex ante collateralized and asset backed securities indexed by the market fundamental.

We believe our methods extend to other set ups in which spot market exchange is desirable or cannot be limited a priori.

A More Proofs

Proof of Lemma 3. The first statement can be proved as follows. First, it is clear that conditions (26)-(31) imply (39)-(40). We now only need to show that (5), (10), and (11) imply (38). Summing up all collateral requirement conditions, (5), (10), and (11), with $p(z_s) = P_{2s}$ gives, for an agent h in state s,

$$\begin{aligned} p(z_s)R_s k^h &+ P_{2s}\max\left(0,\psi_s^h\right) + p(z_s)\max\left(0,\sigma_s^h\right) + \max\left(0,\hat{\psi}_s^h\right) + \max\left(0,\hat{\sigma}_s^h\right) + p(z_s)\nu_s^h + \hat{\nu}_s^h \\ &\geq -\min\left(0,\hat{\psi}_s^h\right) - p(z_s)\min\left(0,\psi_s^h\right) - \min\left(0,\hat{\sigma}_s^h\right) - p(z_s)\min\left(0,\sigma_s^h\right) \end{aligned}$$

which can be rearranged as

$$\begin{aligned} p(z_s)R_s k^h &\geq -\left[\max\left(0,\hat{\psi}_s^h\right) + \min\left(0,\hat{\psi}_s^h\right)\right] - \left[\max\left(0,\hat{\sigma}_s^h\right) + \min\left(0,\hat{\sigma}_s^h\right)\right] - p(z_s)\nu_s^h - \hat{\nu}_s^h \\ &\quad -p(z_s)\left[\max\left(0,\psi_s^h\right) + \min\left(0,\psi_s^h\right)\right] - p(z_s)\left[\max\left(0,\sigma_s^h\right) + \min\left(0,\sigma_s^h\right)\right] \\ &= -\left[\hat{\psi}_s^h + \hat{\sigma}_s^h + \hat{\nu}_s^h\right] - p(z_s)\left[\psi_s^h + \sigma_s^h + \nu_s^h\right] \end{aligned} \tag{134}$$

which is the collateral constraint for an agent h in state s. That is, the collateral constraints for an agent h can be rewritten as desired:

$$p(z_s)R_s k^h + \left[\hat{\psi}_s^h + \hat{\sigma}_s^h + \hat{\nu}_s^h\right] + p(z_s)\left[\psi_s^h + \sigma_s^h + \nu_s^h\right] \geq 0 \tag{135}$$

The second statement is proved as follows. Consider an allocation $\left(k^h, \hat{\theta}_s^h, \theta_s^h\right)_h$ that satisfies (38) and (39)-(40). We will now choose a corresponding allocation $\left(k^h, \hat{\psi}_s^h, \psi_s^h, \hat{\sigma}_s^h, \sigma_s^h, \hat{\nu}_s^h, \nu_s^h\right)_h$ that satisfies $\hat{\theta}_s^h = \hat{\psi}_s^h + \hat{\sigma}_s^h + \hat{\nu}_s^h$, $\theta_s^h = \psi_s^h + \sigma_s^h + \nu_s^h$, the collateral requirement conditions (5), (10), (11), and the market-clearing conditions (26)-(31). Consider the following candidate allocation:

$$\hat{\psi}_s^h = \hat{\theta}_s^h + p(z_s)\theta_s^h \tag{136}$$

$$\psi_s^h = \hat{\nu}_s^h = \nu_s^h = 0 \tag{137}$$

$$\hat{\sigma}_s^h = \hat{\theta}_s^h - \hat{\psi}_s^h = -p(z_s)\theta_s^h \tag{138}$$

$$\sigma_s^h = \theta_s^h \tag{139}$$

Note that agents hold no $\psi_s^h, \hat{\nu}_s^h, \nu_s^h$ securities (see (137)). In particular, they will borrow or lend through directly collateralized contract paying in good-1 $\hat{\psi}_s^h$ only.

First of all, it is clear that $\hat{\theta}_s^h = \hat{\psi}_s^h + \hat{\sigma}_s^h + \hat{\nu}_s^h$ and $\theta_s^h = \psi_s^h + \sigma_s^h + \nu_s^h$. By construction, from (137),

$$\sum_h \alpha^h \psi_s^h = \sum_h \alpha^h \hat{\nu}_s^h = \sum_h \alpha^h \nu_s^h = 0 \tag{140}$$

which are (26), (30), and (31), respectively. (40) and (139) imply that

$$\sum_h \alpha^h \sigma_s^h \;=\; \sum_h \alpha^h \theta_s^h = 0 \tag{141}$$

which is (29). Similarly, (40) and (138) imply that

$$\sum_h \alpha^h \hat{\sigma}_s^h \;=\; -p(z_s) \sum_h \alpha^h \theta_s^h = 0 \tag{142}$$

which is (28). Using (39), (40), and (136), we can also show that

$$\sum_h \alpha^h \hat{\psi}_s^h \;=\; \sum_h \alpha^h \hat{\theta}_s^h + p(z_s) \sum_h \alpha^h \theta_s^h = 0 \tag{143}$$

which is (26).

We now would like to show that collateral requirement conditions (5), (10), (11) also hold. First, we will show that (10) and (11) hold. There are two cases to consider; (i) $\theta_s^h > 0$, (ii) $\theta_s^h < 0$. Case I: Suppose that $\theta_s^h > 0$. Using (139), this implies that $\sigma_s^h > 0$, which in turn leads to $\min\left(0, \sigma_s^h\right) = 0$. On the other hand, it is true that

$$\max\left(0, \hat{\psi}_s^h\right) + \max\left(0, \hat{\sigma}_s^h\right) = \max\left(0, \hat{\psi}_s^h\right) + \max\left(0, \hat{\sigma}_s^h\right) + \hat{\nu}_s^h \geq 0 \tag{144}$$

where the first equality follows from (137). Since $\min\left(0, \sigma_s^h\right) = 0$, we have

$$\max\left(0, \hat{\psi}_s^h\right) + \max\left(0, \hat{\sigma}_s^h\right) = \max\left(0, \hat{\psi}_s^h\right) + \max\left(0, \hat{\sigma}_s^h\right) + \hat{\nu}_s^h \geq -p(z_s) \min\left(0, \sigma_s^h\right) \tag{145}$$

which is (11). On the other hand, (138) implies that $\hat{\sigma}_s^h < 0$ when $\theta_s^h > 0$. As a result, $\min\left(0, \hat{\sigma}_s^h\right) = \hat{\sigma}_s^h$. Using (137), (138), (139), we then can show that

$$\begin{aligned} p(z_s) \max\left(0, \psi_s^h\right) + p(z_s) \max\left(0, \sigma_s^h\right) + p(z_s)\nu_s^h + \min\left(0, \hat{\sigma}_s^h\right) \\ = 0 + p(z_s)\sigma_s^h + 0 + \hat{\sigma}_s^h = p(z_s)\theta_s^h - p(z_s)\theta_s^h = 0 \end{aligned} \tag{146}$$

where the next-to-last equality follows from (138) and (139). This clearly shows that (10) holds.

Case II: Suppose that $\theta_s^h < 0$. (138) and (139) imply that $\max\left(0, \hat{\sigma}_s^h\right) = \hat{\sigma}_s^h = -p(z_s)\theta_s^h$ and $\min\left(0, \sigma_s^h\right) = \sigma_s^h = \theta_s^h$, respectively. We then can write

$$\max\left(0, \hat{\psi}_s^h\right) + \max\left(0, \hat{\sigma}_s^h\right) + \hat{\nu}_s^h = \max\left(0, \hat{\psi}_s^h\right) - p(z_s)\theta_s^h \geq -p(z_s)\theta_s^h = -p(z_s) \min\left(0, \sigma_s^h\right) \tag{147}$$

which is exactly (11). Note that the first equality follows from (137), the second inequality follows from the fact that $\max\left(0, \hat{\psi}_s^h\right) \geq 0$. Similarly, using , we can show that $\max\left(0, \sigma_s^h\right) = \min\left(0, \hat{\sigma}_s^h\right) = 0$. This implies that

$$\begin{aligned} p(z_s) \max\left(0, \psi_s^h\right) + p(z_s) \max\left(0, \sigma_s^h\right) + p(z_s)\nu_s^h + \min\left(0, \hat{\sigma}_s^h\right) \\ = 0 + 0 + 0 + 0 = 0 \end{aligned} \tag{148}$$

which is exactly (10).

Similarly, we can now show that (5) also holds. There are two cases to be considered as well.

Case I: suppose that $\hat{\theta}_s^h + p(z_s)\theta_s^h < 0$. (136) implies that $\hat{\psi}_s^h < 0$, which in turn implies that $\min\left(0, \hat{\psi}_s^h\right) = \hat{\psi}_s^h = \hat{\theta}_s^h + p(z_s)\theta_s^h$. Using (137), we now can show that

$$p(z_s)R_s k^h + \min\left(0, \hat{\psi}_s^h\right) + p(z_s)\min\left(0, \psi_s^h\right) = p(z_s)R_s k^h + \hat{\theta}_s^h + p(z_s)\theta_s^h + 0 \geq 0 \tag{149}$$

where the last inequality follows (38). This implies that (5) holds.

Case II: we can use a similar argument to show that (5) holds when $\hat{\theta}_s^h + p(z_s)\theta_s^h = \hat{\psi}_s^h > 0$. In summary, we have show that all collateral requirement conditions hold. *Q.E.D.*

Proof of Lemma 4. Let $\left(\mathbf{c}_0^h, k^h, \hat{\theta}^h, \theta^h, \hat{\tau}^h, \tau^h\right)_h$ be an attainable allocation. We will show that we can find an equivalent allocation with no spot trade, i.e., $\hat{\tau}_s'^h = \tau_s'^h = 0$. Consider the following candidate allocation (with $'$)

$$\begin{aligned} \mathbf{c}_0'^h &= \mathbf{c}_0^h, \ \forall h && (150) \\ \hat{\theta}_s'^h &= \hat{\theta}_s^h + \hat{\tau}_s^h, \ \forall s, h && (151) \\ \theta_s'^h &= \theta_s^h + \tau_s^h, \ \forall s, h && (152) \\ k'^h &= k^h && (153) \end{aligned}$$

Note that agents here acquire or issue securities on good 1 and good 2 in state s rather than waiting for trade in spot markets.

We will first show that it is attainable; that is, it satisfies the feasibility conditions (24),(25), (39), (40) and the collateral constraints (38).

Since period-0 consumption and collateral are unaltered, it is not difficult to show that the candidate allocation $\left(\mathbf{c}_0'^h, k'^h, \hat{\theta}'^h, \theta'^h, \hat{\tau}'^h, \tau'^h\right)_h$ satisfies conditions (24)-(25), that is using (150) and (153). We also can show from (151) that

$$\begin{aligned} \sum_h \alpha^h \hat{\theta}_s'^h &= \sum_h \alpha^h \left(\hat{\theta}_s^h + \hat{\tau}_s^h\right) \\ &= \sum_h \alpha^h \hat{\theta}_s^h + \sum_h \alpha^h \hat{\tau}_s^h = 0 && (154) \end{aligned}$$

where the last equations follow from the market-clearing conditions (35) and (39) , respectively. This shows that the market-clearing condition for $\hat{\theta}_s'^h$ (39) holds. Similarly, using (37), (40), (152), the market-clearing condition for $\theta_s'^h$ (40) holds;

$$\begin{aligned} \sum_h \alpha^h \theta_s'^h &= \sum_h \alpha^h \left(\theta_s^h + \tau_s^h\right) \\ &= \sum_h \alpha^h \theta_s^h + \sum_h \alpha^h \tau_s^h = 0 && (155) \end{aligned}$$

We now only need to show that the collateral constraints (38) hold. Consider the collateral constraint for an agent h in state s:

$$\begin{aligned} p(z_s)R_s k'^h + \hat{\theta}'^h_s + p(z_s)\theta'^h_s &= p(z_s)\left(R_s k^h + \theta^h_s + \tau^h_s\right) + \left(\hat{\theta}^h_s + \hat{\tau}^h_s\right) \\ &= p(z_s)R_s k^h + \hat{\theta}^h_s + p(z_s)\theta^h_s + \left(\hat{\tau}^h_s + p(z_s)\tau^h_s\right) \\ &= p(z_s)R_s k^h + \hat{\theta}^h_s + p(z_s)\theta^h_s \geq 0, \ \forall s \end{aligned} \tag{156}$$

where the first equality and the fourth equalities follow from (151)-(152), and (35), respectively. The last inequality (156)is the collateral constraint in state s of the original allocation. Thus the alternative allocation satisfies the collateral constraints (38) as well. In summary, the alternative allocation is attainable.

We will now show that the alternative allocation leads to the same consumption allocation and market fundamental. By construction, period-0 consumption allocations are identical. Using (151) and (152), we can show that the corresponding consumption allocations in state s for the agent h are also identical:

$$c'^h_{1s} = e^h_{1s} + \hat{\theta}'^h_s = e^h_{1s} + \hat{\theta}^h_s + \hat{\tau}^h_s = c^h_{1s} \tag{157}$$

$$c'^h_{2s} = e^h_{2s} + R_s k'^h + \theta'^h_s = e^h_{2s} + R_s k^h + \theta^h_s + \tau^h_s = c^h_{2s} \tag{158}$$

We will now show that the alternative allocation also lead to the same market fundamental in each state s. The market-fundamental in state s under the $'$ allocation $\left(\mathbf{c}'^h_0, k'^h, \hat{\theta}'^h, \theta'^h, \hat{\tau}'^h, \tau'^h\right)_h$ is given by

$$z'_s = z\left(\tilde{e}'_s\right) = \frac{\sum_h \alpha^h \left(e^h_{1s} + \hat{\theta}'^h_s\right)}{\sum_h \alpha^h \left(e^h_{2s} + R_s k'^h + \theta'^h\right)} = \frac{\sum_h \alpha^h e^h_{1s}}{\sum_h \alpha^h \left(e^h_{2s} + R_s k\right)} = z_s \tag{159}$$

where the last equality follows from (153), and the market-clearing conditions for $\hat{\theta}'^h$ and θ'^h, respectively. This shows that the resulting market fundamentals are the same. *Q.E.D.*

Proof of Lemma 5. Suppose $\left(\mathbf{c}^h_0, k^h, \hat{\psi}^h_s, \psi^h_s, \hat{\sigma}^h_s, \sigma^h_s, \hat{\nu}^h_s, \nu^h_s, \hat{\tau}^h, \tau^h\right)_h$ is attainable. Consider the following alternative allocation (with $'$) $\left(\mathbf{c}'^h_0, k'^h, \hat{\psi}'^h_s, \psi'^h_s, \hat{\sigma}'^h_s, \sigma'^h_s, \hat{\nu}'^h_s, \nu'^h_s, \hat{\tau}'^h, \tau'^h\right)_h$ such that

$$\mathbf{c}'^h_0 = \mathbf{c}^h_0, \quad k'^h = k^h, \quad \hat{\sigma}'^h_s = \sigma'^h_s = \hat{\nu}'^h_s = \nu'^h_s = 0, \psi^h_s = 0, \ \forall h, s \tag{160}$$

$$\hat{\psi}'^h_s = \left(\hat{\psi}^h_s + \hat{\sigma}^h_s + \hat{\nu}^h_s\right) + p(z_s)\left(\psi^h_s + \sigma^h_s + \nu^h_s\right) \tag{161}$$

$$\hat{\tau}'^h_s = -p(z_s)\left(\psi^h_s + \sigma^h_s + \nu^h_s\right) + \hat{\tau}^h_s \tag{162}$$

$$\tau'^h_s = \left(\psi^h_s + \sigma^h_s + \nu^h_s\right) + \tau^h_s \tag{163}$$

Note that at the alternative allocation, agents will do in spot markets what they might have done in asset-backed security markets. In addition, with active spot markets, there is no need to trade in collateral-backed securities paying in good-2 (trade in the ones paying in numeraire good only).

The rest of the proof is similar to the one of Lemma 4. It is by definition that period-0 consumption allocations are identical. We will now show that the corresponding consumption allocations for the agent h in state s are also identical:

$$\begin{aligned} c_{1s}'^{h} &= e_{1s}^{h} + \hat{\psi}_{s}'^{h} + \hat{\sigma}_{s}'^{h} + \hat{\nu}_{s}'^{h} + \hat{\tau}_{s}'^{h} \\ &= e_{1s}^{h} + \left[\left(\hat{\psi}_{s}^{h} + \hat{\sigma}_{s}^{h} + \hat{\nu}_{s}^{h}\right) + p(z_s)\left(\psi_{s}^{h} + \sigma_{s}^{h} + \nu_{s}^{h}\right)\right] + 0 + 0 + \left[-p(z_s)\left(\psi_{s}^{h} + \sigma_{s}^{h} + \nu_{s}^{h}\right) + \hat{\tau}_{s}^{h}\right] \\ &= e_{1s}^{h} + \hat{\psi}_{s}^{h} + \hat{\sigma}_{s}^{h} + \hat{\nu}_{s}^{h} + \hat{\tau}_{s}^{h} = c_{1s}^{h} \end{aligned}$$

where the first equation follows from (32), the second one results from substituting (160), (161), (162), and the last one follows from (32), and

$$\begin{aligned} c_{2s}'^{h} &= e_{2s}^{h} + \psi_{s}'^{h} + \sigma_{s}'^{h} + \nu_{s}'^{h} + \tau_{s}'^{h} \\ &= e_{2s}^{h} + 0 + 0 + 0 + \left(\psi_{s}^{h} + \sigma_{s}^{h} + \nu_{s}^{h} + \tau_{s}^{h}\right) = c_{2s}^{h} \end{aligned}$$

where the first equation follows from (33), the second one results from substituting (160), (163), and the last one follows from (33).

It is not difficult to show that market-clearing conditions (24)-(31) and (35) hold. For brevity, they are omitted here. Similarly to the proof of Lemma 4, the alternative allocation also lead to the same market fundamental in each state s.

We now only need to show that the collateral constraints hold. Consider the collateral constraint for an agent h in state s:

$$\begin{aligned} p(z_s)R_s k'^{h} &+ \hat{\psi}_{s}'^{h} + \hat{\sigma}_{s}'^{h} + \hat{\nu}_{s}'^{h} + p(z_s)\left(\psi_{s}'^{h} + \sigma_{s}'^{h} + \nu_{s}'^{h}\right) \\ &= p(z_s)R_s k^{h} + \left[\left(\hat{\psi}_{s}^{h} + \hat{\sigma}_{s}^{h} + \hat{\nu}_{s}^{h}\right) + p(z_s)\left(\psi_{s}^{h} + \sigma_{s}^{h} + \nu_{s}^{h}\right)\right] + 0 + 0 + p(z_s)\left(0 + 0 + 0\right) \\ &= p(z_s)R_s k^{h} + \left(\hat{\psi}_{s}^{h} + \hat{\sigma}_{s}^{h} + \hat{\nu}_{s}^{h}\right) + p(z_s)\left(\psi_{s}^{h} + \sigma_{s}^{h} + \nu_{s}^{h}\right) \geq 0 \end{aligned}$$

where the first equation follows from (), and the last inequality is (160)-(163) the collateral constraint of the original allocation. This clearly shows that the alternative allocation satisfies the collateral constraints (38) as well. *Q.E.D.*

Proof of Lemma 6. For simplicity, both state and agent indices will be kept implicit here. Without loss of generality, let $R_s = 1$ for all s. Consider two different allocations $\left(k, \hat{\theta}, \theta\right)$ and $\left(k', \hat{\theta}', \theta'\right)$

with two different market fundamentals z, z', respectively. The collateral constraints (38) for an agent h in state s with these two allocations be binding:

$$
\begin{aligned}
p(z)k + \hat{\theta} + p(z)\theta &= 0 \Longrightarrow \hat{\theta} = -p(z)\left(k+\theta\right) && (164)\\
p(z')k' + \hat{\theta}' + p(z')\theta' &= 0 \Longrightarrow \hat{\theta}' = -p(z')\left(k'+\theta'\right) && (165)
\end{aligned}
$$

Since we are looking for a counter example, we can pick these two allocations to satisfy

$$
\hat{\theta} = \hat{\theta}' < 0 \Longrightarrow p(z)\left(k+\theta\right) = p(z')\left(k'+\theta'\right) > 0 \qquad (166)
$$

The positivity of the prices implies that $k+\theta > 0$ and $k'+\theta' > 0$.

Now consider a convex combination allocation: $k^{\lambda} = \lambda k + (1-\lambda)\,k'$, $\hat{\theta}^{\lambda} = \lambda\hat{\theta} + (1-\lambda)\,\hat{\theta}'$, $\theta^{\lambda} = \lambda\theta + (1-\lambda)\,\theta'$, $\mathbf{c}^{\lambda} = \lambda\mathbf{c} + (1-\lambda)\,\mathbf{c}'$, and $z^{\lambda} = z(\mathbf{c}^{\lambda})$, where $0 < \lambda < 1$. Using equations (164)-(165), we can write

$$
\begin{aligned}
p(z^{\lambda})k^{\lambda} + \hat{\theta}^{\lambda} + p(z^{\lambda})\theta^{\lambda} &= p(z^{\lambda})\left[\lambda k + (1-\lambda)\,k'\right] + \left[\lambda\hat{\theta} + (1-\lambda)\,\hat{\theta}'\right] + p(z^{\lambda})\left[\lambda\theta + (1-\lambda)\,\theta'\right]\\
&= \lambda\left(k+\theta\right)\Big(p(z^{\lambda}) - p(z)\Big) + (1-\lambda)\left(k'+\theta'\right)\Big(p(z^{\lambda}) - p(z')\Big)\\
&= \left(\frac{k+\theta}{p(z')}\right)\left[\lambda p(z^{\lambda})\left(p(z') - p(z)\right) + p(z)\Big(p(z^{\lambda}) - p(z')\Big)\right] && (167)
\end{aligned}
$$

There is no loss of generality to assume that $p(z) < p(z^{\lambda}) < p(z')$. Then, pick λ that is smaller than λ^*:

$$
\lambda^* = \left(\frac{p(z') - p(z^{\lambda})}{p(z') - p(z)}\right)\left(\frac{p(z)}{p(z^{\lambda})}\right) \qquad (168)
$$

Using the condition that $p(z) < p(z^{\lambda}) < p(z')$, we can show that $0 < \lambda^* < 1$. This condition implies that we can pick $0 < \lambda < \lambda^* < 1$ such that

$$
\lambda p(z^{\lambda})\left(p(z') - p(z)\right) + p(z)\Big(p(z^{\lambda}) - p(z')\Big) < 0 \qquad (169)
$$

Using $k+\theta > 0$, the RHS of (167) will be negative. This clearly violates the collateral constraint (38). Therefore, the attainable set is non-convex. *Q.E.D.*

Proof of Lemma 7. The proof is based on first-order conditions for Pareto program 1 and the ones for a collateral equilibrium. Note that the resource constraints in program 1 and the market-clearing constraints in equilibrium are clearly equivalent. In addition, the collateral constraints are the same in both problems as well. Hence, we only need to match all first-order conditions from both problems.

Optimal Conditions for Program 1

The Lagrangian of the Pareto program 1 is given by

$$
\begin{aligned}
\mathscr{L} &= \sum_h \mu_{\bar{u}}^h \left\{ U^h\left(c_{10}^h, c_{20}^h\right) + \beta \sum_s \pi_s U^h\left(e_{1s}^h + \hat{\theta}_{1s}^h, e_{2s}^h + R_s k^h + \theta_s^h\right) \right\} \\
&+ \mu_{10} \sum_h \alpha^h \left[e_{10}^h - c_{10}^h \right] + \mu_{20} \sum_h \alpha^h \left[e_{20}^h - c_{20}^h - k^h \right] - \sum_s \mu_{\hat{\theta}s} \sum_h \alpha^h \hat{\theta}_s^h \\
&- \sum_s \mu_{\theta s} \sum_h \alpha^h \theta_s^h + \sum_h \sum_s \mu_{cc-s}^h \left[p(z_s) R_s k^h + \hat{\theta}_s^h + p(z_s) \theta_s^h \right] + \sum_h \mu_{k0}^h k^h
\end{aligned}
$$

where μ-s are the Lagrange multipliers for the participation, feasibility, non-negativity and collateral constraints.

The first-order conditions with respect to $c_{10}^h, c_{20}^h, k^h, \hat{\theta}_s^h, \theta_s^h$, respectively, are given by

$$\mu_{\bar{u}}^h U_{10}^h = \alpha^h \mu_{10} \tag{170}$$

$$\mu_{\bar{u}}^h U_{20}^h = \alpha^h \mu_{20} \tag{171}$$

$$
\begin{aligned}
\alpha^h \mu_{20} &= \mu_{\bar{u}}^h \sum_s \pi_s \beta U_{2s}^h R_s + \sum_s \mu_{cc-s}^h p(z_s) R_s + \mu_{k0}^h \\
&+ \sum_s \frac{\partial p(z_s)}{\partial k^h} \sum_{\tilde{h}} \mu_{cc-s}^{\tilde{h}} \left(R_s k^{\tilde{h}} + \theta_s^{\tilde{h}} \right)
\end{aligned} \tag{172}
$$

$$\alpha^h \mu_{\hat{\theta}s} = \mu_{\bar{u}}^h \pi_s \beta U_{1s}^h + \mu_{cc-s}^h \tag{173}$$

$$\alpha^h \mu_{\theta s} = \mu_{\bar{u}}^h \pi_s \beta U_{2s}^h + \mu_{cc-s}^h p(z_s) \tag{174}$$

The complementarity slackness conditions for the collateral constraints are given by

$$\mu_{cc-s}^h \left[p(z_s) R_s k^h + \hat{\theta}_s^h + p(z_s) \theta_s^h \right] = 0 \Longrightarrow \mu_{cc-s}^h \left[R_s k^h + \theta_s^h \right] = -\frac{\mu_{cc-s}^h \hat{\theta}_s^h}{p(z_s)}, \ \forall s, h \tag{175}$$

Combining conditions (170)-(175) gives

$$
\begin{aligned}
\frac{\mu_{20}}{\mu_{10}} &= \sum_s \pi_s \frac{\beta U_{2s}^h}{U_{10}^h} R_s + \sum_s \frac{\mu_{cc-s}^h}{\mu_{\bar{u}}^h U_{10}^h} p(z_s) R_s + \frac{\mu_{k0}^h}{\mu_{\bar{u}}^h U_{10}^h} \\
&- \sum_s \frac{1}{\mu_{\bar{u}}^h U_{10}^h} \frac{\partial \ln p(z_s)}{\partial k^h} \sum_{\tilde{h}} \mu_{cc-s}^{\tilde{h}} \hat{\theta}_s^{\tilde{h}} = \frac{U_{20}^h}{U_{10}^h}
\end{aligned} \tag{176}
$$

$$\frac{\mu_{\hat{\theta}s}}{\mu_{10}} = \pi_s \frac{\beta U_{1s}^h}{U_{10}^h} + \frac{\mu_{cc-s}^h}{\mu_{\bar{u}}^h U_{10}^h} \tag{177}$$

$$\frac{\mu_{\theta s}}{\mu_{10}} = \pi_s \frac{\beta U_{2s}^h}{U_{10}^h} + \frac{\mu_{cc-s}^h}{\mu_{\bar{u}}^h U_{10}^h} p(z_s) \tag{178}$$

Note that (176) is exactly the same as (49).

Optimal Conditions for a Collateral Equilibrium

The Lagrangian of a consumer's problem is given by

$$\begin{aligned}\mathscr{L} &= U^h\big(c_{10}^h, c_{20}^h\big) + \beta\sum_s \pi_s U^h\big(e_{1s}^h + \hat{\theta}_s^h, e_{2s}^h + R_s k^h + \theta_s^h\big) + \gamma_{k0}^h k^h \\ &- \gamma_0^h\left[c_{10}^h - e_{10}^h + P_{20}\left(c_{20}^h + k^h - e_{20}^h\right) + \widehat{P}_a\cdot\hat{\theta}^h + P_a\cdot\theta^h\right] + \sum_s \gamma_{cc-s}^h\left[p(z_s)R_s k^h + \hat{\theta}_s^h p(z_s)\theta_s^h\right]\end{aligned}$$

where γ-s are the Lagrange multipliers for the budget, and collateral constraints. The first-order conditions with respect to $c_{10}^h, c_{20}^h, k^h, \hat{\theta}_s^h, \theta_s^h$, respectively, are given by

$$U_{10}^h = \gamma_0^h \tag{179}$$

$$U_{20}^h = \gamma_0^h P_{20} \tag{180}$$

$$\gamma_0^h P_{20} = \sum_s \pi_s\beta U_{2s}^h R_s + \sum_s \gamma_{cc-s}^h p(z_s)R_s + \gamma_{k0}^h \tag{181}$$

$$\gamma_0^h\widehat{P}_{as} = \beta\pi_s U_{1s}^h + \gamma_{cc-s}^h \tag{182}$$

$$\gamma_0^h P_{as} = \beta\pi_s U_{2s}^h + \gamma_{cc-s}^h p(z_s) \tag{183}$$

The complementarity slackness conditions for the collateral constraints are given by

$$\gamma_{cc-s}^h\left[p(z_s)R_s k^h + \hat{\theta}_s^h + p(z_s)\theta_s^h\right] = 0,\ \forall s, h \tag{184}$$

Combining conditions (179)-(183) gives

$$P_{20} = \sum_s \pi_s\frac{\beta U_{2s}^h}{U_{10}^h}R_s + \sum_s \frac{\gamma_{cc-s}^h}{U_{10}^h}p(z_s)R_s + \frac{\gamma_{k0}^h}{U_{10}^h} = \frac{U_{20}^h}{U_{10}^h} \tag{185}$$

$$\widehat{P}_{as} = \pi_s\frac{\beta U_{1s}^h}{U_{10}^h} + \frac{\gamma_{cc-s}^h}{U_{10}^h} \tag{186}$$

$$P_{as} = \pi_s\frac{\beta U_{2s}^h}{U_{10}^h} + \frac{\gamma_{cc-s}^h}{U_{10}^h}p(z_s) \tag{187}$$

($\Longleftarrow$) Suppose that $\gamma_{cc-s}^h = \mu_{cc-s}^h = 0$ for all h and all s. We will show that any collateral equilibrium allocation will also solve the Pareto program 1. First, the collateral equilibrium allocation must satisfy (184)-(187). On the other hand, with no binding collateral constraints, the attainable set is convex. As a result, (175)-(178) are necessary and sufficient. In addition, it is not difficult to show that the collateral equilibrium allocation satisfies (175)-(178) with $\frac{\mu_{20}}{\mu_{10}} = P_{20}$, $\frac{\mu_{\hat{\theta}s}}{\mu_{10}} = \widehat{P}_{as}$, $\frac{\mu_{\theta s}}{\mu_{10}} = P_{as}$, $\gamma_{k0}^h = \frac{\mu_{k0}^h}{\mu_{\bar{u}}^h}$, and $\gamma_{cc-s}^h = \frac{\mu_{cc-s}^h}{\mu_{\bar{u}}^h} = 0$. Therefore, any collateral equilibrium allocation is constrained optimal if $\gamma_{cc-s}^h = \mu_{cc-s}^h = 0$ for all h and all s.

($\Longrightarrow$) Suppose that a collateral equilibrium allocation is constrained optimal, i.e. solves the Pareto program 1. Hence, it must satisfy (175)-(178). Using the same matching, $\frac{\mu_{20}}{\mu_{10}} = P_{20}$, $\frac{\mu_{\hat{\theta}s}}{\mu_{10}} =$

$\widehat{P}_{as}, \frac{\mu_{\theta s}}{\mu_{10}} = P_{as}, \gamma^h_{k0} = \frac{\mu^h_{k0}}{\mu^h_{\bar{u}}}, \gamma^h_{cc-s} = \frac{\mu^h_{cc-s}}{\mu^h_{\bar{u}}}$, this will be true only if the last terms in (176)-(178) are zero. Suppose this can be true even if $\mu^h_{cc-s} \neq 0$ for some h. For expositional result, we will focus on the last term in (176). Suppose that this term is equal to zero:

$$\sum_s \frac{1}{\mu^h_{\bar{u}} U^h_{10}} \frac{\partial \ln p(z_s)}{\partial k^h} \sum_{\tilde{h}} \mu^{\tilde{h}}_{cc-s} \hat{\theta}^{\tilde{h}}_s = \frac{\alpha^h}{\mu^h_{\bar{u}} U^h_{10}} \sum_s \frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \left(\sum_{\tilde{h}} \mu^{\tilde{h}}_{cc-s} \hat{\theta}^{\tilde{h}}_s \right) = 0 \tag{188}$$

This must be true for all h and $\tilde{h}$. We will fist argue that $\sum_{\tilde{h}} \mu^{\tilde{h}}_{cc-s} \hat{\theta}^{\tilde{h}}_s$ has the same sign for every state s. Using (177), we have

$$\begin{aligned} \sum_{\tilde{h}} \mu^{\tilde{h}}_{cc-s} \hat{\theta}^{\tilde{h}}_s &= \sum_{\tilde{h}} \mu_{\hat{\theta}_s} \alpha^{\tilde{h}} \hat{\theta}^{\tilde{h}}_s - \sum_{\tilde{h}} \pi_s \beta \mu^{\tilde{h}}_{\bar{u}} U^{\tilde{h}}_{1s} \hat{\theta}^{\tilde{h}}_s \\ &= \mu_{\hat{\theta}_s} \sum_{\tilde{h}} \alpha^{\tilde{h}} \hat{\theta}^{\tilde{h}}_s - \pi_s \beta \mu_{10} \sum_{\tilde{h}} \frac{U^{\tilde{h}}_{1s}}{U^{\tilde{h}}_{10}} \alpha^{\tilde{h}} \hat{\theta}^{\tilde{h}}_s \\ &= -\pi_s \beta \mu_{10} \sum_{\tilde{h}} \frac{U^{\tilde{h}}_{1s}}{U^{\tilde{h}}_{10}} \alpha^{\tilde{h}} \hat{\theta}^{\tilde{h}}_s \end{aligned} \tag{189}$$

where the second equality follows from (170), and the last one follows from the market-clearing condition of $\hat{\theta}^h_s$ (39). The optimality requires that an agent with a larger IMRS $\frac{U^{\tilde{h}}_{1s}}{U^{\tilde{h}}_{10}}$ will hold positive $\hat{\theta}^{\tilde{h}}_s$ and vice versa. This implies that the positive term of $\alpha^{\tilde{h}} \hat{\theta}^{\tilde{h}}_s$ will be weighted more than the negative one. Combining with the market-clearing condition $\sum_{\tilde{h}} \alpha^{\tilde{h}} \hat{\theta}^{\tilde{h}}_s = 0$, we can conclude that

$$-\pi_s \beta \mu_{10} \sum_{\tilde{h}} \frac{U^{\tilde{h}}_{1s}}{U^{\tilde{h}}_{10}} \alpha^{\tilde{h}} \hat{\theta}^{\tilde{h}}_s \leq 0, \ \forall s \tag{190}$$

With homothetic preferences, $\frac{1}{\mu^h_{\bar{u}} U^h_{10}} \frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} < 0$. This result and (190) imply that

$$\frac{1}{\mu^h_{\bar{u}} U^h_{10}} \frac{\partial \ln p(z_s)}{\partial k^h} \sum_{\tilde{h}} \mu^{\tilde{h}}_{cc-s} \hat{\theta}^{\tilde{h}}_s = -\frac{\pi_s \beta \mu_{10} \alpha^h}{\mu^h_{\bar{u}} U^h_{10}} \frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \sum_{\tilde{h}} \frac{U^{\tilde{h}}_{1s}}{U^{\tilde{h}}_{10}} \alpha^{\tilde{h}} \hat{\theta}^{\tilde{h}}_s \geq 0, \ \forall s \tag{191}$$

As a result, (188) will hold or the sum of (191) over s will be zero only if

$$\sum_{\tilde{h}} \mu^{\tilde{h}}_{cc-s} \hat{\theta}^{\tilde{h}}_s = -\pi_s \beta \mu_{10} \sum_{\tilde{h}} \frac{U^{\tilde{h}}_{1s}}{U^{\tilde{h}}_{10}} \alpha^{\tilde{h}} \hat{\theta}^{\tilde{h}}_s = 0 \tag{192}$$

This condition implies that

$$\frac{U^{\tilde{h}}_{1s}}{U^{\tilde{h}}_{10}} = \frac{U^h_{1s}}{U^h_{10}}, \ \forall h, \tilde{h}, s \tag{193}$$

Using the fact that $\frac{U^h_{2s}}{U^h_{1s}} = p(z_s)$ for all h, we can also show that

$$\frac{U^{\tilde{h}}_{1s}}{U^{\tilde{h}}_{1\hat{s}}} = \frac{U^h_{1s}}{U^h_{1\hat{s}}}, \ \forall h, \tilde{h} \tag{194}$$

In words, the marginal rate of substitutions across times and states are equalized across agents. Under the assumption 1, these equalities are necessary and sufficient conditions for first-best optimality, which in turn implies that all collateral constraints are not binding, i.e. $\gamma^h_{cc-s} = \mu^h_{cc-s} = 0$ for all h and all s. Hence, we can conclude that a collateral equilibrium is constrained optimal only if all collateral constraints are not binding. *Q.E.D.*

Proof of Theorem 2. Let $(\mathbf{x}, \mathbf{y})$, and $\left(P_{20}, P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)\right)$ be a competitive equilibrium. Suppose the competitive equilibrium allocation is not Pareto optimal, i.e. there is an attainable allocation $\tilde{\mathbf{x}} \in X$ such that $\mathscr{U}\left(\tilde{\mathbf{x}}^h\right) \geq \mathscr{U}\left(\mathbf{x}^h\right)$ for all h and $\mathscr{U}\left(\tilde{\mathbf{x}}^{\tilde{h}}\right) > \mathscr{U}\left(\mathbf{x}^{\tilde{h}}\right)$ for some $\tilde{h}$. With local nonsatiation of preferences, we have

$$\sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)} P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \leq \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)} P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \tilde{x}^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$$

for all h, and

$$\sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)} P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) x^{\tilde{h}}\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) < \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)} P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \tilde{x}^{\tilde{h}}\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$$

for some $\tilde{h}$. Summing over all agents with weights $\left(\alpha^h\right)_h$, we have

$$\begin{aligned}\sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)} P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \sum_h \alpha^h x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) < \\ \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)} P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \sum_h \alpha^h \tilde{x}^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)\end{aligned} \tag{195}$$

The optimal condition (94) for the market-maker's profit maximization problem implies that in equilibrium, for any bundle $\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$,

$$P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \leq c_{10} + P_{20}c_{20} + P_{20}k + \widehat{P}_a(\mathbf{z}) \cdot \hat{\theta} + P_a(\mathbf{z}) \cdot \theta + P_\Delta(\mathbf{z}) \cdot \Delta \tag{196}$$

This condition will hold with equality if $y\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) > 0$. As a result, it can be rewritten as

$$P(b)y(b) = \left(c_{10} + P_{20}c_{20} + P_{20}k + \widehat{P}_a(\mathbf{z}) \cdot \hat{\theta} + P_a(\mathbf{z}) \cdot \theta + P_\Delta(\mathbf{z}) \cdot \Delta\right) y(b) \tag{197}$$

for any $b = \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \in \mathscr{B}$. Using the market-clearing condition for lotteries in period-0, (97), we can substitute $\sum_h \alpha^h x^h(b)$ for $y(b)$ for every bundle $b \in \mathscr{B}$ on the left hand side. Then, summing over all bundles b gives

$$\begin{aligned}
\sum_{b\in\mathscr{B}} P(b) \sum_h \alpha^h x^h(b) &= \sum_{b\in\mathscr{B}} \left[c_{10} + P_{20}c_{20} + P_{20}k + \widehat{P}_a(\mathbf{z})\cdot\hat{\theta} + P_a(\mathbf{z})\cdot\theta + P_\Delta(\mathbf{z})\cdot\Delta\right] y(b) \\
&= \sum_{b\in\mathscr{B}} y(b)\, c_{10} + P_{20} \sum_{b\in\mathscr{B}} y(b)\,(c_{20}+k) \\
&\quad + \sum_s \sum_{z_s} \widehat{P}_a(z_s,s) \sum_{(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta)} y(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},z_s,\Delta)\hat{\theta}_s \\
&\quad + \sum_s \sum_{z_s} P_a(z_s,s) \sum_{(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta)} y(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},z_s,\Delta)\theta_s \\
&\quad + \sum_s \sum_{z_s} P_\Delta(z_s,s) \sum_{(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta)} y(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},z_s,\Delta)\Delta_s \\
&= \sum_h \alpha^h e_{10}^h + P_{20} \sum_h \alpha^h e_{20}^h \qquad (198)
\end{aligned}$$

where the last equality follows the market-clearing conditions (89)-(91) and (95)-(96).

Similarly, given that $\sum_h \alpha^h \tilde{x}^h(b) \geq 0$, multiplying (196) by $\sum_h \alpha^h \tilde{x}^h(b)$ and then summing over $b \in \mathscr{B}$ give

$$\begin{aligned}
\sum_{b\in\mathscr{B}} P(b) \sum_h \alpha^h \tilde{x}^h(b) &\leq \sum_{b\in\mathscr{B}} \left[c_{10} + P_{20}c_{20} + P_{20}k + \widehat{P}_a(\mathbf{z})\cdot\hat{\theta} + P_a(\mathbf{z})\cdot\theta + P_\Delta(\mathbf{z})\cdot\Delta\right] \sum_h \alpha^h \tilde{x}^h(b) \\
&= \sum_{b\in\mathscr{B}} \sum_h \alpha^h \tilde{x}^h(b)\, c_{10} + P_{20} \sum_{b\in\mathscr{B}} \sum_h \alpha^h \tilde{x}^h(b)\,(c_{20}+k) \\
&\quad + \sum_s \sum_{z_s} \widehat{P}_a(z_s,s) \sum_{(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta)} \sum_h \alpha^h \tilde{x}^h(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},z_s,\Delta)\hat{\theta}_s \\
&\quad + \sum_s \sum_{z_s} P_a(z_s,s) \sum_{(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta)} \sum_h \alpha^h \tilde{x}^h(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},z_s,\Delta)\theta_s \\
&\quad + \sum_s \sum_{z_s} P_\Delta(z_s,s) \sum_{(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta)} \sum_h \alpha^h \tilde{x}^h(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},z_s,\Delta)\Delta_s \\
&\leq \sum_h \alpha^h e_{10}^h + P_{20} \sum_h \alpha^h e_{20}^h \qquad (199)
\end{aligned}$$

where the last inequality follows the feasibility conditions (68)-(71) and (74).

Using (198) and (199), (195) can be rewritten as

$$\sum_h \alpha^h e_{10}^h + P_{20} \sum_h \alpha^h e_{20}^h < \sum_h \alpha^h e_{10}^h + P_{20} \sum_h \alpha^h e_{20}^h$$

It is clear that the left-hand side is exactly equal to the right-hand side of the inequality. This is a contradiction! *Q.E.D.*

Proof of Theorem 3. Given that the optimization problems are well-defined concave problems, Kuhn-Tucker conditions are necessary and sufficient. The proof are divided into three steps

(i) Kuhn-Tucker conditions for Pareto Optimal allocations: We will first characterize solutions to the Pareto program using Kuhn-Tucker conditions. Let $\widetilde{P}_{10}, \widetilde{P}_{20}$ be the dual variables on the resource constraints of good-1 and good-2 in period-0 (68)-(69), respectively, and $\alpha^h \widetilde{P}_l^h$ be the dual variable on the probability constraint (64). Let $\widetilde{\widetilde{P}}_a(\mathbf{z}) = \left(\widetilde{\widetilde{P}}_a(z_s, s)\right)_{s=1}^S$, $\widetilde{P}_a(\mathbf{z}) = \left(\widetilde{P}_a(z_s, s)\right)_{s=1}^S$, and $\widetilde{P}_\Delta(\mathbf{z}) = \left(\widetilde{P}_\Delta(z_s, s)\right)_{s=1}^S$ be the dual variables on the resource constraints for contracts paying in good-1 (70), those for contracts paying in good-2 (71), and those for the consistency constraints (74), respectively.

The Lagrangian of the Pareto program is given by

$$\begin{aligned} \mathscr{L} &= \sum_h \lambda^h \alpha^h \sum_{b \in \mathscr{B}} x^h(b) \left\{ U(c_{10}, c_{20}) + \beta V^h\left(k, \hat{\theta}, \theta, \mathbf{z}\right) \right\} + \sum_h \alpha^h \widetilde{P}_l^h \left[1 - \sum_{b \in \mathscr{B}} x^h(b)\right] \\ &- \sum_h \alpha^h \sum_{b \in \mathscr{B}} x^h(b) \left[\widetilde{P}_{10} c_{10} + \widetilde{P}_{20} c_{20} + \widetilde{P}_{20} k + \widetilde{\widetilde{P}}_a(\mathbf{z}) \cdot \hat{\theta} + \widetilde{P}_a(\mathbf{z}) \cdot \theta + \widetilde{P}_\Delta(\mathbf{z}) \cdot \Delta\right] \end{aligned}$$

where all non-negativity constraints are kept implicit for brevity. A solution to the Pareto program satisfies the following condition for $x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$

$$\begin{aligned} \lambda^h \left[U(c_{10}, c_{20}) + \beta V^h\left(k, \hat{\theta}, \theta, \mathbf{z}\right)\right] &\leq \widetilde{P}_{10} c_{10} + \widetilde{P}_{20} c_{20} + \widetilde{P}_{20} k \\ &+ \widetilde{\widetilde{P}}_a(\mathbf{z}) \cdot \hat{\theta} + \widetilde{P}_a(\mathbf{z}) \cdot \theta + \widetilde{P}_\Delta(\mathbf{z}) \cdot \Delta + \widetilde{P}_l^h \end{aligned} \tag{200}$$

where the inequality holds with equality if $x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) > 0$. For any bundle $\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ with $x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) > 0$ for some h, we can show that

$$\frac{\widetilde{P}_a(z_s, s)}{\widetilde{\widetilde{P}}_a(z_s, s)} = \frac{U_{2s}^h}{U_{1s}^h} = p(z_s) \tag{201}$$

This result is derived using a variational principle with respect to $\hat{\theta}_s$ and θ_s, and using the fact that the agent can trade in spot markets at price $p(z_s)$, which implies that $\frac{U_{2s}^h}{U_{1s}^h} = p(z_s)$. This result in fact it is the counterpart of the result in lemma 8.

(ii) Kuhn-Tucker conditions for equilibrium allocations: We will characterize solutions to the consumers' and intermediary's problems in equilibrium using Kuhn-Tucker conditions. Let $\gamma^h(0)$ and $\gamma^h(l)$ be the Lagrange multiplier for constraint (100), and for the probability constraint (64), respectively. The Lagrangian for the expenditure minimization problem of agent h is given by

$$\begin{aligned} \mathscr{L} &= \sum_{\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)} \hat{x}^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) + \gamma^h(l)\left[1 - \sum_{b \in \mathscr{B}} \hat{x}^h(b)\right] \\ &- \gamma^h(0)\left[U(c_{10}, c_{20}) + \beta V^h\left(k, \hat{\theta}, \theta, \mathbf{z}\right) - \mathscr{U}\left(\mathbf{x}^h\right)\right] \end{aligned}$$

where all non-negativity constraints terms are kept implicit for brevity. The optimal condition for $x^h\left(c,k,\hat{\theta},\theta,\mathbf{z}\right)$ is given by

$$\gamma^h(0)\left[U\left(c_{10},c_{20}\right)+\beta V^h\left(k,\hat{\theta},\theta,\mathbf{z}\right)\right] \;\le\; P\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)+\gamma^h(l) \tag{202}$$

where the inequality holds with equality if $x^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)>0$.

Recall that the optimal condition of the market-maker's profit maximization problem (93), for each bundle $\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)$, is

$$P\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)\le c_{10}+P_{20}c_{20}+P_{20}k+\widehat{P}_a(\mathbf{z})\cdot\hat{\theta}+P_a(\mathbf{z})\cdot\theta++P_\Delta(\mathbf{z})\cdot\Delta \tag{203}$$

where the condition holds with equality if $y\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)>0$.

(iii) Matching dual variables and prices: we will show that the optimal conditions of the Pareto program are equivalent to the optimal conditions of consumers' and market-maker's problems. Recall that good-1 is the numeraire. To match let $\gamma^h(0)=\frac{\lambda^h}{\widetilde{P}_{10}}\ \forall h$, $P_{20}=\frac{\widetilde{P}_{20}}{\widetilde{P}_{10}}$, $\widehat{P}_a(\mathbf{z})=\frac{\widetilde{\widehat{P}}_a(\mathbf{z})}{\widetilde{P}_{10}}\ \forall\mathbf{z}$, $P_a(\mathbf{z})=\frac{\widetilde{P}_a(\mathbf{z})}{\widetilde{P}_{10}}\ \forall\mathbf{z}$, and $\gamma^h(l)=\frac{\widetilde{P}_l^h}{\widetilde{P}_{10}}\ \forall h$.

Recall that the optimal condition for the constrained optimality (200) is

$$\begin{aligned}\lambda^h\left[U\left(c_{10},c_{20}\right)+\beta V^h\left(k,\hat{\theta},\theta,\mathbf{z}\right)\right] &\le \widetilde{P}_{10}c_{10}+\widetilde{P}_{20}c_{20}+\widetilde{P}_{20}k\\ &+\widetilde{\widehat{P}}_a(\mathbf{z})\cdot\hat{\theta}+\widetilde{P}_a(\mathbf{z})\cdot\theta+\widetilde{P}_\Delta(\mathbf{z})\cdot\Delta+\widetilde{P}_l^h\end{aligned}$$

Using the matching conditions specified above, this condition becomes

$$\begin{aligned}\gamma^h(0)\left[U\left(c_{10},c_{20}\right)+\beta V^h\left(k,\hat{\theta},\theta,\mathbf{z}\right)\right] &\le c_{10}+P_{20}c_{20}+P_{20}k\\ &+\widehat{P}_a(\mathbf{z})\cdot\hat{\theta}+P_a(\mathbf{z})\cdot\theta+P_\Delta(\mathbf{z})\cdot\Delta+\gamma^h(l)\end{aligned} \tag{204}$$

On the other hand, using (203), the optimal condition for the equilibrium (202) becomes

$$\begin{aligned}\gamma^h(0)\left[U\left(c_{10},c_{20}\right)+\beta V^h\left(k,\hat{\theta},\theta,\mathbf{z}\right)\right] &\le c_{10}+P_{20}c_{20}+P_{20}k\\ &+\widehat{P}_a(\mathbf{z})\cdot\hat{\theta}+P_a(\mathbf{z})\cdot\theta+P_\Delta(\mathbf{z})\cdot\Delta+\gamma^h(l)\end{aligned} \tag{205}$$

which is exactly the same as (204). This shows that a solution to the Pareto program also solves the consumer's and market-maker's problems.

Recall that the resource constraints in the Pareto program are identical to the market-clearing conditions in equilibrium. In addition, the consistency constraints in the Pareto program are equivalent to the ones in the market-maker's problem. Hence, we have shown that any Pareto optimal allocation is also a compensated equilibrium allocation.

Q.E.D.

Proof of Theorem 4. Let $\mathbf{x} = \left(\mathbf{x}^h\right)_h$ be a Pareto optimal allocation. According to Theorem 3, any Pareto optimal allocation can be supported as a compensated equilibrium. We only need to show that any compensated equilibrium, corresponding to $\lambda > 0$, is a competitive equilibrium with transfers. In particular, we will use a cheaper-point argument to show that the expenditure minimization (99) is equivalent to the utility maximization (102). In order to do so, we shall show that there exists an allocation $\widehat{\mathbf{x}}^h \in X^h$ that costs less than $\mathbf{x}^h$, for every agent h.

Let $(\mathbf{x}, \mathbf{y})$ be a compensated equilibrium allocation. Redistributed wealth for an agent h is defined by $w^h = \sum_{b\in\mathscr{B}} P(b)x^h(b)$. Accordingly, we can show that

$$\begin{aligned}
\sum_h \alpha^h w^h &= \sum_{b\in\mathscr{B}}\sum_h \alpha^h x^h(b)P(b) = \sum_{b\in\mathscr{B}} y(b)P(b) \\
&= \sum_{b\in\mathscr{B}} y(b)\left[c_{10} + P_{20}c_{20} + P_{20}k + \widehat{P}_a(\mathbf{z})\cdot\hat{\theta} + P_a(\mathbf{z})\cdot\theta + P_\Delta(\mathbf{z})\cdot\Delta\right] \\
&= \sum_{b\in\mathscr{B}} y(b)\left[c_{10} + P_{20}\left(c_{20}+k\right)\right] \\
&\quad + \sum_s\sum_{z_s} \widehat{P}_a(z_s,s) \sum_{(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta)} y(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},z_s,\Delta)\hat{\theta}_s \\
&\quad + \sum_s\sum_{z_s} P_a(z_s,s) \sum_{(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta)} y(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},z_s,\Delta)\theta_s \\
&\quad + \sum_s\sum_{z_s} P_\Delta(z_s,s) \sum_{(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta)} y(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},z_s,\Delta)\Delta_s \\
&= \sum_{b\in\mathscr{B}} y(b)\left[c_{10} + P_{20}\left(c_{20}+k\right)\right] = \sum_h \alpha^h\left[e^h_{10} + P_{20}e^h_{20}\right]
\end{aligned}$$

where the second equality follows condition (97), the third one follows condition (197), the fifth one uses conditions (89)-(91), and the last one results from conditions (95)-(96). This result shows that the wealth allocation $\left(w^h\right)_h$ is feasible, i.e. satisfying condition (101).

With $\lambda^h > 0$, for every h, an Inada condition ($\lim_{c\to 0} U^h_i(c) = \infty$ for $i = 1, 2$) guarantees that a solution to the Pareto program, which is a compensated equilibrium allocation, will not have a strictly positive mass on $c = 0$.

We will then show that there is an alternative allocation, $\hat{\mathbf{x}}$, with $\mathbf{c}_0 = 0$ that is cheaper than the compensated equilibrium allocation, $\mathbf{x}$. In particular, let $0 \in C$; that is, the zero consumption allocation in period-0 is on the grid. Consider an alternative allocation for an agent h, $\widehat{\mathbf{x}}^h$, such that

$$\hat{x}^h\left(0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) = \sum_{\mathbf{c}_0} x^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) \tag{206}$$

$$\hat{x}^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) = 0, \quad \text{for any } \mathbf{c}_0 \neq 0 \tag{207}$$

Note that the alternative allocation put strictly positive masses on bundles with $c = 0$. The strictly increasing utility function implies that $P_{20} > 0$. Consequently, the optimal condition of the market-maker (94) implies that, for a given bundle $\left(k,\hat{\theta},\theta,\mathbf{z},\Delta\right)$,

$$P\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) > P\left(0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right), \quad \text{for any } c \neq 0 \tag{208}$$

We now would like to compare the equilibrium values of $\mathbf{x}^h$ and $\widehat{\mathbf{x}}^h$.

$$\begin{aligned}
\sum_{b\in\mathscr{B}} P(b)\,x^h(b) &= \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)} P\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) x^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) \\
&> \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)} P\left(0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) x^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) \\
&= \sum_{\left(k,\hat{\theta},\theta,\mathbf{z},\Delta\right)} P\left(0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) \sum_{c} x^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) \\
&= \sum_{\left(k,\hat{\theta},\theta,\mathbf{z},\Delta\right)} P\left(0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) \hat{x}^h\left(0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) \\
&= \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)} P\left(0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) \hat{x}^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) = \sum_{b\in\mathscr{B}} P(b)\,\hat{x}^h(b)
\end{aligned}$$

where the second inequality follows (208), the fourth equality results from (206), and the last equality follows (207).

This shows that there exists an allocation $\widehat{\mathbf{x}}^h$ that is cheaper than the compensated equilibrium allocation, $\mathbf{x}^h$, for every agent h. As a result, using the cheaper-point argument, a compensated equilibrium is a competitive equilibrium with transfers. *Q.E.D.*

Proof of Theorem 5. For notational convenience, we redefine the grid to include the endowment profiles; the period-0 endowment of an agent h is given by probability measure on $\mathscr{B}$ such that

$$\begin{aligned}
\mathbf{e}^h(b) &= 1, \quad \text{for } b = \left(\mathbf{e}_0^h, k=0, \hat{\theta}=0, \theta=0, \mathbf{z}=0, \Delta=0\right) \\
&= 0, \quad \text{otherwise}
\end{aligned}$$

In addition, the optimal condition of the market-maker (94) implies that the price of bundle $\left(\mathbf{e}_0^h, k=0, \hat{\theta}=0, \theta=0, \mathbf{z}=0, \Delta=0\right)$ is $P\left(\mathbf{e}_0^h,0,0,0,0,0\right) = e_{10}^h + P_{20}e_{20}^h$. Therefore, the total value of period-0 endowment lottery of an agent h, $\mathbf{e}^h$, is given by

$$\sum_{b\in\mathscr{B}} P(b)\,\mathbf{e}^h(b) = P\left(\mathbf{e}_0^h, 0, 0, 0, 0, 0\right) = e_{10}^h + P_{20}e_{20}^h \tag{209}$$

which is exactly income in the budget constraint (84).

Let $\mathbf{P} = (P(b))_{b\in\mathscr{B}}$ be the prices of all bundles. In addition, we also add the price of good-2 in period-0, P_{20} into the price space as $P_{20} = P(c = (0,1), 0, 0, 0, 0, 0)$. In other words, P_{20} is embedded in $\mathbf{P}$. As in Prescott and Townsend (2005), with the possibility of negative prices, we restrict prices $\mathbf{P}$ to the closed unit ball;

$$D = \left\{\mathbf{P} \in \mathbb{R}^n | \sqrt{\mathbf{P}\cdot\mathbf{P}} \leq 1\right\} \tag{210}$$

Note that the set D is compact and convex.

Consider the following mapping $(\lambda, \mathbf{x}, \mathbf{P}) \rightarrow (\lambda', \mathbf{x}', \mathbf{P}')$, where $\lambda, \lambda' \in S^{H-1}$, $\mathbf{x}^h \in X^h$. Recall that the consumption possibility set X^h is non-empty, convex, and compact. Let $\overline{X}$ be the cross-product over h of X^h: $\overline{X} = X^1 \times \ldots \times X^H$.

The first part of the mapping is given by $\lambda \longrightarrow (\mathbf{x}', \mathbf{P}')$, where $\mathbf{x}'$ is the solution to the Pareto program given the Pareto weight λ, and $\mathbf{P}'$ is the renormalized prices. With the second welfare theorem, the solution to the Pareto program for a given Pareto weight λ also gives us (compensated) equilibrium prices $\mathbf{P}^*$, where $P^*(c, k, \hat{\theta}, \theta, \mathbf{z}, \Delta) = \widetilde{P}_{10}c_{10} + \widetilde{P}_{20}c_{20} + \widetilde{P}_{20}k + \widetilde{\widetilde{P}}_a(\mathbf{z})\cdot\hat{\theta} + \widetilde{P}_a(\mathbf{z})\cdot\theta + \widetilde{P}_\Delta(\mathbf{z})\cdot\Delta$. The nonlocal satiation of preferences implies that $\mathbf{P}^* \neq 0$. The normalized prices are given by

$$\mathbf{P}' = \frac{\mathbf{P}^*}{\mathbf{P}^*\cdot\mathbf{P}^*}$$

where "$\cdot$" is the inner product operator. Note that $\mathbf{P}'\cdot\mathbf{P}' = 1$. In order to preserve the convexity of the mapping while prices in the unit ball D, we define the convex hull of the normalized prices. Let $\widetilde{D}$ be the sets of all normalized prices, and accordingly $co\widetilde{D}$ be its convex hull. Since $\mathbf{P}' \in \widetilde{D}$, $\mathbf{P}' \in co\widetilde{D}$, which is compact and convex. Note that extending $\widetilde{D}$ to its convex hull does not add any new relative prices. It is not too difficult to show that this mapping, $\lambda \longrightarrow (\mathbf{x}', \mathbf{P}')$, is non-empty, compact-valued, convex-valued. By the Maximum theorem, it is upper hemi-continuous. In addition, the upper hemi-continuity is preserved under the convex-hull operation.

The second part of the mapping is given by $(\lambda, \mathbf{x}, \mathbf{P}) \longrightarrow \lambda'$. The new weight can be formed as follows:

$$\hat{\lambda}^h = \max\left\{0, \lambda^h + \frac{\mathbf{P}\cdot\left(\mathbf{e}^h - \mathbf{x}^h\right)}{A}\right\} \tag{211}$$

$$\lambda'^h = \frac{\hat{\lambda}^h}{\sum_h \hat{\lambda}^h} \tag{212}$$

where A is a positive number such that $\sum_h \left|\mathbf{P}\cdot\left(\mathbf{e}^h-\mathbf{x}^h\right)\right| \leq A$. It is clear that this mapping is also non-empty, compact-valued, convex-valued, and upper hemi-continuous. In conclusion, $(\lambda,\mathbf{x},\mathbf{P}) \rightarrow (\lambda',\mathbf{x}',\mathbf{P}')$ is a mapping from $S^{H-1}\times\overline{X}\times S^{n-1}\longrightarrow S^{H-1}\times\overline{X}\times S^{n+1}$. Since each set is non-empty, compact, and convex, so does its cross-product. In addition, the overall mapping is non-empty, compact-valued, compact-value, and upper hemi-continuous since these properties are preserved under the cross product operation. By Kakutani's fixed point theorem, there exists a fixed point $(\lambda,\mathbf{x},\mathbf{P})$.

Proved in Theorem 3, any Pareto optimal allocation can be supported as a compensated equilibrium. In addition, the strictly increasing utility function implies that $P_{20}>0$. Hence, with positive endowments, an agent h's wealth at the fixed point is strictly positive;

$$w^h=\mathbf{P}\cdot\mathbf{e}^h=e_{10}^h+P_{20}e_{20}^h>0$$

With strictly positive wealth, a compensated equilibrium is a competitive equilibrium with transfers (using a cheaper-point argument as in the proof of Theorem 4).

We now need to show that the budget constraint (without transfers);

$$\mathbf{P}\cdot\left(\mathbf{e}^h-\mathbf{x}^h\right)=0$$

holds for every agent h. Summing $\mathbf{P}\cdot\left(\mathbf{e}^h-\mathbf{x}^h\right)$ over h with weights $\left[\alpha^h\right]_h$ gives

$$\begin{aligned}
\sum_h \alpha^h\mathbf{P}\cdot\left(\mathbf{e}^h-\mathbf{x}^h\right) &= \sum_h\alpha^h\sum_{b\in\mathscr{B}}P(b)\mathbf{e}^h(b)-\sum_{b\in\mathscr{B}}P(b)\sum_h\alpha^h x^h(b)\\
&= \sum_h\alpha^h\sum_{b\in\mathscr{B}}P(b)\mathbf{e}^h(b)-\sum_{(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta)}P(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta)y(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta)\\
&= \sum_h\alpha^h\left[e_{10}^h+P_{20}e_{20}^h\right]-\sum_{(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta)}y(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta)\left[c_{10}+P_{20}c_{20}+P_{20}k\right]\\
&- \sum_{(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta)}y(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta)\left[\widehat{P}_a\left(\mathbf{z}\right)\cdot\hat{\theta}+P_a\left(\mathbf{z}\right)\cdot\theta+P_\Delta\left(\mathbf{z}\right)\cdot\Delta\right]\\
&= \sum_h\alpha^h e_{10}^h-\sum_{(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta)}y(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta)c_{10}\\
&+ \sum_h\alpha^h e_{20}^h-\sum_{(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta)}y(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta)\left[c_{20}+k\right]\\
&+ \sum_s\sum_{z_s}\widehat{P}_a(z_s,s)\sum_{(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta)}\sum_h\alpha^h x^h(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},z_s,\Delta)\hat{\theta}_s\\
&+ \sum_s\sum_{z_s}P_a(z_s,s)\sum_{(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta)}\sum_h\alpha^h x^h(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},z_s,\Delta)\theta_s\\
&+ \sum_s\sum_{z_s}P_\Delta(z_s,s)\sum_{(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta)}\sum_h\alpha^h x^h(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},z_s,\Delta)\Delta_s
\end{aligned} \quad (213)$$

where the second equality follows from the market-clearing condition for lotteries (97), the third equality uses conditions (94) and (209). Using the market-clearing constraints (89)-(90) and (95)-(97), and the consistency constraint for the market-maker (91), we can show that the right hand side of equation (213) is exactly zero:

$$\sum_h \alpha^h \mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right) \quad = \quad 0$$

In addition, at a fixed point $\mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right)$ must be the same sign for every h. Hence, $\mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right) = 0$ for every agent h. This clearly confirms that the budget constraint (without transfers) of every agent h holds. Hence, a competitive equilibrium (without transfers) exists. Note that supply of the market-maker, $\mathbf{y}$, can be recovered from $\mathbf{x}$, and similarly other prices, $\left(P_{20}, \widehat{P}_a(\mathbf{z}), P_a(\mathbf{z}), P_\Delta(\mathbf{z})\right)$, can be inferred from the dual variables from the Pareto program at the fixed point. *Q.E.D.*

B Detailed Derivations

B.1 Derivation of a Competitive Equilibrium with the Externality in Example 3

First of all, the price of good-2 in period-0 is given by

$$P_{20} = \left(\frac{2}{2-k}\right)^2 \tag{214}$$

Similarly, the market fundamental in each state s is $z_s = \frac{2}{2+k}$. Hence, the spot price of good-2 in each state s is given by

$$p(z_s) = \left(\frac{2}{2+k}\right)^2, \ \forall s \tag{215}$$

Further, the price of a (collateralized) security paying in good-2 in state s is given by

$$P_s = \max_h \left(\frac{\pi_s U_{2s}^h}{U_{10}^h}\right), \ \forall s \tag{216}$$

The endowment structure implies that agents type 2 will have higher MRS $\frac{\pi_s U_{2s}^h}{U_{10}^h}$ in state 1, and vice versa. Hence, (216) can be rewritten as

$$P_1 = \frac{\pi_s U_{21}^2}{U_{10}^2} = \frac{1}{2}\left(\frac{2}{1+k+\theta}\right)^2 = \frac{\pi_s U_{22}^1}{U_{10}^1} = P_2 \tag{217}$$

Note that the symmetry also implies that $P_1 = P_2$. Using the optimal conditions with respect to k^h and θ_s^h, we can show that

$$P_{20} = P_1 + P_2 \Longrightarrow \left(\frac{2}{2-k}\right)^2 = \left(\frac{2}{1+k+\theta}\right)^2 \tag{218}$$

Next, with the homotheticity of preferences, the ratio of consumption in each state of each agent must be equal to the market fundamental; that is,

$$\frac{1+\hat{\theta}}{1+k+\theta} = z_s = \frac{2}{2+k} \tag{219}$$

Furthermore, the collateral constraint in state $s=1$ of an agent type $h=1$ is binding, i.e.

$$p(z_1)k - \hat{\theta} - p(z_1)\theta = 0 \Longrightarrow \hat{\theta} = \left(\frac{2}{2+k}\right)^2 (k-\theta) \tag{220}$$

Note that the same equation can be derived from the binding collateral constraint in state $s=2$ for an agent type $h=2$.

We can compute a collateral equilibrium using (218), (219), and (220) to solve for $\left(k,\theta,\hat{\theta}\right)$. We can rewrite (218) as

$$2-k = 1+k+\theta \Longrightarrow \theta = 1-2k \tag{221}$$

In addition, Substituting (220) into (219) gives

$$1+\left(\frac{2}{2+k}\right)^2 (k-\theta) = \left(\frac{2}{2+k}\right)(1+k+\theta) \tag{222}$$

Then, substituting (221) into (222) will give

$$1+\left(\frac{2}{2+k}\right)^2 (k-1+2k) = \left(\frac{2}{2+k}\right)(1+k+1-2k) \Longrightarrow 3k^2+16k-8=0 \tag{223}$$

The unique feasible (positive) solution to the above quadratic equation is $k \approx 0.4603$.

References

Alvarez, F. and U. Jermann (2000). Efficiency, Equilibrium, and Asset Pricing with Risk of Default. *Econometrica 68*(4), 775–797.

Bewley, T. (1986). Stationary monetary equilibrium with a continuum of independently fluctuating consumers. *Contributions to Mathematical Economics in Honor of Gérard Debreu: In Honor of Gerard Debreu*, 79.

Biais, B. and T. Mariotti (2005). Strategic Liquidity Supply and Security Design. *Review of Economic Studies 72*(3), 615–649.

Bisin, A. and P. Gottardi (2006). Efficient Competitive Equilibria with Adverse Selection. *Journal of Political Economy 114*(3), 485–516.

Caballero, R. and A. Krishnamurthy (2004). Smoothing Sudden Stops. *Journal of Economic Theory 119*(1), 104–127.

DeMarzo, P. (2005). The Pooling and Tranching of Securities: A Model of Informed Intermediation. *Review of Financial Studies 18*(1), 1–35.

Demarzo, P. and D. Duffie (1999). A Liquidity-based Model of Security Design. *Econometrica 67*(1), 65–99.

Geanakoplos, J. (2003). Liquidity, Default, and Crashes. *Advances in Economics and Econometrics: Theory and Applications: Eighth World Congress*.

Geanakoplos, J. and W. Zame (2002). Collateral and the Enforcement of Intertemporal Contracts. Technical report, UCLA Working Paper.

Golosov, M. and A. Tsyvinski (2007). Optimal Taxation With Endogenous Insurance Markets. *The Quarterly Journal of Economics 122*(2), 487–534.

Greenwald, B. and J. Stiglitz (1986). Externalities in Economies with Imperfect Information and Incomplete Markets. *Quarterly Journal of Economics 101*(2), 229–264.

Hansen, G. (1985). Indivisible labor and the business cycle. *Journal of Monetary Economics 16*, 309–327.

Holmström, B. and J. Tirole (1998). Private and Public Supply of Liquidity. *Journal of Political Economy 106*(1), 1–40.

Jacklin, C. (1987). Demand Deposits, Trading Restrictions, and Risk Sharing. *Contractual Arrangements for Intertemporal Trade*, 26–47.

Kehoe, T. and D. Levine (1993). Debt-Constrained Asset Markets. *Review of Economic Studies 60*(4), 865–888.

Kilenthong, W. T. (2006). Collateral Premia and Risk Sharing under Limited Commitment. *UCSB Working Paper*.

Kilenthong, W. T. (2008). A Walrasian Equilibrium Model with Asset-Backed Securities. *UCSB Working Paper*.

Kiyotaki, N. and J. Moore (1997). Credit Cycles. *Journal of Political Economy 105*(2), 211–248.

Kocherlakota, N. R. (1996). Implications of efficient risk sharing without commitment. *Review of Economic Studies 63*(4), 595–609.

Lorenzoni, G. Inefficient Credit Booms. *Review of Economic Studies 75*(3), 809–833.

Lustig, H. (2007). The Market Price of Aggregate Risk and the Wealth Distribution. *UCLA Working Paper*.

Negishi, T. (1960). Welfare Economics and Existence of an Equilibrium for a Competitive Economy. *Metroeconomica 12*(2-3), 92–97.

Prescott, E. C. and R. M. Townsend (1984a). General Competitive Analysis in an Economy with Private Information. *International Economic Review 25*(1), 1–20.

Prescott, E. C. and R. M. Townsend (1984b). Pareto Optima and Competitive Equilibria with Adverse Selection and Moral Hazard. *Econometrica 52*(1), 21–45.

Prescott, E. S. and R. M. Townsend (2005). Firms as Clubs in Walrasian Markets with Private Information: Technical Appendix. *mimeo Federal Reserve Bank of Richmond*.

Prescott, E. S. and R. M. Townsend (2006). Firms as Clubs in Walrasian Markets with Private Information. *Journal of Political Economy 114*(4), 644–671.

Rogerson, R. (1988). Indivisible Labor, Lotteries and Equilibrium. *Journal of Monetary Economics 21*(1), 3–16.

Steinert, M. and J. Torres-Martínez (2007). General equilibrium in CLO markets. *Journal of Mathematical Economics 43*(6), 709–734.

Uzawa, H. (1958). The Kuhn-Tucker Theorem in Concave Programming. *Stanford Mathematical Studies in the Social Sciences*.