Intermediary Asset Pricing

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Abstract

We present a model to study the dynamics of risk premia during crises in asset markets where the marginal investor is a financial intermediary. Intermediaries face a constraint on raising equity capital. When the constraint binds, so that intermediaries' equity capital is scarce, risk premia rise to reflect the capital scarcity. We calibrate the model and show that it does well in matching two aspects of crises: the nonlinearity of risk premia during crisis episodes; and, the speed of adjustment in risk premia from a crisis back to pre-crisis levels. We use the model to quantitatively evaluate the effectiveness of a variety of central bank policies, including reducing intermediaries' borrowing costs, infusing equity capital, and directly intervening in distressed asset markets. All of these policies are effective in aiding the recovery from a crisis. Infusing equity capital into intermediaries is particularly effective because it attacks the equity capital constraint that is at the root of the crisis in our model.

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1 Introduction

The performance of many asset markets – e.g., prices of mortgage-backed securities, corporate bonds, etc. – depend on the financial health of the intermediary sector, broadly defined to include traditional commercial banks as well as investment banks and hedge funds. The subprime crisis and the 1998 hedge fund crisis are two compelling data points in support of this claim.[1] However, traditional approaches to asset pricing ignore intermediation by invoking the assumption that intermediaries' actions reflect the preferences of their client-investors. With this assumption, the traditional approach treats intermediaries as a "veil," and instead posits that a representative household is marginal in pricing all assets. Thus, the pricing kernel for the S&P500 stock index is the same as the pricing kernel for mortgage-backed securities. Yet, many crises, such as the 1998 episode, play out primarily in the more complex securities that are the province of the intermediary sector. The traditional approach cannot speak to this relationship between financial intermediaries and asset prices. It sheds no light on why "intermediary capital" is important for asset market equilibrium. It also does not allow for a meaningful analysis of the policy actions, such as increasing intermediaries' equity capital or discount window lending, which are commonly considered during crises.

We offer a framework to address these issues. We develop a model in which the intermediary sector is not a veil, and in which its capital plays an important role in determining asset market equilibrium. We calibrate the model to data on the intermediation sector and show that the model performs well in replicating asset market behavior during crises.

The striking feature of financial crises is the sudden and dramatic increase of risk premia. For example, in the hedge fund crisis of the fall of 1998, many credit spreads and mortgage-backed security spreads doubled from their pre-crisis levels. Our baseline calibration can replicate this dramatic behavior. When intermediary capital is low, losses within the intermediary sector have significant effects on risk premia. However, when capital is high, losses have little to no effect on risk premia. The asymmetry in our model captures the non-linearity that is present in asset market crises. Simulating the model, we find that the average risk premium is near 5%. Using this number to reflect a pre-crisis normal level, we find that the probability of the risk premium exceeding 7.5% is 4%. The probability of the risk premium exceeding 10%, which is twice the "normal" level,

[1] There is a growing body of empirical evidence documenting the effects of intermediation constraints (such as capital or collateral constraints) on asset prices. These studies include, research on mortgage-backed securities (Gabaix, Krishnamurthy, and Vigneron, 2005), corporate bonds (Collin-Dufresne, Goldstein, and Martin, 2001), default swaps (Berndt, et. al., 2004), catastrophe insurance (Froot and O'Connell, 1999, 2001), and index options (Bates, 2003; Garleanu, Pedersen, and Poteshman, 2005).

is 1%. The 1998 episode saw risk premia and Sharpe ratios rise considerably, in the range of $1.5X$ to $2X$. Our model puts the probability of such an event between 1% and 4%.

Another important feature of financial crises is the pattern of recovery of spreads. In the 1998 crisis, most spreads took about 10 months to halve from their crisis-peak levels to pre-crisis levels. As we discuss later in the paper, half-lives of between 6 months and extending over a year have been documented in a variety of asset markets and crisis situations. We note that these types of recovery patterns are an order of magnitude slower than the daily mean reversion patterns documented in the market microstructure literature (e.g., Campbell, Grossman, and Wang, 1993). A common wisdom among many observers is that this recovery reflects the slow movement of capital into the affected markets (Froot and O'Connell, 1999, Berndt, et. al., 2004, Mitchell, Pedersen, and Pulvino, 2007). Our baseline calibration of the model can replicate these speeds of capital movement. We show that simulating the model starting from an extreme crisis state (risk premium of 20%), the half-life of the risk premium back to the unconditional average risk premium is about 5.5 months. From a risk premium of 10%, which is about twice the unconditional average risk premium in our simulations, the half-life is close to 19 months.

We also use the model as a laboratory to quantitatively evaluate government policies. Beginning from the extreme crisis state with risk premium of 20%, we trace the crisis recovery path conditional on three government policies. The policies we consider are: (1) Infusing equity capital into the intermediaries during a crisis; (2) Lowering borrowing rates to the intermediary, as with a decrease in the central bank's discount rate; and, (3) Direct purchase of the risky asset by the government, financed by debt issuance and taxation of households. These three policies are chosen because they are among those undertaken by central banks in practice, and have been used in varying degrees in the current crisis. Both the equity infusion and risky asset purchase policies have an immediate impact of lowering the risk premium. Moreover, in comparing \$250bn of equity infusion to \$500bn of risky asset purchase, we find that the equity infusion is more effective in reducing the risk premium. This occurs in our model because the problem friction in the model is an equity capital constraint. Thus infusing equity capital attacks the problem at its heart. The interest rate policy is also effective, but the effect is more gradual in speeding up the crisis recovery.

Finally, we study government bond policies as suggested by Woodford (1990) and Holmstrom and Tirole (1998). These authors suggest that the government should issue safe debt to the private sector, as a form of liquidity provision. Our model is non-Ricardian and such a policy can be meaningfully evaluated. The interesting result from our analysis is that this form of liquidity provision uniformly lowers the risk premium confirming the benefits of debt policy.

The contribution of our paper is to work out an equilibrium model of intermediation that is dynamic, parsimonious, and can be realistically calibrated. Previous models in the literature have almost exclusively been static and designed to highlight qualitative effects. Allen and Gale in a number of papers show how the financial structure of intermediaries plays an important role in financial crises (see Allen and Gale, 2005).[2] Holmstrom and Tirole (1997) show how capital constraints in intermediation can affect the equilibrium interest rate as well as interest rate spreads. Shleifer and Vishny (1997) argue that the tendency for investors to withdraw funds from intermediaries following negative performance limits the ability of intermediaries to exploit high returns.[3] Our paper draws on the ideas from this prior literature, incorporating these ideas into a fully dynamic and quantitative general equilibrium model.[4]

Our paper is also related to a companion paper, He and Krishnamurthy (2008). We solve for the optimal intermediation contract in that paper, while we assume the (same) form of contract in the current analysis. That paper also solves for the equilibrium asset prices in closed form, while we rely on numerical solutions in the present paper. On the other hand, that paper has a degenerate steady state distribution which does not allow for a meaningful simulation or the other quantitative exercises we perform in the present paper. In addition, the present paper models households with labor income and an intermediation sector which always carries some leverage. Both aspects of the model are important in realistically calibrating the model. However, these same features of the model require us to rely on numerical solutions. Apart from these differences, the analysis in He and Krishnamurthy (2008) provides theoretical underpinnings for some of the assumptions we make in this paper.

The paper is organized as follows. Sections 2 and 3 outline the model and its solution. Section 4 explains how we calibrate the model. Section 5 presents the results of the crisis calibration. Sections 6 and 7 study policy actions. Section 8 concludes and is followed by an Appendix with details of the model solution.

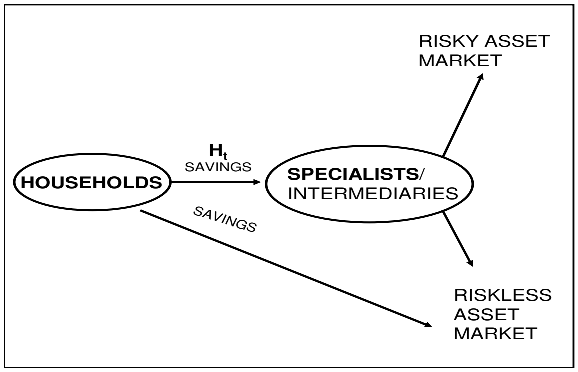
[2] The work of Allen and Gale builds on the Diamond and Dybvig (1983) framework. See also Diamond (1997) and Diamond and Rajan (2005) for models with linkages between the asset market and financial intermediaries.

[3] Other papers in the literature on asset pricing and intermediation include Allen and Gorton (1993), Allen and Gale (1994), Brennan (1993), Grossman and Zhou (1996), Holmstrom and Tirole (1997), Shleifer and Vishny (1997), Dasgupta, Prat and Verardo (2005), and Vayanos (2005).

[4] Related papers in the asset market crisis literature include Xiong (2001), Kyle and Xiong (2001), Gromb and Vayanos (2002), Brunnermeier and Pedersen (2008), He and Krishnamurthy (2008), and Gabaix, Krishnamurthy, and Vigneron (2007).

2 The Model: Intermediation and Asset Prices

Figure 1: The Economy



This figure depicts the agents in the economy and their investment opportunities.

Figure 1 lays out the building blocks of our model. There is a risky asset that represents complex assets (i.e. mortgage-backed securities) where investment requires some sophistication (i.e. prepayment modeling). **Households** cannot invest directly in the risky asset market. There is limited market participation, as in Mankiw and Zeldes (1991), Allen and Gale (1994), Basak and Cuoco (1998), or Vissing-Jorgensen (2002). **Specialists** have the knowledge to invest in the risky assets, and unlike in the limited market participation literature, the specialists can invest in the risky asset on behalf of the households. This investment conduit is the intermediary of our model. In our model, the households demand intermediation services while the specialists supply these services. We are centrally interested in describing how this intermediation relationship affects and is affected by the market equilibrium for the "intermediated" risky asset.

We assume that if the household does not invest in the intermediary, it can only invest in a riskless short-term bond. This is clearly counterfactual (i.e. households invest in the S&P 500 index), but simplifies the analysis considerably.

Households thus face a portfolio choice decision of allocating funds between the intermediaries and the riskless bond. The intermediaries accept H_t of the household funds and then allocate their total funds under management between the risky asset and the riskless bond. We elaborate on each of the elements of the model in the next sections.

2.1 Assets

The assets are modeled as in the Lucas (1978) tree economy. The economy is infinite-horizon, continuous-time, and has a single perishable consumption good, which we will use as the numeraire. There are two assets, a riskless bond in zero net supply, and a risky asset that pays a risky dividend. We normalize the total supply of risky assets to be one unit.

The risky asset pays a dividend of D_t per unit time, where $\{D_t\}$ follows a geometric Brownian motion,

$$\frac{dD_t}{D_t} = gdt + \sigma dZ_t \quad \text{given} \quad D_0. \tag{1}$$

$g > 0$ and $\sigma > 0$ are constants. Throughout this paper $\{Z_t\}$ is a standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. We denote the processes $\{P_t\}$ and $\{r_t\}$ as the risky asset price and interest rate processes, respectively. We also define the total return on the risky asset as,

$$dR_t = \frac{D_t dt + dP_t}{P_t}. \tag{2}$$

2.2 Specialists and intermediation

There is a unit mass of identical specialists who manage the intermediaries in which the households invest. The specialists represent the insiders/decision-makers of a hedge fund or a mutual fund. We collapse all of an intermediary's insiders into a single agent, following the device of modeling entrepreneur-managers of firms in the corporate finance literature (e.g. Holmstrom and Tirole, 1997).

Formally, we assume that the specialists are infinitely-lived and maximize an objective function,

$$E\left[\int_0^\infty e^{-\rho t} u(c_t) dt\right] \qquad \rho > 0; \tag{3}$$

where c_t is the date t consumption rate of the specialist. We consider a CRRA instantaneous utility function with parameter γ for the specialists, $u(c_t) = \frac{1}{1-\gamma} c_t^{1-\gamma}$.

Each specialist manages one intermediary. We denote the date t wealth of specialists as w_t and assume that this is wholly invested in the intermediary. We think of w_t as the specialist's "stake" in the intermediary, possibly capturing financial wealth at risk in the intermediary. Although outside the scope of the model, we may imagine that w_t also captures reputation that is at stake in the intermediary and the future income from being an insider of the intermediary.

We envision the following to describe the interaction between specialists and households. At every t, each specialist is randomly matched with a household to form an intermediary. These interactions occur instantaneously and result in a continuum of (identical) bilateral relationships.[5] The household allocates some funds H_t to the intermediary. Specialists then execute trades for the intermediary in a Walrasian risky asset and bond market, and the household trades in only the bond market. At $t + dt$ the match is broken, and the intermediation market repeats itself.

Consider one of the intermediary relationships between specialist and household. The specialist manages an intermediary whose total capital is the sum of the specialist's wealth, w_t, and the wealth that the household allocates to the intermediary, H_t. The specialist makes all investment decisions on this capital and faces no portfolio restrictions in buying or short-selling either the risky asset or the riskless bond. Suppose that the specialist chooses to invest a fraction α_t^I of the portfolio in the risky asset and $1 - \alpha_t^I$ in the riskless asset. Then, the return delivered by the intermediary is,

$$\widetilde{dR_t} = r_t dt + \alpha_t^I (dR_t - r_t dt), \tag{4}$$

where dR_t is the total return on the risky asset.

2.3 Intermediation constraint

The key assumption of our model is that the household is unwilling to invest more than mw_t of funds in the intermediary ($m > 0$ is a constant). That is, if the specialist has one dollar of wealth invested in the intermediary, the household will only invest up to m dollars of his own wealth in the intermediary. He and Krishnamurthy (2008) derive this sort of capital constraint by assuming moral hazard by the specialist. In their model, the household requires that the specialist have a sufficient stake in the intermediary to prevent shirking.[6] Here we adopt the constraint in reduced form.

The wealth requirement implies that the *supply of intermediation* facing a household is at most,

$$H_t \leq mw_t. \tag{5}$$

[5]Why the matching structure instead of a Walrasian intermediation market? In the Walrasian case, when intermediation is supply constrained, specialists charge the households a fee for managing the intermediary that depends on the tightness of the intermediation constraint. In the matching structure the fee is always zero which makes solving the model somewhat easier. Introducing a constant fee into the model is both easy and does not alter results appreciably. See He and Krishnamurthy (2008) where we study the Walrasian case.

[6]The He and Krishnamurthy (2008) model is adapted from Holmstrom and Tirole (1997). One feature of the contract derivation in He and Krishnamurthy worth noting is that it assumes that the household cannot observe and dictate the specialist's portfolio choice within the intermediary. Without this assumption, it is possible that the household will bribe the specialist to make a particular portfolio choice. Of course, such a strategy violates the spirit of the model: namely that only the specialist has the expertise to invest in the risky asset.

If either m is small or w_t is small, the household's ability to indirectly participate in the risky asset market will be restricted.

We may interpret the wealth requirement in two ways. First, as noted above, we can think of w_t as the specialist's stake in the intermediary, and this stake must be sufficiently high for household's to feel comfortable with their investment in the intermediary. Thus, one interpretation is that w_t reflects the capital base of a hedge fund. The managers of a hedge fund typically have some of their wealth tied up in the investments of the hedge fund. Such an arrangement ensures that the incentives of the hedge fund's managers and investors are aligned. However, if a hedge fund loses a lot of money then the capital of the hedge fund will be depleted. In this case, investors will be reluctant to contribute money to the hedge fund, fearing mismanagement or further losses. A hedge fund "capital shock" is one phenomena that we can capture with our model.

Another interpretation, which is more in keeping with regularities in the mutual fund industry, is that the wealth of a specialist summarizes his past success in making investment decisions. Low wealth then reflects poor past performance by a mutual fund, which makes households reluctant to delegate investment decisions to the specialist. The relation between past performance and mutual fund flows is a well-documented empirical regularity (see, e.g., Warther (1995)). As w_t falls, reflecting poor past performance, investors reduce their portfolio allocation to the mutual fund. Shleifer and Vishny (1997) present a model with a similar feature: the supply of funds to an arbitrageur in their model is a function of the previous period's return by the arbitrageur.

Since we adopt constraint (5) in reduced form, we do not take a stand on the interpretation of the constraint. Indeed, in our calibration scenarios, we match the specialist-intermediary to the entire intermediary sector – including hedge funds, banks, and mutual funds. From this standpoint, it is useful that the constraint may be appropriate across a variety of intermediaries.

The novel feature of our model is that w_t, and the supply of intermediation, evolve endogenously as a function of shocks and the past decisions of specialists and households. In both the hedge fund and the mutual fund example, if the intermediation constraint (5) binds, a fall in w_t causes households to reduce their allocation of funds to intermediaries and invest in the riskless bond. Of course, the risky asset still has to be held in equilibrium. As households indirectly reduce their exposure to the risky asset, via market clearing, the specialist increases his exposure to the risky asset. To induce the specialist to absorb more risk, the risky asset price falls and its expected return rises. This dynamic effect of w_t on the equilibrium is the central driving force of our model. We think it arises naturally when considering the equilibrium effects of intermediation.

We note that both the household and specialist receive the return $\widetilde{dR_t}$ (see (4)) on their contributions to the intermediary; that is, both household and specialist invest in the **equity** of the intermediary. Constraint (5) limits the equity contribution by the household to the intermediary as a function of the specialist's equity contribution. It is important to point out that this constraint is not the usual constraint in the literature on corporate investment and credit rationing (see as an example, Holmstrom and Tirole, 1997). In that literature, firms face a restriction on the quantity of funds they can borrow using *either* equity securities or debt securities. Constraint (5) in our model does not restrict the amount of debt issued by an intermediary and therefore does not restrict the total funds that an intermediary can raise. Intermediaries can short an instantaneous (maturity dt) bond in our model in the Walrasian bond market. There is no default on such debt contracts in our continuous time model. Constraint (5) restricts how risk is shared between specialists and households. It is the dynamics of risk sharing that drives the behavior of asset prices in our model.[7,8]

To close this section, we write the decision problem of the specialist. The specialist chooses his consumption rate and the portfolio decision of the intermediary to solve,

$$\max_{\{c_t,\alpha_t^I\}} E\left[\int_0^\infty e^{-\rho t} u(c_t)\, dt\right] \quad s.t. \quad dw_t = -c_t dt + w_t r_t dt + w_t\left(\widetilde{dR}_t\left(\alpha_t^I\right) - r_t dt\right). \tag{6}$$

We can also rewrite the budget constraint in terms of the underlying return:

$$dw_t = -c_t dt + w_t r_t dt + \alpha_t^I w_t \left(dR_t - r_t dt\right).$$

Note that α_t^I is effectively the specialist's portfolio share in the risky asset.

2.4 Households: The demand for intermediation

We model the household sector as an overlapping generation (OG) of agents. This keeps the decision problem of the household fairly simple.[9] On the other hand, we enrich the model to include household labor income and

[7] In practice, financial institutions use the repo market to borrow funds via debt contracts. They also borrow from investors via equity contracts. It is plausible that during crisis episodes hedge funds, for example, are also restricted in their debt borrowings through tighter margin requirements or haircuts. Gromb and Vayanos (2002) and Brunnermeier and Pedersen (2008) study models in which agents face a constraint on the amount of debt financing available to an agent, where the constraint is a function of w_t. In that paper, by assumption, agents are unable to raise funds via equity contracts. For simplicity, we choose to only model restrictions on the equity borrowing based on the following rationale: because equity is junior to debt any constraints are likely to be tighter on equity than debt. The inability of financial institutions to raise equity capital figures prominently in discussions of the ongoing subprime crisis.

[8] In He and Krishnamurthy (2008), we allow for all forms of contracts and derive an optimal contract that places a constraint on equity capital contributions but no constraint on debt contributions.

[9] Note the specialists are infinitely lived while households are modeled using the OG structure. As we will see, specialists play the key role in determining asset prices. Our modeling ensures that their choices reflect the forward-looking dynamics of the

introduce heterogeneity within the household sector. Both enrichments are useful in realistically calibrating the model.

For the sake of clarity in explaining the OG environment in a continuous time model, we index time as $t, t+\delta, t+2\delta, \ldots$ and consider the continuous time limit when δ is of order dt. A unit mass of generation t agents are born with wealth w_t^h and live in periods t and $t+\delta$. They maximize utility:

$$\rho\delta \; ln \; c_t^h + (1-\rho\delta)\, E_t[ln \; w_{t+\delta}^h]. \tag{7}$$

c_t^h is the household's consumption in period t and $w_{t+\delta}^h$ is a bequest for generation $t+\delta$. Note that both utility and bequest functions are logarithmic.

In addition to wealth of w_t^h, we assume that generation t households receive labor income at date t of $l \; D_t \; \delta$. $l > 0$ is a constant and D_t is the dividend on the risky asset at time t. Labor income is assumed proportional to dividends in order to preserve some useful homogeneity properties of the equilibrium. We introduce labor income to more realistically match the consumption-savings profile of households. Providing the households some labor income also ensures that the economy never reaches a state where households "die" out, as often happens in two-agent models (see, for example, Dumas (1989) and Wang (1996)).

It is easy to verify that as $\delta \to dt$ in the continuous time limit, the household's consumption rule is,

$$c_t^h = \rho w_t^h. \tag{8}$$

In particular, note that the labor income does not affect the consumption rule because the labor income flow is of order dt. Interpreting $\rho > 0$ as the household's rate of time preference, we note that this is the standard consumption rule for logarithmic agents. The household is "myopic" and his rule does not depend on his investment opportunity set.

A household invests its wealth from t to $t+\delta$ in financial assets. As noted earlier, households are not directly able to save in the risky asset and can only directly access the riskless bond market. We assume that the household can choose any positive level of bond holdings when saving in the riskless bond (note that short-selling of the bond is rule out). The household must use an intermediary when accessing the risky asset market.

We consider a further degree of heterogeneity in the intermediation investment restriction. We assume that a fraction λ of the households can ever only invest in the riskless bond. The remaining fraction, $1-\lambda$,

economy. We treat households in a simpler manner for tractability reasons. We deem the cost of the simplification to be low since households play a secondary role in the model.

may enter the intermediation market and save a fraction of their wealth with intermediaries which indirectly invest in the risky asset on their behalf. We refer to the former as "debt households" and the latter as "risky asset households."[10]

The heterogeneity among households is realistic. Clearly, there are many households that only save in a bank account. In the literature cited earlier on limited market participation, all households are "debt households." The demand for intermediation in our model stems from the risky asset households. Introducing this degree of heterogeneity allows for a better model calibration.

2.5 Household decisions

To summarize, a debt and risky asset household are born at generation t with wealth of w_t^h. The households receive labor income and choose a consumption rate of ρw_t^h. They also make savings decisions, respecting the restriction on their investment options.

The debt household's consumption decision, given wealth of w_t^h, is described by (8). The savings decision is to invest w_t^h in the bond market at the interest rate r_t.

The risky asset household's consumption is also described by (8). His portfolio decision is how much wealth to allocate to intermediaries. We denote $\alpha_t^h \in [0,1]$ as the fraction of the household's wealth in the intermediary and recall that the intermediary's return is $\widetilde{dR}_t$ in (4). The remaining $1-\alpha_t^h$ of household wealth is invested in the riskless bond and earns the interest rate of $r_t dt$. The risky asset household chooses α_t^h to maximize (7). Given the log objective function, this decision solves,

$$\max_{\alpha_t^h \in [0,1]} \alpha_t^h E_t[\widetilde{dR}_t] - \frac{1}{2}\left(\alpha_t^h\right)^2 Var_t[\widetilde{dR}_t] \quad s.t. \quad \alpha_t^h(1-\lambda)w_t^h \equiv H_t \leq m w_t. \tag{9}$$

Note the constraint here, which corresponds to the intermediation constraint we have discussed earlier.

Given the decisions by the debt household and the risky asset household, the evolution of w_t^h across generations is described by,

$$dw_t^h = (lD_t - \rho w_t^h)dt + w_t^h r_t dt + \alpha_t^h(1-\lambda)w_t^h\left(\widetilde{dR}_t - r_t dt\right). \tag{10}$$

[10] The wealth of the debt household and risky asset household evolve differently between t and $t+\delta$. We assume that this wealth is pooled together and distributed equally to all agents of generation $t+\delta$. The latter assumption ensures that we do not need to keep track of the distribution of wealth over the households when solving for the equilibrium of the economy.

2.6 Equilibrium

Definition 1 *An* equilibrium *is a set of price processes* $\{P_t\}$ *and* $\{r_t\}$*, and decisions* $\{c_t, c_t^h, \alpha_t^I, \alpha_t^h\}$ *such that,*

1. *Given the price processes, decisions solve the consumption-savings problems of the debt household, the risky asset household (9) and the specialist (6);*
2. *Decisions satisfy the intermediation constraint of (5);*
3. *The risky asset market clears:*
$$\frac{\alpha_t^I(w_t + \alpha_t^h(1-\lambda)w_t^h)}{P_t} = 1; \tag{11}$$
4. *The goods market clears:*
$$c_t + c_t^h = D_t(1+l). \tag{12}$$

Given market clearing in risky asset and goods markets, the bond market clears by Walras' law. The market clearing condition for the risky asset market reflects that the intermediary is the only direct holder of risky assets and has total funds under management of $w_t + \alpha_t^h(1-\lambda)w_t^h$, and the total holding of risky asset by the intermediary must equal the supply of risky assets.

Finally, an equilibrium relation that proves useful when deriving the solution is that,

$$w_t + w_t^h = P_t.$$

That is, since bonds are in zero net supply, the wealth of specialists and households must sum to the value of the risky asset.

3 Solution

We outline the main steps in deriving the solution in this section. For detailed derivations, see the Appendix A. We begin with an example that illustrates the main features of our model and helps in understanding the steps in the solution.

3.1 Example

Suppose that $m = 1$ and $\lambda = 0$. Moreover, suppose we are in a state where $w_t = 100$ and $w_t^h = 200$. Then it is clear that since $mw_t < w_t^h$, this is a state where intermediation is constrained by (5). Since the riskless asset is in zero net supply, the value of the risky asset is equal to the sum of w_t and w_t^h (i.e. 300). Suppose that households saturate the intermediation constraint by investing 100 in intermediaries. Then intermediaries have total equity contributions of 200 (the households' 100 plus the specialists' w_t). Since intermediaries hold all of the risky asset worth 300, their portfolio share in the risky asset must be equal to 150%. Their portfolio share in the bond is -50%. That is, the intermediary holds a levered position in the risky asset. The household's portfolio shares are $0.5 \times 150\% = 75\%$ in risky asset; and, 25% in debt. The households and specialists have different portfolio exposures to the risky asset. But since the specialist drives the pricing of the risky asset, risk premia must adjust to make the 150% portfolio share optimal.

From this situation, suppose that dividends on the risky asset fall. Then, since the specialists are more exposed to the risky asset than households, w_t falls relative to w_t^h. The shock then further tightens the intermediation constraint, which creates an amplified response to the shock.

Contrast this situation with one in which there is no intermediation constraint. Suppose that households invest all of their wealth with the intermediaries. Since intermediaries now have 300 and the risky asset is worth 300, the portfolio share of both specialists and households is equal to 100%. Both agents share equally in the asset's risk and shocks do not affect the distribution of wealth between the agetns.

3.2 State Variables and Specialists' Euler Equation

We look for a stationary Markov equilibrium where the state variables are (y_t, D_t), where $y_t \equiv \frac{w_t^h}{D_t}$ is the dividend scaled wealth of the household. As the example illustrates, the intermediation frictions depend on the distribution of wealth between households and specialists. We capture this relative distribution by y_t.

As standard in any CRRA/GBM economy, our economy is homogeneous in dividends D_t. We conjecture that the equilibrium risky asset price is,

$$P_t = D_t F(y_t), \tag{13}$$

where $F(y)$ is the price/dividend ratio of the risky asset.

Now we use the agents' optimal decisions and market clearing conditions to derive the equation for F. While the household faces investment restrictions on his portfolio choices, the specialist (intermediary) is unconstrained in his portfolio choices. This important observation implies that the specialist is always the marginal investor in determining asset prices, while the household may not be. Standard arguments then tell us that we can express the pricing kernel in terms of the specialist's equilibrium consumption process.

We have noted in (8) that the household's optimal consumption given w_t^h is $c_t^h = \rho w_t^h$, which we can rewrite as $c_t^h = \rho y_t D_t$. Now the market clearing condition for goods (from (12)) is,

$$c_t + \rho y_t D_t = D_t(1 + l).$$

Thus, in equilibrium, the specialist consumes:

$$c_t = D_t(1 + l - \rho y_t). \tag{14}$$

We thereby express specialist consumption as a function of the state variables D_t and y_t.

Optimality for the specialist gives us the standard consumption-based asset pricing relations (Euler equation):[11]

$$-\rho dt - \gamma E_t\left[\frac{dc_t}{c_t}\right] + \frac{1}{2}\gamma(\gamma+1)Var_t\left[\frac{dc_t}{c_t}\right] + E_t\left[dR_t\right] = \gamma Cov_t\left[\frac{dc_t}{c_t}, dR_t\right]; \tag{15}$$

and for interest rate, we have

$$r_t dt = \rho dt + \gamma E_t\left[\frac{dc_t}{c_t}\right] - \frac{\gamma(\gamma+1)}{2}Var_t\left[\frac{dc_t}{c_t}\right]. \tag{16}$$

Using (14) and (13), we can express dR_t and $\frac{dc_t}{c_t}$ as a function of the derivatives of $F(y)$, and the unknown drift and diffusion of y_t. These unknown drift and diffusion will depend on the households' equilibrium portfolio choices, which is the focus of the next section. Combining these results, we arrive at a differential equation that must be satisfied by $F(y)$ (see Appendix A).

3.3 Dynamics of Household Wealth

Given the wealth dynamics of the household in (10) and the intermediary return $\widetilde{dR}_t - r_t dt = \alpha_t^I\left(dR_t - r_t dt\right)$, we have

$$dw_t^h = (lD_t - \rho w_t^h)dt + w_t^h r_t dt + \left(\alpha_t^h \alpha_t^I\right)(1-\lambda)w_t^h\left(dR_t - r_t dt\right).$$

We now determine the household's exposure to the risky asset return $\left(\alpha_t^h \alpha_t^I\right)(1-\lambda)$. First, note that when the intermediation constraint of equation (5) binds, the household choice must satisfy

$$\alpha_t^{h,const}(1-\lambda)w_t^h = m w_t, \quad \text{which implies,} \quad \alpha_t^{h,const} = \frac{m\left(F\left(y\right) - y\right)}{(1-\lambda)y}. \tag{17}$$

That is, the binding constraint pins down the household's portfolio share in the intermediary. Moreover, since all risky assets are held through the intermediary, the equilibrium market clearing condition (11) gives,

$$\frac{\alpha_t^{I,const}(w_t + m w_t)}{P_t} = 1.$$

[11] The Euler equation is a necessary condition for optimality. In Appendix B, we prove sufficiency.

Using the fact that $w_t + w_t^h = P_t$, we find,

$$\alpha_t^{I,const} = \frac{1}{1+m}\frac{F(y)}{F(y)-y}. \tag{18}$$

The logic in arriving at this expression is the same as in the example.

When the intermediation constraint does not bind, the household is unconstrained in choosing α_t^h. We make an assumption that implies that $\alpha_t^h = 1$ in this case:

Parameter Assumption 1 *We focus on parameters of the model such that in the absence of any portfolio restrictions, the risky asset household will choose to have at least* 100% *of his wealth invested in the intermediary, i.e.,* $\alpha_t^h = 1$.

Although we are unable to provide a precise mathematical condition for this parameter restriction, in our calibration it appears that $\gamma \geq 1$ is a sufficient condition. Loosely speaking, if the specialist is more risk averse than the household, the household will hold more risky assets than the specialist. But given market clearing in the risky asset market, the specialist always holds more than 100% of his wealth in the risky asset. Recall that we assume that the household cannot short bonds. Thus, the household allocates the maximum of 100% of his wealth to the intermediary. Using the market clearing condition for risky assets, we find,

$$\alpha_t^{I,unconst} = \frac{F(y)}{F(y)-\lambda y}. \tag{19}$$

3.4 Constraint Threshold

We now characterize the conditions under which the intermediation constraint binds. Setting $\alpha_t^h = 1$ in (5) yields that the constraint binds when,

$$(1-\lambda)w_t^h \geq m w_t.$$

Using $w_t^h + w_t = P_t$, we rewrite the inequality to find an expression that gives a cutoff for the constrained states:

$$y^c = \frac{m}{1+m-\lambda}F(y^c).$$

This equation has a unique solution in all of our parameterizations.

In summary, when $y < y^c$, the intermediation constraint is binding, and we have the expressions for α_t^h and α_t^I as in (17) and (18). When $y < y^c$, the household chooses $\alpha_t^h = 1$ and α_t^I is given by (19).

3.5 Boundary Condition

The model has a natural upper boundary condition on y that is determined by the goods market clearing condition. Since

$$c_t = D_t \left(1 + l - \rho y_t\right),$$

and the specialist's consumption c_t must be positive, y_t has to be bounded by

$$y^b \equiv \frac{1+l}{\rho}.$$

In Appendix B, we show that y^b is an entrance-no-exit boundary, and that y_t never reaches y^b.

On an equilibrium path in which y approaches y^b, the specialist's equilibrium consumption c goes to zero. Since the specialist's wealth is $w = D\left(F\left(y\right) - y\right)$, one natural guess for the boundary condition at this singular point y^b is

$$F\left(y^b\right) = y^b. \tag{20}$$

In words, when the specialist's consumption approaches zero, his wealth also converges to zero. In the argument for verification of optimality of the specialist's equilibrium strategy which is detailed in Appendix B, we see that this condition translates to the transversality condition for the specialist's budget equation. Therefore the boundary condition (20) is sufficient for the equilibrium presented in this paper to be well-defined.

4 Calibration

4.1 m and λ

Table 1 provides data on the main intermediaries in the US economy. Households hold wealth through a variety of intermediaries including banks, retirement funds, mutual funds, and hedge funds.[12] This section explains how we map the institutions of Table 1 into our model, and how we calibrate m and λ.

Our model treats the entire intermediary sector as a group of identical institutions, while it is clear from Table 1 that there is heterogeneity across the modes of intermediation. The model takes a broad-brush approach at the effects of intermediation on asset prices. One aspect of intermediary heterogeneity which is important to discuss further is that some of the intermediaries have no debt positions and never take on debt while other do take on debt (see the last column in Table 1).

[12] We need to be careful in interpreting these numbers because there is some amount of double counting – i.e. pension funds invest in hedge funds.

Table 1: Intermediation Data[a]

Group	Assets[b]	Debt	Leverage
Commercial Banks	9,156	8,240	0.90
Savings & Loans	1,749	1,670	0.95
Property & Casualty Insurance	1,242	803	0.65
Life Insurance	4,351	4,076	0.94
Private Pensions	4,527	0	0.00
Government Ret Funds	3,69	0	0.00
Mutual Funds (excluding Money Funds)	5,882	0	0.00
Hedge Funds	3,406	2,433	0.71

[a] Most data is from the Flow of Funds Q3 2005 Levels Tables. The Hedge Fund data is based on an estimate of total hedge fund capital of $973 billion from Fung and Hsieh (2006) and an estimate that the average fund leverages up its capital base 3.5 times (taken from McGuire, Remolona and Tsatsaronis (2005))

[b] Assets and Debt are in billions of Dollars

In our model, when the intermediation constraint (5) binds, losses among intermediaries lead households to reduce their equity exposure to these intermediaries. If the intermediaries scale down their asset holdings proportionately, the asset market will not clear – i.e. the intermediary sector's assets still have to be held in equilibrium. In our model, the equilibrium is one where the [identical] intermediaries take on debt and hold a riskier position in the asset.

In practice, if households withdraw money from mutual funds, then mutual funds do not take on debt. Rather, they reduce their holdings of financial assets and some other entity buys their financial assets. The other entity may be a hedge fund that temporarily provides liquidity to the mutual fund, or it may be another mutual fund that buys the liquidated assets. If on the one hand, the buyers have no trouble raising equity to finance this purchase, then the leverage of the financial sector is unchanged. Only the identity of who holds the assets change. This case corresponds to our unconstrained region. On the other hand, if all potential buyers have difficulty raising equity to fund the purchase – e.g., they themselves have suffered losses and had withdrawals – then they have to raise the funds via debt. This is the case of our constrained region. In practice, such buyers will likely be hedge funds who temporarily increase leverage rather than a mutual fund that does not operate through leverage.

Thus, mapping our model to practice, we see that heterogeneity plays a role in dictating who among the intermediary sector increases their exposure to the risky asset during a period of liquidation. However, we note that when condition (5) binds, the marginal investor, both in the model and practice, prices assets based on concentrated risk exposure/leverage considerations. In this sense, our model captures the marginal investor's preferences well despite omitting heterogeneity.[13] Moreover, as our discussion suggests, the leveraged investor who plausibly is the marginal buyer during times of crises is an investor like a hedge fund.

We are interested in matching crises and particularly asset price behavior in the constrained region. We therefore choose the intermediation multiplier m to match data on hedge funds.

We note that m can be interpreted to measure the specialist's inside stake in the intermediary relative to the household's. That is, our constraint dictates that a hedge fund manager must receive at least $\frac{1}{1+m}$ of the hedge fund's returns. Hedge fund contracts typically pay the manager 20% of the fund's return in excess of a benchmark, plus $1-2\%$ of funds under management (Fung and Hsieh, 2006). A value of $m = 4$ implies that the specialist's inside stake is $1/5 = 20\%$. The 20% is an option contract so it is not a full equity stake. The 1% is on funds under management and therefore grows as the fund is successful and garners more inflows. Thus, a 20% stake is in the range of parameters that may reasonably capture a hedge fund manager's inside stake. We also show an $m = 6$ case to provide a sense as to the sensitivity of the results to the choice of m.

Our choice of m affects the dynamics of leverage and risk concentration *conditional on being in the constrained region*. In practice, we can see from Table 1 that the intermediary sector always has some leverage. Matching leverage even in the unconstrained region is important, because leverage affects how dividend shocks lead the state to transit from unconstrained to constrained region.

We choose the parameter λ to match leverage in the unconstrained region. In this region, we interpret the marginal investor as being an amalgam of all intermediaries. Within the model, when $\lambda > 0$ some households only demand debt, and the intermediaries supply the debt and thereby achieve leverage even when intermediation is not constrained.

Across all of the intermediaries of Table 1, the Total Debt/Total Assets ratio is 0.50. However, Banks and Savings & Loans do not only hold traded assets, which are the subject of our model; they hold non-traded assets (e.g., commercial loans) as well as traded assets (e.g., mortgage backed securities). If we say that Banks and Savings & Loans have 50% of their assets in traded or securitized assets and compute aggregate leverage based on 50% of the debt and assets of Banks and Savings & Loans, the aggregate leverage is 0.43. If we

[13] It is worth pausing and also considering the effect of debt constraints on the dynamic we describe. In 1998, when LTCM ran into trouble, their credit lines were reduced, preventing them from increasing leverage to hold risky assets. Of course, their risky assets still had to be held by someone in equilibrium. In practice, the investment banking community and trading desks absorbed these assets; risk exposures became more concentrated and levered in the few players that remained, exactly as in our model. In 1998, AIG and Warren Buffet offered to buy LTCM's portfolio. Presumably their bid prices were dictated by the next best bid for the assets represented by the investment banking community.

exclude Banks and Savings & Loans completely, the aggregate leverage is 0.31.

We set $\lambda = 0.5$, which produces leverage in the unconstrained region around 0.42, and an unconditional average leverage ratio around 0.49.

4.2 σ and g

The asset payoffs we price are ones where investment requires some expertise. To be concrete, these intermediated payoffs may stem from mutual-fund/hedge-fund trading of individual stocks, mutual-fund/hedge-fund/bank investments in mortgage-backed securities, mutual-fund/hedge-fund/bank/insurance company investments in corporate debt or credit derivatives, hedge-fund trading of portfolios of risky assets based on statistical analysis, or intermediaries' provision of short-term liquidity to financial markets.

Ideally, we would like information on the cash-flows of the amalgam of these intermediated payoffs to set g and σ. We do not have such data. Instead, we base g and σ on the aggregate stock market. Our logic here is that the aggregate stock market is likewise an amalgam of payoffs, and may have similar cash-flow characteristics. We set $\sigma = 12\%$ and $g = 1.84\%$ (see, e.g., Barberis, Huang, and Santos, 2001).

Of the two parameters, σ is the critical one because it is closely related to the amount of risk borne by the specialist and the volatility of the pricing kernel. The choice of $\sigma = 12\%$ produces an equilibrium return volatility in our model between 12% and 13%.

Mortgage-backed securities (MBS), as we have noted, are a good example of an intermediated payoff and a relatively large asset class. The MBS market is close to \$10tn in size. Based on data from Lehman Brothers from 1976 to 2005, we find that the annual return volatility on the universe of mortgage-backed securities is 8%. Note that real estate prices primarily rose over this period, suggesting that the 8% number understates the true volatility (as we are seeing currently, as prices decline). Thus, our 12% calibration is within the range of numbers suggested by the MBS market.[14]

[14] Our choice of $\sigma = 12\%$ is an order of magnitude higher than aggregate consumption volatility of close to 3%. In standard general equilibrium approaches to asset pricing, exemplified by Campbell and Cochrane (1999) or Barberis, Huang, and Santos (2001), models assume a representative agent whose consumption is equal to NIPA aggregate consumption and price a payoff with a dividend stream that matches properties of aggregate stock market dividends.

The marginal investor in our model is the specialist-intermediary rather than a representative agent because intermediaries are not a veil. As our analysis shows, the specialist's marginal utility is endogenously affected by fluctuations in the value of assets that the specialist holds. Thus, we do not exogenously specify the marginal investor's consumption process based on aggregate consumption, but endogenously derive the joint behavior of specialist consumption and the prices of intermediated assets. For this reason, we choose the volatility of the risky asset's dividends to match those of financial payoffs rather than that of aggregate consumption. Indeed, we see the endogenous relationship between financial wealth fluctuations and the pricing kernel as an

Table 2: Parameters

Panel A: Intermediation		
m	Intermediation multiplier	4, 6
λ	Debt ratio	0.5
Panel B: Preferences and Cashflows		
g	Dividend growth	1.84%
σ	Dividend volatility	12%
ρ	Time discount rate	8%
γ	RRA of specialist	2
l	Household labor income ratio	1

4.3 l, γ, and ρ

We choose l to match the income profile of typical household. In our model, households receive expected capital income of $E\left[w_t r_t dt + (1-\lambda)\alpha_t^h \left(\widetilde{dR}_t - r_t dt\right)\right]$ and expected labor income of $E[lD_t dt]$. In NIPA data, capital income as a share of GDP is about 30%. Malloy, Moskowitz, and Vissing-Jorgenson (2006) report that for the top one-third of households in terms of wealth, the share of capital income in total income in 2001 was 34%. We choose l based on these considerations. We set $l = 1$ which produces a capital income to total income share around 35%.

We choose $\gamma = 2$ as risk aversion of the specialist. As noted earlier, the household has logarithmic preferences. Allowing for $\gamma > 1$ for the specialist allows us to capture dynamic hedging effects that would be absent if we set $\gamma = 1$. We choose ρ to match an average riskless interest rate of between 0% and 1%. This leads us to a value of ρ equal to 0.08. These numbers are all typical in the literature. Finally, our parameter choices are also dictated by the restriction that, $\rho + g(\gamma-1) - \frac{\gamma(\gamma-1)\sigma^2}{2} - \frac{l\gamma\rho}{1+l} > 0$. This restriction is necessary to ensure that the economy is well-behaved at $t = \infty$ (see Appendix A).

4.4 Numerical method

We present numerical solutions based on the calibration of Table 2. We use one of *MATLAB*'s built-in ODE solvers to derive solutions for $F(y)$, μ_y, and σ_y. Further details are provided in the Appendix A.

With these solutions in hand, we numerically simulate the model to obtain the steady state distribution of

important reason to model intermediaries rather than treat them as a veil.

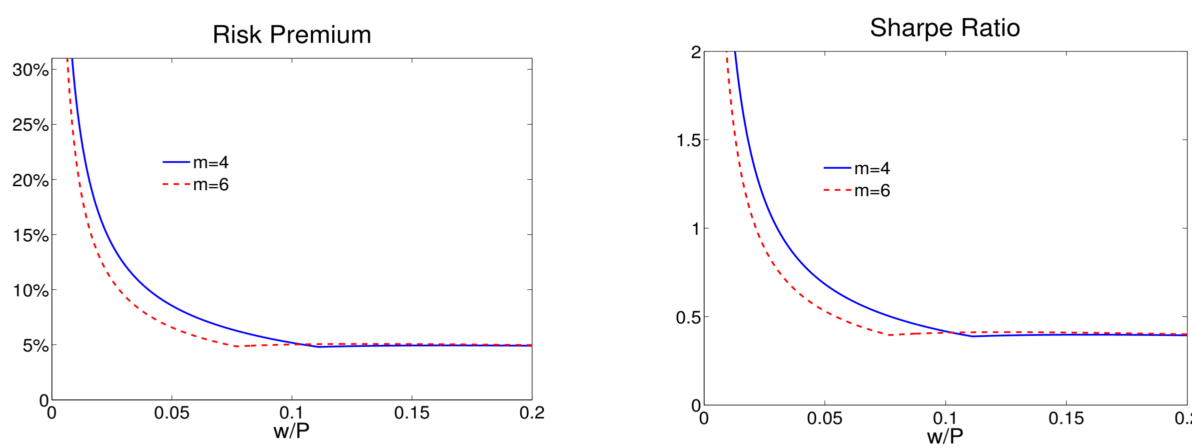
Finally, in principle it seems possible to reconcile the low aggregate consumption volatility we observe in practice with the 12% dividend volatility of the model by assuming that the household sector's labor income is weakly correlated with dividends (as in the data). Unfortunately, such a model will no longer be homogeneous with respect to dividends which will considerably complicate the analysis.

the state variable y as well as a number of asset price measurements that we report in the next sections. We begin the economy at a state ($y_0 = y^c, D_0 = 1$) and simulate the economy for 5000 years. That is we obtain a sequence of independent draws from the normal distribution and use these draws to represent innovations in our shock process Z_t. The path of Z_t can then be mapped into a path of the state variable. We compute the time-series averages of a number of relevant asset price measurements from years 1000 to 5000 of this sample. The simulation unit is monthly, and based on those monthly observations we compute annual averages. We repeat this exercise 5000 times, averaging across all of the simulated Z_t paths. We find that changing the starting value y_0 does not affect the computed distribution or any of the asset price measurements, indicating that the distribution truly represents the steady state distribution of the economy.

5 Crisis Behavior

5.1 Risk Premium and Sharpe Ratio

Figure 2: Risk Premium and Sharpe Ratio



Risk premium (left panel) and Sharpe ratio (right panel) are graphed against w/P, the specialist wealth as a percentage of the assets held by the intermediation sector. Parameters are $m = 4, \lambda = 0.5$ and $m = 6, \lambda = 0.5$ and those given in Table 2. The cutoff for the constrained region for the two cases are 0.11 ($m = 4$ case) and 0.08 ($m = 6$) case.

Figure 2 graphs the risk premium and Sharpe ratio for the two calibrations ($m = 4, 6$) as a function of the specialist wealth relative to the value of the risky asset (w/P). The latter ratio can be interpreted as the inside capital of the intermediation sector as a percentage of the assets held by the intermediation sector.

Even though we solve our model based on the household's scaled wealth $y = w^h/D$, we decide to illustrate our results using w/P in order to more clearly discuss the role of intermediation capital.

The prominent feature of our model, clearly illustrated by the graphs, is the asymmetric behavior of the risk premium and Sharpe ratio. The right hand side of the graphs represent the unconstrained states of the economy, while the left hand side represent the constrained states. Risk premia and Sharpe ratio rise as specialist wealth falls in the constrained region, while being relatively constant in the unconstrained region.

This asymmetric behavior is intuitively what one would expect from the model: the model's intermediation constraint is by its nature asymmetric, and binding only when specialist wealth is low. To sharpen understanding of the mapping between the constraint and risk premia, consider the following calculation. As noted above, the pricing kernel in our model can be expressed in terms of the specialist's consumption. Thus, the risk premium on the risky asset is equal to:

$$\gamma \; cov_t \left(\frac{dc_t}{c_t}, dR_t \right)$$

To a first-order approximation, the volatility of the specialist's consumption growth is equal to the volatility of the return on his wealth (the approximation is exact if $\gamma = 1$). Thus,

$$var_t \left(\frac{dc_t}{c_t} \right) \approx \left(\alpha_t^I \right)^2 \; var_t(dR_t),$$

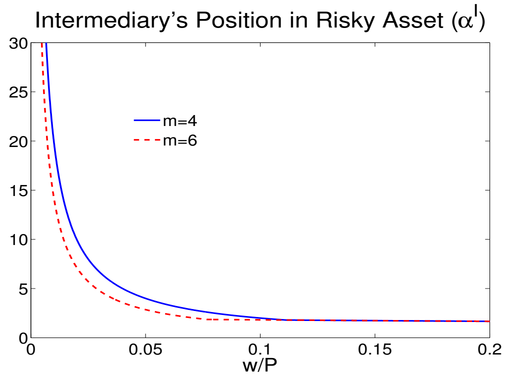
where α_t^I is the portfolio exposure to the risky asset in the intermediary's (and specialist's) portfolios. Therefore, the risk premium is approximately,

$$\gamma \left(\alpha_t^I \right)^2 var_t \left(dR_t \right).$$

In our model, the variance of returns is roughly constant as a function of state (see the discussion of this point below). Most of the action in the risk premium comes from the changing α_t^I. We have noted before that in the constrained region, as households withdraw from intermediaries and limit their participation in the risky asset market, the specialists increase their exposure to the risky asset (see equation (18)). This dynamic, driven through α_t^I, explains the behavior of the risk premium. Figure 3 graphs α^I. We note the close correspondence between this graph and those in Figure 2.

Figures 2 and 3 are graphed for the two cases, $m = 4$ and $m = 6$. The cutoff for the constrained region for these two cases are 0.11 ($m = 4$ case) and 0.08 ($m = 6$) case. The larger m leads to a narrower constrained region ("constraints" effect). As discussed in Section 3.4, when m is larger the specialist is able to raise more external capital based on any given level of his own wealth. Thus his wealth has to be lower in order to fall into the constrained region. Comparing the slopes of the risk premium graph, in the constrained region, for

Figure 3: Portfolio Holdings



The intermediary's portfolio share in the risky asset (α^I) is graphed in the left panel against w/P. Parameters are $m = 4, m = 6$ and those given in Table 2.

the $m = 4$ and $m = 6$ cases, the higher m case resembles a convex transformation of the lower m case. In particular, deep into the constrained region, the risk premium is more sensitive to changes in specialist wealth when m is larger. This is a "sensitivity" effect. When m is higher and the constraint is binding, a \$1 fall in specialist wealth leads to an \$m reduction in household contributions to the intermediary, creating the sharper response of the risk premium.

5.2 Discussion: Leverage

Figure 3 may also be read as showing that the rise in the risk premium in the constrained region is closely related to the rise in leverage. This association seems counterintuitive when viewed in light of the financial press, where crises typically accompany reports of financial institutions selling assets at fire-sale prices in order to pay down debts. Papers such as Kiyotaki and Moore (1997), Gromb and Vayanos (2005) and Brunnermeier and Pedersen (2008) develop models in which falling asset prices causes agents to reduce their borrowings.

The reason for this discrepancy is that our model operates under the logic that, even after a negative shock, in equilibrium the risky asset must be held by the intermediary sector. Moreover, because our model has identical intermediaries, and because negative shocks reduce the ability of intermediaries to issue equity relative to debt, market clearing implies that these identical intermediaries borrow in the debt market to hold the risky asset.[15]

In contrast to our paper, both Kiyotaki and Moore (1997) and Brunnermeier and Pedersen (2008) posit a second-best buyer, modeled as a downward sloping demand function, for the agent's liquidated assets. Thus, agents shed debt in equilibrium and sales of the assets to the second-best buyer result in prices falling. The main problem with this approach is that the second-best buyer is left unmodeled and yet is central to price determination.

In a sense, the ideal model is one in which there is heterogeneity within the intermediary sector. With such heterogeneity, it is likely that constraints bind more tightly for some institutions than others. The institutions with the tight constraints delever and sell with other institutions buying the sold assets. Note that if one is primarily interested in the pricing of assets, it is the marginal condition for the buyers that needs to be analyzed. Our paper focuses on this marginal pricing condition and links it to leverage and concentrated risk exposure. In practice, leverage surely must rise when an institution such as J.P. Morgan takes over Bear Sterns, or when Goldman Sachs invests some resources to bailout one of their funds. Of course, as is widely appreciated, precisely measuring the economic leverage of a financial institution is difficult. Nevertheless, the theoretical logic that someone in the intermediary sector has to hold the assets in equilibrium is hard to counter in a general equilibrium model.

The second point of difference between our model and the earlier literature lies in nature of the financing constraint. In our model, intermediaries face financing constraints in raising equity from households. In Kiyotaki and Moore (1997), Gromb and Vayanos (2005) and Brunnermeier and Pedersen (2008), a lower net worth w restricts the amount of debt that an agent can contract, and thereby leads to less debt. Note that these models implicitly rule out equity contracts – debt is the only margin that adjusts with w. Again, it is worth imagining an ideal model with both debt and equity contracts. Borrowing through both contracts are affected by negative shocks to w. However, since equity is junior to debt, it is likely that the equity constraint is more severely affected than the debt constraint. If the intermediary sector has to hold the risky asset, it is likely in equilibrium that adjustment occurs on the less constrained margin. Our model studies the extreme case of no constraint on debt borrowing and only constraints on equity borrowing, and leverage rises following negative shocks.

[15] Alternatively, a tightening of the constraint does, ceteris paribus, induce a reduction in the portfolio position of the intermediary. However, since the risky asset must in equilibrium be held by the intermediary sector, this reduction cannot occur, and prices must adjust.

5.3 Discussion: Asymmetry

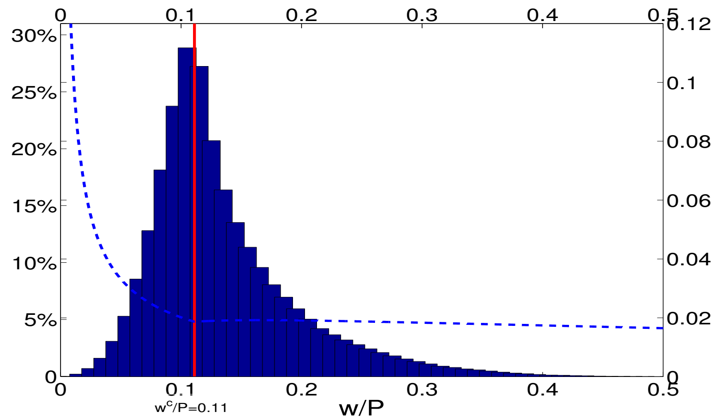
An interesting point of comparison for our results is to the literature on state-dependent risk premia, notably, Campbell and Cochrane (1999), Barberis, Huang, and Santos (2001), and Kyle and Xiong (2001). In these models, as in ours, the risk premium is increasing in the adversity of the state. Campbell and Cochrane and Barberis, Huang, and Santos modify the utility function of a representative investor to exhibit state-dependent risk aversion. We work with a standard CRRA utility function, but generate state dependence endogenously as a function of the frictions in the economy. For empirical work, our approach suggests that measures of intermediary capital/capacity will help to explain risk premia. In this regard, our model is closer in spirit to Kyle and Xiong who generate a risk premium that is a function of "arbitrageur" wealth. The main theoretical difference between Kyle and Xiong and our model is that the wealth effect in their model comes from assuming that the arbitrageur has log utility, while in our model it comes because the intermediation constraint is a function of intermediary capital. One notable distinction of our model is the sharp asymmetry of our model's risk premia: a muted dependence on capital in the unconstrained region and a strong dependence in the constrained region. In Kyle and Xiong, the log utility assumption delivers a risk premium that is a much smoother function of arbitrageur wealth. Plausibly, to explain a crisis episode, one needs the type of asymmetry delivered by our model.

5.4 Steady State Risk Premia

Quantitatively, as one can see from Figure 2, the calibration produces a risk premium in the unconstrained region of around 5%. The numbers for the risk premium are higher in the constrained region; however, without knowing the probability that a given specialist-wealth state may occur, it is not possible to interpret a statement about how much higher. To provide some sense for the values of the risk premium we may be likely to observe in practice, we simulate the model as described in Section 4.4 and compute the equilibrium probability of each state. The resulting steady state distribution over the specialist wealth as a percentage of the assets held by the intermediation sector (w/P) is graphed in Figure 4 (for the $m = 4$ case). Also superimposed on the figure in a dashed line is the risk premium from the previous graph.

There are two forces driving the center-peaked distribution in Figure 4. First, as w/P falls, the risk premium rises. This in turn means that the specialist, who is holding a levered position in the risky asset, increases his wealth on average. This force is stronger as the risk premium rises, which is why the distribution places almost no weight on risk premia as high as 30%. At the other end, when w/P is large so that w^h is

Figure 4: Steady State Distribution



The steady state distribution of w/P is graphed for the $m = 4$ case. The vertical line gives the state where the intermediation constraint starts binding. The dashed line graphs the risk premium in order to illustrate the actual range of variation of the risk premium. Risk premium is indicated on the left scale, while the distribution is indicated on the right scale.

small, the households are poor and consuming little but still receive labor income. Thus, their wealth grows as they save the labor income, which shifts the wealth distribution back towards the constrained region.

Table 3 provides further information on the range of variation of the state variable. The economy spends most of the time in the unconstrained region (59% and 68% for the two cases). We may think of the unconstrained region as a "normal" non-crisis period. The average risk premium and Sharpe ratio, conditional on being in the unconstrained region, are around 4.9% and 0.40, respectively (Panel A). In the constrained region, the risk premium rises averaging close to 6%. The probability that the risk premium will exceed 7.5% is 4.35%. For the risk premium to exceed 10%, which is about double the unconstrained region average in terms of both risk premium and Sharpe ratio, the probability is 1.04% (Panel B). An extreme crisis that increases risk premia and Sharpe ratio about 4.5X to 20% is very unlikely, in keeping with the historical record. Our model puts this probability around 0.05%.

To put these numbers in perspective, consider the 1998 crisis. Figure 5 graphs the behavior of the high grade credit spread (AAA bonds minus Treasuries), the spread on FNMA mortgage backed securities relative to Treasuries, and the option adjusted spread on volatile interest-only mortgage derivative securities (data are from Gabaix, Krishnamurthy, and Vigneron, 2007). The spreads are graphed over a period from 1997 to

Table 3: Measurements

Panel A: Constrained and Unconstrained Regions

This panel presents a number of average measurements for the economy, broken down into conditional on being in the constrained region, conditional on being in the unconstrained region, and unconditional average. Parameters for the two cases reported are given in Table 2.

	$m = 4$ Case			$m = 6$ Case		
	Avg.	Unconst.	Const.	Avg.	Unconst.	Const.
Probability		59.09	40.91		67.75	32.25
Risk Premium (%)	5.34	4.87	6.01	5.32	4.97	6.03
Sharpe Ratio (%)	42.91	39.22	48.24	43.06	40.34	48.97
Interest Rate (%)	0.50	0.88	−0.045	0.50	0.79	−0.099
Leverage Ratio (%)	47.56	41.43	56.42	47.67	42.86	57.72
Income Ratio (%)	36.89	40.58	31.55	40.07	41.76	36.36

Panel B: Measures at Different Risk Premia

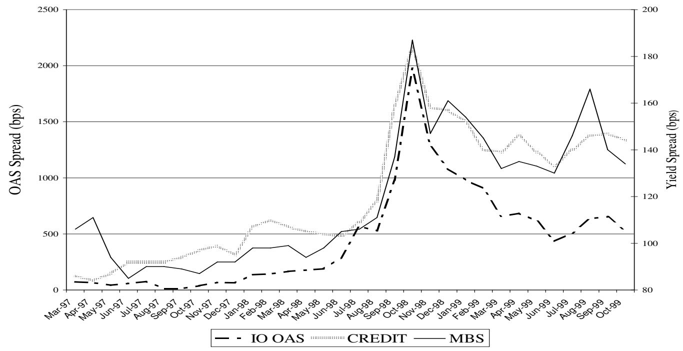
The different risk premia at which we compute the various measures are given in the first row (denoted π). The second row reports the probability that the economy will ever reach a value of risk premium greater than the given π. The rest of the rows report measures at the given π.

	$m = 4$				$m = 6$			
Risk Premium (%) $\equiv \pi$	5%	7.5%	10%	20%	5%	7.5%	10%	20%
Prob (Risk Premium$> \pi$)	34.37	4.35	1.04	0.05	63.31	3.33	0.81	0.04
Sharpe Ratio at π	40.35	59.93	80.60	170.82	40.75	60.76	81.21	169.63
Interest Rate at π	0.84	−1.36	−3.71	−13.82	0.84	−1.46	−3.81	−13.92
Leverage Ratio at π	47.55	70.02	79.92	92.24	48.56	70.94	80.33	92.10

1999 and includes the fall of 1998 hedge fund crisis. During 1997 and upto the middle of 1998 spreads move in a fairly narrow range. If we interpret the unconstrained states of our model as this "normal" period, then the muted response of risk premia to the state can capture this pre-crisis period. In a short period around October 1998 spreads on these securities increase sharply. The credit spreads and MBS spreads double from their pre-crisis level. The mortgage derivative spread increases by many multiples. Although it is hard to estimate precisely how much Sharpe ratios increase during the episode, a doubling is plausibly within the range of estimates. Certainly from the standpoint of standard representative household asset pricing models, even a 50% increase during the 1998 event is difficult to understand as aggregate consumption was barely at risk. In our model, the asymmetry in the intermediation constraint calibrated to hedge fund data can generate the dramatic increase in risk premia around crises. Many observers comment on the reduction in intermediation capital in the fall of 1998 crisis, and refer to the episode as a tail event. Both statements make sense from our calibration.

Table 3 also provides a sense as to the effect of varying m, by comparing the two cases represented. There are two – but almost offsetting – effects of m. Raising m lowers the probability of the constrained region from 41% to 32%. However it increases the probability that the risk premium will exceed 5% from 34% to 63%,

Figure 5: Crisis Spreads



The spreads between the Moody's index of AAA corporate bonds and the 10 year Treasury rate (grey line), the spreads between FNMA 6% TBA mortgage-backed securities and the 10 year Treasury rate (black line), and the option-adjusted spreads on a portfolio of interest-only mortgage-backed securities relative to Treasury bonds (dashed line) are graphed monthly from 1997 to 1999.

while leaving the probabilities at the more extreme points relatively unaffected. The constrained region gets smaller, but the probability mass becomes concentrated at a slightly higher risk premium. The net effect is almost a wash, and the average risk premium across the two cases is within two basis points.[16]

5.5 Flight to quality

The row in Table 3, Panel A corresponding to the interest rate shows that the interest rate falls from an average of 0.88% in the unconstrained region to -0.045% in the constrained region. There are two intuitions behind this fall in interest rates. First, as the specialist's consumption volatility rises with the tightness of the intermediation constraint, the precautionary savings effect increases specialist demand for the riskless bond. Second, as specialist wealth falls, households withdraw equity from intermediaries, increasing their demand for the riskless bond. To clear the bond market, the equilibrium interest rate has to fall. Both the behavior of the interest rate and the disintermediation-driven demand for bonds is consistent with a flight to quality.

However, we can also see from the table that the interest rate is over-sensitive to the state in our model. At the 7.5% risk premium state, the interest rate is around -1.40%, falling to -3.7% at the 10% risk premium

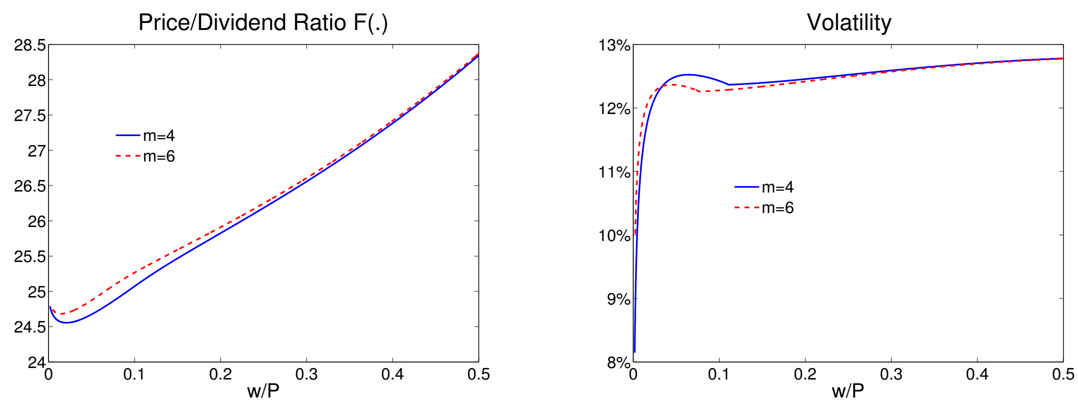
[16] Table 3, Panel A also includes measures on which we calibrate λ and l. The average income ratio from the data is about $30 \sim 35\%$; our numbers in the table are closer to 37% and 40%. The leverage ratio suggested by the data is around 0.43. This number is close to the leverage ratio conditional on being in the unconstrained region.

state (for the $m = 4$ case).

The main reason for this over-sensitive interest rate is that we are pushing the general equilibrium of our model too far. Our model-economy consists of only an intermediation sector and therefore ascribes all movements in interest rates to shocks within that sector. In practice, part of the demand for bonds in the economy is from sectors that are unaffected by the intermediation constraint, so that it is likely that our model overstates the interest rate effect. However, it also does not seem appropriate to fix the interest rate exogenously, since interest rates do fall during a crisis episode. Thus, while the qualitative prediction of our model for interest rates seems correct, the quantitative implications are the least credible results of our analysis.

5.6 Price/Dividend Ratio and Volatility

Figure 6: P/D Ratio and Volatility



Price/Dividend ratio (left panel) and risky asset return volatility (right panel) are graphed against the specialist wealth as a percentage of the assets held by the intermediation sector (w/P). Parameters are $m = 4$ and $m = 6$ and those given in Table 2.

The left-hand panel of Figure 6 graphs the price/dividend ratio $F(\cdot)$ against w/P. Consistent with intuition, over most of the range, $F(\cdot)$ falls as specialist wealth falls. There is a non-monotonicity that arises when the specialist wealth is very small – although this occurs for values of w/P for which the steady state distribution places very little weight (see Figure 4). The non-monotonicity arises because interest rates diverge to negative infinity when the specialist wealth approaches zero. There are two forces affecting the discount rates applied to dividends in determining $F(\cdot)$: On the one hand, the risk premium is high when the specialist wealth is low;

on the other hand, the interest rate is low for higher specialist wealth. These two effects combine to produce the non-monotonicity of $F(\cdot)$.

The right-hand panel of Figure 6 gives the pattern of the risky asset return volatility when the specialist wealth varies. Over most of the relevant range of variation of the state variable, the volatility is constant between 12% and 13%. In particular, the model fails to replicate the observed increase in conditional volatility accompanying a crisis period.

The non-monotonicity in $F(\cdot)$ also causes volatility to fall in the region where w/P approaches zero. The risky asset price is equal to $D_t \times F(y_t)$. The non-monotonicity means that a shock that causes a fall in D_t leads to a rise in F (y_t is negatively correlated with D_t). We stress again that the steady state distribution places almost no weight on these small values of w/P.

5.7 Capital movement and recovery from crisis

Referring to Figure 5, the corporate bond spread and MBS spread widen from 90 bps in July 1998 to a high of 180 bps in October 1998 before coming down to 130 bps in June 1999. Thus, the half-life — that is, the time it takes the spread to fall halfway to the pre-crisis level — is about 10 months. The interest-only mortgage derivative spread, which is very sensitive to market conditions, widens from 250 bps in July 1998 to a high of 2000 bps before coming back to 500 bps in June 1999. We note that this timescale for mean reversion, on the order of months, is much slower than the daily mean-reversion patterns commonly addressed in the market micro-structure literature (e.g., Campbell, Grossman, and Wang, 1994).

A common wisdom among many observers is that this pattern of recovery reflects the slow movement of capital into the affected markets (Froot and O'Connell, 1999, Berndt, et. al., 2004, Mitchell, Pedersen, and Pulvino, 2007, Duffie, Garleanu, and Pedersen, 2007). Our model captures this slow movement. We will show in this section that our baseline calibration can also replicate these speeds of capital movement.

In the crisis states of our model, risk premia are high and the specialists hold leveraged positions on the risky asset. Over time, profits from this position increase w_t, thereby increasing the capital base of the intermediaries. The increase in specialist capital is mirrored by an m-fold increase in the allocation of households' capital to the intermediaries, as the intermediation constraint is relaxed. Together these forces reflect a movement of capital back into the risky asset market and lead to increased risk-bearing capacity and lower risk premia. Note, however, that one dimension of capital movement that plausibly occurs in practice but is not captured by our model is the entry of "new" specialists into the risky asset market.

We can use the model simulation to gauge the length and severity of a crisis within our model. Table 4 presents data on how long it takes to recover from a crisis in our model. We fix a state (y, D) corresponding to an instantaneous risk premium in the "Transit from" row. Simulating the model from that initial condition, we compute and report the first passage time that the state hits the risk premium corresponding to the "Transit to" column. The time is reported in years.

Table 4: Crisis Recovery

This table presents transition time data from simulating the model for the $m = 4$ and $m = 6$ cases. The rest of the parameters are those given in Table 2. We fix a state corresponding to an instantaneous risk premium of 20% ("Transit from"). Simulating the model from that initial condition, we compute and report the first passage time that the state hits the risk premium corresponding to that in the "Transit to" column. Time is reported in years. The column "Increment time" reports the time between incremental "Transit to" rows.

	$m = 4$		$m = 6$	
	Transit from 20	Increment time	Transit from 20	Increment time
Transit to				
15	0.22	0.22	0.21	0.21
12.5	0.47	0.25	0.43	0.22
10	1.00	0.53	0.93	0.50
7.5	2.61	1.61	2.46	1.53
6	5.81	3.20	5.53	3.07
5	12.95	7.15	12.26	6.74

If we start from the extreme crisis state of 20% and compute how long it takes to recover to 12.5% — i.e. halfway back to the unconditional average levels we report earlier of around 5% — the time is 0.47 years (5.5 months) for the $m = 4$ case. The transit times are uniformly faster for the $m = 6$ case because as specialist capital increases, households react more strongly in bringing their own capital back to intermediaries.

From the 10% crisis state to the 7.5% state takes 1.6 years in the $m = 4$ case. For the fall of 1998 episode, the half-life we suggested was around 10 months. The model half-life from 10% is larger, but is in the order of magnitude of the empirical observation. It is worth keeping in mind that it is difficult to measure exactly how high risk premia or Sharpe ratios were at the peak of the crisis. For example, if we judge that the peak risk premia corresponded to 13% in our model, then the half-life down to 8% is close to 10 months (not reported in the table).

One failing of the model that we see from Table 4 is the extremely slow recovery from 6% to 5%. Our simulations put these numbers around 7 years. Essentially, the specialist profits when the risk premium is 6% are so small that specialist capital grows at a modest speed, and hence our model predicts a correspondingly slow recovery time. In practice, the final stages of recovery from crisis may also lead to other "specialists" moving into the affected market. Our model does not capture such an effect. It is also worth noting that in practice, statistically distinguishing a 6% risk premium from a 5% risk premium is difficult, so that the recovery time from 6% to 5% may not be a meaningful measurement.

The slow adjustment of risk premia, in timescales of many months, during the 1998 episode is also consistent with other studies of crisis episodes. Berndt, et. al. (2005) study the credit default swap market from 2000 to 2004 and note a dramatic market-wide increase in risk premia (roughly a quadrupling) in July 2002 (see Figures 1 and 2 of the paper). Risk premia gradually fall over the next two years: From the peak in July 2002, risk premia halve by April 2003 (9 months). The authors argue that dislocations beginning with the Enron crisis led to a decrease in risk-bearing capacity among corporate bond investors. Mirroring the decreasing risk-bearing capacity, risk premia rose before slowly falling as capital moved back into the corporate bond market and expanded risk bearing capacity. Gabaix, Krishnamurthy, and Vigneron (2007) note a dislocation in the mortgage-backed securities in late 1993 triggered by an unexpected wave of consumer prepayments. A number of important hedge fund players suffered losses and went out of business during this period, leading to a reduction in risk bearing capacity. Figure 3 in the paper documents that risk premia reached a peak in December 1993 before halving by April 1994 (5 months). Froot and O'Connell (1999) study the catastrophe insurance market and demonstrate similar phenomena. When insurers suffer losses that deplete capital they raise the price of catastrophe insurance. Prices then gradually fall back to long-run levels as capital moves back into the catastrophe insurance market. Froot and O'Connell show that the half-life in terms of prices can be well over a year.[17]

Each of these markets are intermediated markets that fit our model well. Investors are institutions who have specialized expertise in assessing risk in their markets. Our theory explains the slow movement of risk bearing capacity and risk premia documented in these case-studies. The calibrated model also captures the frequency of the slow adjustment of risk premia.

6 Crisis Policy Experiments

We now study the effect of policy interventions in the crisis of the model. We study three policies: (1) Infusing equity capital into the intermediaries during a crisis; (2) Lowering borrowing rates to the intermediary, as with

[17] Mitchell, Pedersen, and Pulvino (2007) document similar effects in the convertible bond market in 1998 and again in 2005. In both cases, crisis recovery times are in the order of months. They also note that spreads in merger arbitrage strategies took several months to recover following the October 1987 risky asset-market crash.

a decrease in the central bank's discount rate; and, (3) Direct purchase of the risky asset by the government, financed by debt issuance and taxation of households. These three policies are chosen because they are among those undertaken by central banks in practice. Our aim is to quantify the effects of these policies on the equilibrium of our model. The analysis is purely positive, and we make no claims as to optimality.

To evaluate these policies, we compute two equilibria, one with the policy and one without the policy. We compare across these equilibria to provide a quantitative sense of the effect of policy.[18]

6.1 Capital infusion

A number of crisis interventions are aimed at increasing the equity capital of intermediaries. For example, in the Great Depression, the government directly acquired preferred shares in banks, thereby increasing their equity capital. In the ongoing subprime crisis, the U.S. Treasury has devoted \$250bn towards purchasing equity capital in the intermediary sector.

Within our model, we can investigate the impact of a capital infusion by comparing the economic conditions prevailing at two states with different specialist wealth. This comparison corresponds to evaluating the effect of a capital "gift," with the government not requiring any repayment (unlike actual proposals).

In our benchmark case, the extreme crisis state with risk premium of 20% corresponds to intermediary capital of $w/D = 0.38$, or $w/P = 1.55\%$. We think of P as corresponding to the aggregate investments of the intermediary sector. From the Flow of Funds tables of the Federal Reserve, the total household investment in intermediaries is on the order of \$15tn. This comes from summing bank deposits, mutual funds, pension funds, and life insurance reserves.

A transfer to the intermediary sector from the households of 0.5% of $P = 15tn$ (or \$75bn) moves the state from $w/D = 0.38$ to $w/D = 0.50$, lowering the risk premium to 16.28%. A larger transfer of 1% of P (\$150bn), which moves the state from $w/D = 0.38$ to $w/D = 0.63$, causes the risk premium to fall to 13.77%. In terms of dynamics, the risk premium falls immediately upon the transfer. From the lower risk premium the recovery follows the same path as that indicated in Table 4.

These numbers provide a sense of how much equity capital needs to be infused in order to lower the risk premium. The U.S. Treasury's proposed \$250bn equity infusion (1.67% of P) will lower the risk premium in

[18] This policy analysis is really a comparative static exercise. A more accurate policy experiment would be to study a policy that is expected to be enacted given some value of the state variable – say the government infuses equity capital if the risk premium touches 20%. Such a policy would be anticipated by agents within the equilibrium of the model. Analyzing such a policy does not pose any difficulty for our modeling structure, but requires further explanation and development of the model.

our model to 11.2%.

6.2 Borrowing Subsidy

During financial crises, the central bank lowers its discount rate and its target for the overnight interbank interest rate. Financial intermediaries rely heavily on rolling over one-day loans for their operation (see, for example, Adrian and Shin (2008) on the overnight repurchase market). Because of this dependence, intermediaries are perhaps the most sensitive sector within the economy to overnight interest rates. Commercial and investment banks have access to overnight funds at the discount window of the central bank. Thus, to the extent that the central bank lowers overnight rates, including the discount rate, it reduces the borrowing costs of financial intermediaries.

While our model does not have a monetary side within which to analyze how a central bank alters the equilibrium overnight interest rate, we can go some way towards examining the effect of this policy by studying the following transfer. The debt position of intermediaries at date t is $(\alpha_t^I - 1)w_t$. Suppose that the government makes a lumpsum transfer of $\Delta r \times (\alpha_t^I - 1)w_t dt$ from households to intermediaries, where Δr measures the size of the transfer. The transfer is proportional to the debt of the intermediary.

The subsidy experiment can be thought of as a reduction in the central bank's discount rate. In practice, when the central bank makes funds available more cheaply to the financial sector through the discount window it is transferring real resources from taxpaying households to the financial sector. However, since our model is cast in real terms, the subsidy is only a stand-in for a reduction in something like the overnight Federal Funds rate.

Formally, we examine an equilibrium where Δr is paid only if $w > w^c$. For $w < w^c$ there is no subsidy. We express this transfer of $\Delta r \times (\alpha_t^I - 1)w_t dt$ in terms of the primitive state variables y_t and D_t. Then, the dynamic budget constraints of household and specialist are altered to account for the transfer (see equation 10), and this change is traced through to rederive the ODE for the price/dividend ratio (see Appendix C for details).

Table 5 presents the results. We start the economy in the state corresponding to the 20% risk premium. The subsidy of Δr is provided to the intermediaries as long as the economy is in the constrained region. The table reports the recovery times from the 20% extreme crisis state for different levels of Δr. Consistent with intuition, a higher subsidy speeds up the recovery process. The 200 bps subsidy speeds up the recovery to 7.5% by 0.75 years, which is roughly 30% faster than the case of $\Delta r = 0$. Note that from the inception of the

Table 5: Borrowing Subsidy

This table presents transition time data from simulating the model for the $m = 4$ case. We begin in the 20% risk premium state and report the first passage time for the state to reach that in the first column of the table ("Transit to" column). Time is reported in years. We report the case of no subsidy ($\Delta r = 0$), as well as subsidies of $0.005, 0.02$, and 0.045. A subsidy of 0.005 corresponds to 50 *bps*.

Transit to	$\Delta r = 0$	$\Delta r = 0.005$	$\Delta r = 0.02$	$\Delta r = 0.045$
15	0.22	0.21	0.17	0.07
12.5	0.47	0.43	0.34	0.19
10	1.00	0.93	0.73	0.46
7.5	2.61	2.40	1.86	1.28
6	5.81	5.30	4.10	2.66
5	12.95	11.66	8.93	5.67

subprime crisis in August 2007 to October 2008, the discount rate decreased by 450 bps. The last column in the table indicates the effect of this policy within our model.

Another way to look at these numbers is to compare them to the capital infusion experiment. There, about \$150*bn* has to be spent in order to instantly decrease the risk premium to 13.44%. The subsidy route indicates that 200*bps* is required to move 1.5 months faster in reaching the 12.4% risk premium.

6.3 Direct Asset Purchase

In both the subprime crisis as well as the Great Depression the government directly entered the asset market to purchase distressed assets. The U.S. Treasury's Troubled Asset Relief Program proposes purchasing over \$500*bn* of distressed mortgage-backed securities. We can evaluate the impact of this policy as follows. Suppose that in state (y, D) the government purchases $s(y, D)$ shares of the risky asset, financing this purchase by issuing $s(y, D)P$ of instantaneous debt (P is the price of the risky asset). The cash-flow, after repaying debt, from this transaction is,

$$s(y, D)P(dR_t - r_t dt).$$

We assume that the government raises lumpsum taxes from (or rebates to) the households to balance this cash-flow. The tax/rebate on households is

$$\tau dt = -s(y, D)P(dR_t - r_t dt).$$

The tax affects the household's dynamic budget constraint.

$$dw_t^h = (lD_t - \rho w_t^h)dt + w_t^h r_t dt + \alpha_t^h (1 - \lambda) w_t^h \left(\widetilde{dR}_t - r_t dt\right) - \tau_t dt.$$

We derive the ODE for the price/dividend ratio with this new dynamic budget constraint.

To provide some intuition into how this policy affects equilibrium, we substitute the tax into the household's budget constraint,

$$dw_t^h = (lD_t - \rho w_t^h)dt + w_t^h r_t dt + \alpha_t^h (1-\lambda) w_t^h \left(\widetilde{dR}_t - r_t dt\right) + s(y,D)P(dR_t - r_t dt).$$

The equation shows that the asset purchase increases the household's exposure to the risky asset (the last term). In turn, this means that specialists bear less risk and hence the risk premium falls. Effectively this policy puts less risk on the limited risk-bearing capacity of the intermediary sector. Of course, in doing so, the policy forces the risk onto the household sector, which they may not prefer, given the limited market participation assumption of the model. Indeed, some of the discussion of the Troubled Asset Relief Program surrounded the fact that neither the Treasury nor taxpayers would have the expertise to value and manage the risk on a portfolio of complex mortgage-backed securities.

We consider the following policy rule:

$$s(y) = s \times 1_{\{y-y^*>0\}} \tag{21}$$

for positive constants s and y^*. We set y^* equal to 10 in the simulation. For technical reasons, so that the policy is well defined as household wealth approaches zero, we require that $y^* > 0$. The probability that $y > 10$ is around 99.6%.

Table 5 reports the results for three values of s, which is the share of the intermediated risky asset market that the government purchases. The \$500bn that the Treasury proposes corresponds to roughly $s = 0.033$, based on the \$15tn market size. We assume that the policy is initiated in the state corresponding to 20% risk premium and then trace the recovery path from this state.

Upon its inception, the policy causes the risk premium to jump downwards, reflecting the lower risk held by the intermediary sector. The first row of the table (in italics) reports this initial jump. The jump is the substantial component of policy in all of the cases. After this initial jump the recovery path is almost the same as the case of no intervention. For example, if we compare the incremental time it takes the economy to move from 15% to 10%, we see that the time for the no intervention case is 0.78 years, while it is 0.77 years for the case of $s = 0.12$. Intuitively the purchase has no further effect because there is a countervailing force: the specialist holds a smaller position in the risky asset (since the taxpayer holds a larger share) and hence less of the risk premium accrues to it, which causes intermediary capital to recover more slowly.

Comparing \$250bn of equity infusion to \$500bn of risky asset purchase, it is clear that the equity infusion

Table 6: Asset Purchase

This table presents transition time data from simulating the model for the $m = 4$ case. We begin in the 20% risk premium state and report the first passage time for the state to reach that in the first column of the table ("Transit to" column). Time is reported in years. We report the case of no purchase ($s = 0$), as well as purchases of $0.04, 0.08$, and 0.12. A purchase with $s = 0.04$ corresponds to the government buying 4% of the outstanding risky asset of intermediated risky assets. The first row of the table (in italics) reports the instantaneous jump downwards in the risk premium when the government begins its purchase.

Transit to	$s = 0$	$s = 0.04$	$s = 0.08$	$s = 0.12$
		18.92%	*17.84%*	*16.77%*
15	0.22	0.20	0.16	0.12
12.5	0.47	0.44	0.39	0.35
10	1.00	0.97	0.92	0.89
7.5	2.61	2.57	2.48	2.46
6	5.81	5.65	5.71	5.56
5	12.95	12.67	12.15	12.03

is more effective in reducing the risk premium. This should not be surprising because the basic problem our model studies is a lack of equity capital. Thus infusing equity capital attacks the problem at its heart. But it is also worth noting, that all of the policies we have studied are effective in speeding up the crisis recovery. Finally, our analysis does not compute the costs of these policies to taxpayers.

7 Government Debt Liquidity

Woodford (1990) and Holmstrom and Tirole (1998) suggest that government bonds (i.e. Treasury Bills) offer special liquidity services, and that the state can play a role in "liquidity provision" by managing the outstanding stock of government debt. Krishnamurthy and Vissing-Jorgensen (2008) empirically show that U.S. Treasury debt carries a sizeable liquidity premium, reporting that it has historically averaged close to 1%. They also show that the liquidity premium falls as the stock of debt rises, which is a finding our model replicates.

Our model shares some important features with those of Woodford (1990) and Holmstrom and Tirole (1998), making it possible to quantitatively explore the liquidity provision issues that they raise. Suppose that the government in our model issues short-term debt financed by lumpsum taxes on households. Since households are modeled in an overlapping generations framework, such debt issuance will not be Ricardian. Government debt will also be especially liquid in the following sense: all agents participate in the riskless debt market, while only the specialists participate in the risky asset market. Thus the government debt is more liquid than the risky asset.[19] Changing the supply of government debt adds to the stock of liquid assets held by the agents in the economy, which is the essential feature of the policy in the Woodford/Holmstrom-Tirole models.

Consider the following policy. Suppose that the government rolls over a short-term riskless bond, whose interest cost is financed partly by levying lumpsum taxes on the households. At date t the government has bonds outstanding of $B(y_t, D_t)$ on which it pays the interest rate of r_t. Then, the flow budget constraint for the government is,

$$dB_t - r_t B_t dt + \tau_t dt = 0.$$

In this equation, dB_t is the net increase in bond issuance (i.e., moneys received from issuance in excess of moneys paid to redeem the existing issue), $r_t B_t$ is the interest cost on the outstanding risky asset of debt, and τ_t is the lumpsum taxes raised on households. The tax affects the wealth dynamics for the households. Equation (10) is altered to,

$$dw_t^h = (lD_t - \rho w_t^h)dt + w_t^h r_t dt + \alpha_t^h (1-\lambda) w_t^h \left(\widetilde{dR}_t - r_t dt\right) - \tau_t dt.$$

We consider the following class of bond policies that facilitates analysis in our setting,

$$B(y_t, D_t) = D_t \times b \times 1_{\{y_t - y^* \geq 0\}} \tag{22}$$

for positive constants b and y^*.[20] This policy scales the bond issue linearly with dividends to ensure that the bond policy does not either dominate or vanish from the economy. Roughly speaking, we can think of this policy as one that maintains a constant debt-to-GDP ratio.

We also consider a policy of the following form:

$$B(y_t, D_t) = D_t \times b \times \max[y_t - y^*, 0] \tag{23}$$

for positive constants b and y^*. Relative to (22), this policy has the government issuing more bonds as y rises and the economy is more constrained. Such an increase is interesting because it is consistent with the notion that the government provides more liquidity during crises.

[19] This notion of liquidity, deriving from market segmentation, is similar to Allen and Gale (1994) and Huang and Wang (2007), or to models of the liquidity of money in the monetary economics literature (e.g., Alvarez, Atkeson, and Kehoe, 2002).

[20] As with the risky asset purchase policy of the previous section, we impose the technical restriction that the policy is implemented only for states y that are sufficiently positive.

We analyze the policy by altering the budget constraint for the household (see Appendix C). The bond intervention also requires us to modify the boundary condition (see (24) below).

The results are shown in Table 7 and Figure 7. In Table 7, we present results for both the constant debt policy and the increasing debt policy ($y^* = 10$). For each policy we present three scenarios. We choose b so that the average value of debt to total assets (debt plus the risky asset) is 5%, 10% and 15%. If we think of total intermediated assets as being around 15 *tn*, then the 15% scenario corresponds to introducing about 2.65 *tn* of government debt. The measures are broken down into conditional on being in the constrained region, conditional on being in the unconstrained region, and unconditional average.

Table 7: Government Debt Liquidity

This presents the average values of the risk premium and interest rate, for different debt liquidity scenarios. The first row of the table (in italics) is the case of no debt. We present the constant debt policy and the increasing debt policy case, for average levels of debt to total assets of 5%, 10%, and 15%. The measures are broken down into conditional on being in the constrained region, conditional on being in the unconstrained region, and unconditional average. Parameters are the $m = 4$ case and those given in Table 2. For the increasing debt case, we set $y^* = 10$.

		Risk Premium (%)			Interest Rate (%)		
	$E\left[\frac{B}{P+B}\right]$ (%)	Avg.	Unconst.	Const.	Avg.	Unconst.	Const.
No Debt	0	*5.34*	*4.87*	*6.01*	*0.50*	*0.88*	*-0.055*
Constant Debt	5	5.24	4.67	5.80	0.82	1.28	0.35
Constant Debt	10	5.11	4.47	5.56	1.19	1.72	0.82
Constant Debt	15	4.99	4.26	5.33	1.58	2.19	1.29
Increasing Debt	5	5.29	4.76	5.84	0.78	1.20	0.35
Increasing Debt	10	5.24	4.61	5.70	1.08	1.56	0.74
Increasing Debt	15	5.15	4.44	5.50	1.45	1.96	1.20

There are two effects to keep in mind in looking at these results. First, issuing bonds has a direct effect on the price of the risky asset. As one would expect, interest rates are higher with the policy, which tends to lower the present value of the risky asset. In Table 7, the average interest is uniformly higher across all of the debt scenarios. For the 15% cases, the average interest rate is close to 1% higher than the case of no debt. Algebraically, the easiest way to see this valuation effect is to examine the boundary condition for the economy. Previously, at the upper boundary for y we have:

$$y^b = F(y^b) = \frac{1+l}{\rho}.$$

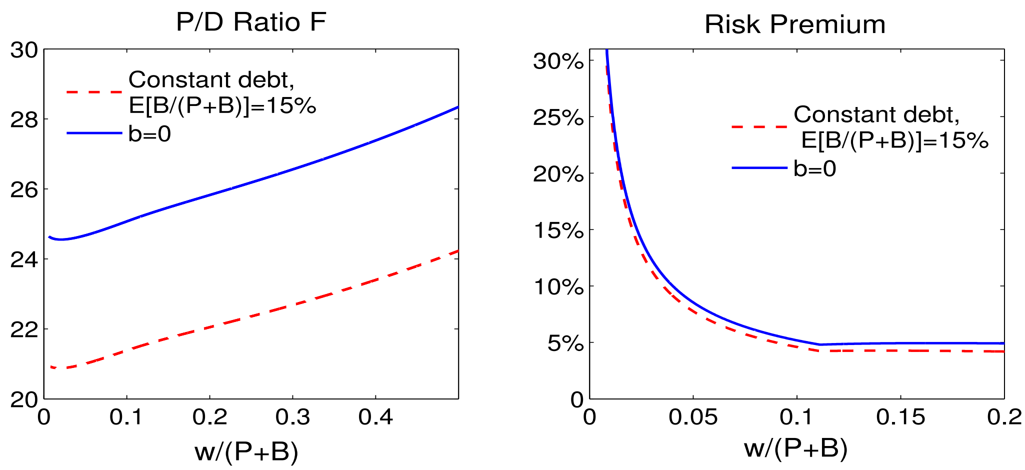
That is, the household owns all of the wealth of the economy, amounting to $F(y)$. With the introduction of government bonds, the household's wealth at the upper boundary must include both bonds and the risky asset. Thus, the boundary changes to:

$$y^b = F(y^b) + B(y^b)/D = \frac{1+l}{\rho}. \tag{24}$$

Increasing $B(y^b)$ causes $F(y^b)$ to fall.

The left-panel of Figure 7 illustrates this point. We plot $F(\cdot)$ for the constant debt policy (15%) as well as the case of no debt. Introducing debt lowers $F(\cdot)$ uniformly.

Figure 7: Debt Liquidity



The left panel graphs the Price/Dividend ratio as a function of $w/(P+B)$. The right panel graphs the risk premium. We present two cases: (1) zero debt and (2) constant debt with average value of $\frac{B}{P+B}$ of 15%.

The second effect of the policy is to reduce intermediary leverage. In our model, some households only demand debt. Via market clearing, the intermediary sector provides this debt and therefore takes on leverage. When the government issues debt it effectively crowds out the privately issued debt, causing intermediary leverage to fall. The reduced leverage then decreases the effective risk borne by the intermediary sector and in turn lowers the risk premium. The right panel of Figure 7 graphs the risk premium for 15% debt and no-debt cases, illustrating the drop in risk premium.

Table 7 reports how much the risk premium falls. The risk premium in the constant debt case of 15% is 35 *bps* lower than the no debt case. The drop is across both constrained and unconstrained regions, albeit larger in the constrained region. Note also the fall in the risk premium is increasing in the the size of the debt. Since part of the premium on the risky asset is due to its illiquidity relative to the riskless debt, we interpret this result as showing that the liquidity premium on debt falls as the stock of debt rises, consistent with the findings of Krishnamurthy and Vissing-Jorgensen mentioned above.

It is also interesting to see that the risk premium falls less in the increasing debt case. This occurs through the effect that debt has on the level of the price/dividend ratio. In the increasing debt case, the price/dividend ratio falls faster as w/P falls. Dynamically, this makes the asset price more volatile in the constrained region. The volatility effect outweighs the countervailing leverage effect, causing the risk premium to rise more than in the constant debt case.

The comparison between the increasing debt and constant debt case offers an additional insight relative to Woodford and Holmstrom-Tirole on the workings of debt liquidity policy. It suggests that debt issuance works best as an ex-ante preventive policy rather than a crisis policy. That is, if during a non-crisis period, the government issues bonds, then it lowers asset values immediately but also reduces financial sector leverage. The benefit of the lower leverage is that crises, if they occur, will be of lower magnitude. If on the other hand, the government issues more debt in the midst of a crisis; then, it does reduce leverage, but it also lowers asset values. This second effect, by tightening the intermediation constraint, can compound the crisis. Concretely, this analysis suggests that if the US Treasury had increased government debt issuance from 2000 to 2004, then the financial sector would have been carrying less leverage entering the subprime crisis.

8 Concluding Remarks

We have presented a model to study the dynamics of risk premia in a crisis episode where intermediaries' equity capital is scarce. We calibrate the model and show the model does well in matching two aspects of crises: the nonlinearity of risk premia in crisis episodes; and, the recovery from crises in the order of many months. We also use the model to evaluate the effectiveness of central bank policies, finding that infusing equity capital into intermediaries is the most effective policy in our model.

A limitation of our model is that it does not shed any light on the connection between the performance of intermediated asset markets we model (i.e. the mortgage-backed securities market) and the aggregate stock market. Yet, as we have seen during the subprime crisis, the deterioration in intermediation does spillover to the S&P500. It will be interesting to explore such a connection by introducing a second asset, in positive supply, that the households invest in directly. Such an asset can represent the S&P500 and may shed light on the equity premium puzzle. Introducing such an asset is also likely to dampen the over-sensitive interest rate effect that is present in our model.

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A ODE Solution

In this appendix, we detail the ODE that characterizes the equilibrium. We analyze our ODE based on state variable y, i.e., the scaled households wealth. Denote the dynamics of y_t as,

$$dy_t = \mu_y dt + \sigma_y dZ_t, \tag{25}$$

for unknown functions μ_y and σ_y

We write $\frac{dc_t}{c_t}$ and dR_t as functions of μ_y,σ_y and the derivatives of $F(y)$. Because $c_t = D_t\left(1 + l - \rho y_t\right)$, we have

$$\begin{aligned}\frac{dc_t}{c_t} &= \frac{dD_t}{D_t} - \frac{\rho dy}{1+l-\rho y} - \frac{\rho}{1+l-\rho y} Cov_t\left[dy, \frac{dD}{D}\right] \\ &= \left(g - \frac{\rho}{1+l-\rho y}(\mu_y + \sigma_y\sigma)\right)dt + \left(\sigma - \frac{\rho\sigma_y}{1+l-\rho y}\right)dZ_t.\end{aligned}$$

We also have

$$dR_t = \frac{dP_t + D_t dt}{P_t} = \left[g + \frac{F'}{F}\mu_y + \frac{1}{2}\frac{F''}{F}\sigma_y^2 + \frac{1}{F} + \frac{F'}{F}\sigma_y\sigma\right]dt + \left(\sigma + \frac{F'}{F}\sigma_y\right)dZ_t.$$

Substituting these expressions into (16) we obtain the following ODE,

$$\begin{aligned}g + \frac{F'}{F}\mu_y + \frac{1}{2}\frac{F''}{F}\sigma_y^2 \quad + \quad & \frac{1}{F} + \frac{F'}{F}\sigma_y\sigma = \rho + \gamma g - \frac{\gamma\rho^h}{1-\rho^h y}\left(\mu_y + \sigma_y\sigma\right) \\ & + \gamma\left(\sigma - \frac{\rho^h}{1-\rho^h y}\sigma_y\right)\left(\sigma + \frac{F'}{F}\sigma_y\right) - \frac{1}{2}\gamma(\gamma+1)\left(\sigma - \frac{\rho^h}{1-\rho^h y}\sigma_y\right)^2.\end{aligned} \tag{26}$$

A.1 Derivation of μ_y and σ_y

We rewrite equation (10) which describes the wealth dynamics (budget constraint) of the household sector as:

$$dw^h = \theta_s dP + D\theta_s dt + r\hat{\theta}_b dt + lD_t dt - \rho w^h dt. \tag{27}$$

In this equation,

$$\theta_s = \alpha^I \alpha^h (1-\lambda)\frac{w^h}{P} \tag{28}$$

are the number of shares that the risky asset household owns, and

$$\hat{\theta}_b D = w^h - \theta_s P \tag{29}$$

is the amount of funds that the risky asset and debt households together have invested in the riskless bond. α^h and α^I are defined in the text and depends on whether the economy is constrained or not.

We apply Ito's Lemma to $P = DF(y)$ to find expressions for the drift and diffusion of dP. We can then substitute back into equation (27) to find expressions for the drift and diffusion of dw^h.

Now, we have defined $w^h = Dy$. We apply Ito's Lemma to this equation to arrive at a second expression for the drift and diffusion of dw^h. Matching the drift and diffusion terms from these two ways of writing dw^h, we solve to find μ_y and σ_y.

The result of this algebra is that:

$$\sigma_y = -\frac{\hat{\theta}_b}{1-\theta_s F'}\sigma,$$

and,

$$\mu_y = \frac{1}{1-\theta_s F'}\left(\theta_s + l + (r + \sigma^2 - g)\hat{\theta}_b - \rho y + \frac{1}{2}\theta_s F''\sigma_y^2\right).$$

A.2 ODE

Substituting for μ_y derived in (A.1) into (26), we find,

$$
\begin{aligned}
&\left(\frac{F'}{F}+\frac{\gamma\rho}{1+l-\rho y}\right)\left(\frac{1}{1-\theta_s F'}\right)\left(\theta_s+l+\hat{\theta}_b(r-g)-\rho y\right)+\frac{1}{F}\\
&\qquad+\frac{1}{2}F''\sigma_y^2\left(\frac{1}{1-\theta_s F'}\right)\left(\frac{1}{F}+\theta_s\frac{\gamma\rho}{1+l-\rho y}\right)\\
=\;&\rho+g(\gamma-1)+\gamma\left(\sigma-\frac{\rho}{1+l-\rho y}\sigma_y\right)\left(\sigma+\frac{F'}{F}\sigma_y\right)\\
&\qquad-\frac{1}{2}\gamma(\gamma+1)\left(\sigma-\frac{\rho}{1+l-\rho y}\sigma_y\right)^2
\end{aligned}
$$

where,

$$
\begin{aligned}
r \;=\;& \rho+g\gamma-\frac{\rho\gamma}{1+l-\rho y}\frac{\theta_s+l+(r-g)\,\hat{\theta}_b-\rho y+\frac{\sigma^2}{2}\theta_s F''\frac{\hat{\theta}_b^2}{(1-\theta_s F')^2}}{1-\theta_s F'}\\
&-\frac{\gamma\,(\gamma+1)\,\sigma^2}{2}\left(1+\frac{\rho\hat{\theta}_b}{1+l-\rho y}\frac{1}{1-\theta_s F'}\right)^2
\end{aligned}
$$

We define a function, $G(y)\equiv\frac{1}{1-\theta_s F'}$; with this definition, we can write $G'=\theta_s G^2 F''$, and

$$
\sigma_y=-\frac{\hat{\theta}_b}{1-\theta_s F'}\sigma=-\hat{\theta}_b\sigma G.
$$

Therefore we have

$$
\begin{aligned}
G'\ \frac{(\hat{\theta}_b\sigma)^2}{2}\ \ G\ \left(\frac{1}{\theta_s F}+\frac{\gamma\rho^h}{1+l-\rho y}\right)=\;&\rho+g(\gamma-1)-\frac{1}{F}\\
&+\frac{1}{2}\gamma\sigma^2\left(1+\frac{\rho}{1+l-\rho y}\hat{\theta}_b G\right)\left(\frac{2\left(y-G\hat{\theta}_b\right)}{\theta_s F}-(1+\gamma)\,\frac{1+l-\rho y+\rho G\hat{\theta}_b}{1+l-\rho y}\right)\\
&-\left(\frac{G-1}{\theta_s F}+\frac{\gamma\rho^h}{1+l-\rho y}G\right)\left(\theta_s+l+\hat{\theta}_b(r-g)-\rho y\right)
\end{aligned}
$$

and

$$
r=\frac{\rho+g\gamma-\frac{\rho\gamma G}{1+l-\rho y}\left(\theta_s+l-g\hat{\theta}_b-\rho y+\frac{\sigma^2}{2}G'\hat{\theta}_b^2\right)-\frac{\gamma(\gamma+1)\sigma^2}{2}\left(1+\frac{\rho\hat{\theta}_b G}{1+l-\rho y}\right)^2}{1+\frac{\rho\gamma G\hat{\theta}_b}{1+l-\rho y}}.
$$

We combine these two pieces, using the relation, $\hat{\theta}_b\left(\frac{G-1}{\theta_s F}+\frac{\gamma\rho^h G}{1+l-\rho y}\right)=-\frac{y-G\hat{\theta}_b}{\theta_s F}+\frac{1+l-\rho y+\gamma\rho^h G\hat{\theta}_b}{1+l-\rho y}$, and arrive at a final expression of the ODE:

$$
\begin{aligned}
&G'\frac{(\hat{\theta}_b\sigma)^2}{2}\frac{G}{\theta_s F}\left(\frac{1+l+\rho y(\gamma-1)}{1+l-\rho y+\rho\gamma G\hat{\theta}_b}\right)\\
=\;&\rho+g(\gamma-1)-\frac{1}{F}+\frac{\gamma\,(1-\gamma)\,\sigma^2}{2}\left(1+\frac{\rho G\hat{\theta}_b}{1+l-\rho y}\right)\frac{y-G\hat{\theta}_b}{\theta_s F}\left[\frac{1+l-\rho y-\rho G\hat{\theta}_b}{1+l-\rho y+\rho\gamma G\hat{\theta}_b}\right]\\
&-\left(\frac{(1+l-\rho y)\,(G-1)}{\theta_s F}+\gamma\rho^h G\right)\frac{\theta_s+l+\hat{\theta}_b(g(\gamma-1)+\rho)-\rho y}{1+l-\rho y+\rho\gamma G\hat{\theta}_b}.
\end{aligned}
\tag{30}
$$

The expressions for the bond holding $\hat{\theta}_b$ and risky asset holding θ_s depend on whether the economy is constrained or not. In the unconstrained region, as shown in Section 3.3, $\alpha^h=1$, and $\alpha^I=\frac{F}{F-\lambda y}$. Utilizing (29) and (28), we have $\theta_s=\frac{(1-\lambda)y}{F-\lambda y}$, and $\widehat{\theta}_b=\lambda y\frac{F-y}{F-\lambda y}$. In the constrained region $\alpha^h=\frac{m(F-y)}{(1-\lambda)y}$, $\alpha^I=\frac{1}{1+m}\frac{F}{F-y}$, therefore $\theta_s=\frac{m}{1+m}$, and $\widehat{\theta}_b=y-\frac{m}{1+m}F$. Finally, as illustrated in Section 3.3, the cutoff for the constraint satisfies $y^c=\frac{m}{1-\lambda+m}F(y^c)$, and the economy is in the unconstrained region if $0<y\le y^c$.

A.3 Boundary conditions and technical parameter restriction

The upper boundary condition is described in Section 3.5. A lower boundary condition occurs when $y \to 0$. This case corresponds to one where specialists hold the entire financial wealth of the economy. Using L'Hopital's rule, it is easy to check that $\frac{G-1}{\theta_s F} \to \frac{F'(0)}{F(0)}$. Plugging this result into (30), and noting that both θ_s and $\widehat{\theta}_b$ go to zero as y goes to zero, we obtain,

$$F(0) = \frac{1 + F'(0)\, l}{\rho + g(\gamma - 1) + \frac{\gamma(1-\gamma)\sigma^2}{2} - \frac{l\gamma\rho}{1+l}}. \tag{31}$$

When $l = 0$, one can check that $F(0)$ is the equilibrium Price/Dividend ratio for the economy with the specialists as the representative agent. However because in our model the growth of the household sector affects the pricing kernel, this boundary P/D ratio $F(0)$ also depends on the household's labor income l. As in the case where $l = 0$, for the P/D ratio to be well defined we require that parameters satisfy,

$$\rho + g(\gamma - 1) + \frac{\gamma(1-\gamma)\sigma^2}{2} - \frac{l\gamma\rho}{1+l} > 0. \tag{32}$$

Furthermore, a straightforward calculation yields that $F'\left(y^b\right) = 1$ if $F(y^b) = y^b$. This result also ensures that the mapping from the scaled household's wealth y to the scaled specialist wealth $w/D = F(y) - y$ is strictly decreasing in the scaled household's wealth y (this monotone relation clearly fails if $F\left(y^b\right) > y^b$.) As a result, it is equivalent to model either agent's wealth as our state variable.

A.4 Numerical Method

In our ODE (30) both boundaries are singular, causing difficulties in directly applying the built-in ODE solver *ode*15*s* in *Matlab*. To overcome this issue, we approximate the upper-end boundary $\left(y^b, F\left(y^b\right) = y^b\right)$ by $\left(y^b - \eta, y^b - \eta\right)$ (where η is sufficiently small), and adopt a "forward-shooting and line-connecting" method for the lower–end boundary. Take a small $\epsilon > 0$ and call $\widetilde{F}$ as the attempted solution. For each trial $\phi \equiv \widetilde{F}'(\epsilon)$, we set $\widetilde{F}'(0) = \phi$, solve $\widetilde{F}(0)$ based on (31), and let $\widetilde{F}(\epsilon) = \widetilde{F}(0) + \phi\epsilon$. Since $\left(\epsilon, \widetilde{F}(\epsilon)\right)$ is away from the singularity, by trying different ϕ's we apply the standard shooting method to obtain the desired solution F that connects at $\left(y^b - \eta, y^b - \eta\right)$. For $y < \epsilon$, we simply approximate the solution by a line connecting $(0, F(0))$ and $(\epsilon, F(\epsilon))$. In other words, we solve F on $\left[\epsilon, y^b\right]$ with a smooth pasting condition for $F'(\epsilon) = \frac{F(\epsilon) - F(0)}{\epsilon}$ and a value matching condition for $F\left(y^b\right) = y^b$.

We use $\epsilon = 0.1$ and $\eta = 0.001$ which give ODE errors bounded by 3×10^{-5} for $y > \epsilon$. Different ϵ's and η's deliver almost identical solutions for $y > 1$. Because we are mainly interested in the solution behavior near y^c (which takes a value of 14 even in the $m = 1$ case) and onwards, our main calibration results are free of the approximation errors caused by the choice of ϵ and η. Finally we find that, in fact, these errors are at the same magnitude as those generated by the capital constraint around y^c (3.5×10^{-5}).

B Verification of optimality

In this section we take the equilibrium Price/Dividend ratio $F(y)$ as given, and verify that the specialist's consumption policy $c = D_t(1 + l - y_t)$ is optimal subject to his budget constraint. Our argument is a variant of the standard one: it uses the strict concavity of $u(\cdot)$ and the specialist's budget constraint to show that the specialist's Euler equation is necessary and sufficent for the optimality of his consumption plan.

Specifically, fixing $t = 0$ and the starting state (y_0, D_0), define the pricing kernel as

$$\xi_t \equiv e^{-\rho t} c_t^{-\gamma} = e^{-\rho t} D_t^{-\gamma} (1 + l - \rho y_t)^{-\gamma}.$$

Consider another consumption profile $\widehat{c}$ which satisfies the budget constraint $E\int_0^\infty \widehat{c}_t \xi_t dt \le \xi_0 D_0 (F_0 - y_0)$ (recall that the specialist's wealth is $D_0(F_0 - y_0)$; here we require that the specialist's feasible trading strategies be well-behaved, e.g., his wealth process remains non-negative). Then we have

$$\begin{aligned} E\int_0^\infty e^{-\rho t} u(c_t)\, dt &\ge E\int_0^\infty e^{-\rho t} u(\widehat{c}_t)\, dt + E\int_0^\infty e^{-\rho t} u'(c_t)(c_t - \widehat{c}_t)\, dt \\ &= E\int_0^\infty e^{-\rho t} u(\widehat{c}_t)\, dt + E\int_0^\infty \xi_t c_t dt - E\int_0^\infty \xi_t \widehat{c}_t dt. \end{aligned}$$

If the specialist's budget equation holds in equality for the equilibrium consumption process c, i.e., if

$$E\int_0^\infty \xi_t c_t dt = \xi_0 D_0 \left(F_0 - y_0\right),$$

then the result follows. Somewhat surprisingly, for our model this seemingly obvious claim requires an involved argument because of the singularity at $y^b = \frac{1+l}{\rho}$.

One can easily check that, for $\forall T > 0$, we have

$$\xi_0 D_0 \left(F_0 - y_0\right) = \int_0^T c_t \xi_t dt + \int_0^T \sigma\left(D_t, y_t\right) dZ_t + \xi_T D_T \left(F_T - y_T\right), \tag{33}$$

where $\sigma\left(D_t, y_t\right)$ corresponds to the specialist's equilibrium trading strategy (which involves terms such as $(1+l-\rho y)^{-\gamma-1}$ and is NOT uniformly bounded as $y \to y^b$). Our goal in the following steps is to show that in expectation, the latter two terms vanishes when $T \to \infty$.

Step 1: Limiting Behavior of y at y^b The critical observation regarding the evolution of y is that when y approaches y^b, it approximately follows a Bessel process with a dimension $\delta = \gamma + 2 > 2$. (Given a δ-dimensional Brownian motion Z, a Bessel process with a dimension δ is the evolution of $\|Z\| = \sqrt{\sum_{i=1}^{\delta} Z_i^2}$, which is the Euclidean distance between Z and the origin.) According to standard results on Bessel processes, y^b is an entrance-no-exit point, and is not reachable if the starting value $y_0 < y^b$ (if $\delta > 2$). Intuitively, when y is close to y^b, the dominating part of μ_y is proportional to $\frac{1}{y-y^b} < 0$, while the volatility σ_y is bounded— therefore a drift that diverges to negative infinity keeps y away from the singular point y^b. This result implies that our economy never hits y^b.

To show that for y close to y^b, y's evolution can be approximated by a Bessel Process, one can easily check that when $y \to y^b$,

$$r \simeq -\frac{(\gamma+1)\sigma^2}{2}\frac{\rho^h \hat{\theta}_b G}{1+l-\rho^h y}, \mu_y \simeq -\frac{(\gamma+1)\sigma^2}{2}\frac{\rho^h \hat{\theta}_b^2 G^2}{1+l-\rho^h y}, \sigma_y = -G\sigma\hat{\theta}_b;$$

and therefore

$$dy = -\frac{(\gamma+1)\sigma^2}{2}\frac{\rho\hat{\theta}_b^2 G^2}{1+l-\rho y}dt - G\sigma\hat{\theta}_b dZ_t.$$

Utilizing the result $F'\left(y^b\right) = 1$ established in Section 3.5, we know that when $y \to y^b$, $\hat{\theta}_b \simeq F - \theta_s y \simeq \frac{1}{1+m}y^b = \frac{1}{1+m}\frac{1+l}{\rho}$, and $G \simeq 1+m$. Let

$$x_t = 1 + l - \rho y_t;$$

then it is easy to show that $q = \frac{x}{G\sigma\hat{\theta}_b\rho} = \frac{x}{\sigma(1+l)}$ evolves approximately according to

$$dq = -\frac{1}{G\sigma\hat{\theta}_b}dy = \frac{(\gamma+1)}{2q}dt + dZ_t,$$

which is just a standard Bessel process with a dimension $\delta = \gamma + 2$. Therefore, x is also a scaled version of a Bessel process, and can never reach 0 (or, y cannot reach y^b). In the following analysis, we focus on the limiting behavior of x.

Step 2: Localization Note that in (33), due to the singularity at $x = 0$ (or, $y = y^b$), both the local martingale part $\int_0^T \sigma\left(D_t, y_t\right) dZ_t$ and the terminal wealth part $\xi_T D_T \left(F_T - y_T\right)$ are not well-behaved. To show our claim, we have to localize our economy, i.e., stop the economy once y is sufficiently close to y^b (or, once D is sufficiently close to 0). Specifically, we define

$$T_n = \inf\left\{t : x_t = \frac{1}{n} \text{ or } D_t = \frac{1}{n^h}\right\}$$

where h is a positive constant (as we will see, the choice of h, which is around 1, gives some flexibility for γ other than 2). Since y and x have a one-to-one relation ($x = 1 + l - \rho y$), for simplicity we localize x instead.

Clearly this localization technique ensures that the local martingale part $\int_0^{T_n} \sigma\left(D_t, y_t\right) dZ_t$ is a martingale (one can check that $\sigma\left(D_t, y_t\right)$ is continuous in D_t and y_t, in turn D_t and x_t; therefore $\sigma\left(D_t, y_t\right)$ is locally bounded). As $T_n \to \infty$ when $n \to \infty$, for our claim we need to show

$$\lim_{n\to\infty} E\left[\xi_{T_n} D_{T_n} \left(F_{T_n} - y_{T_n}\right)\right] = 0$$

We substitute from the definition of ξ:

$$E\left[e^{-\rho T_n} D_{T_n}^{1-\gamma} x_{T_n}^{-\gamma}\left(F\left(y_{T_n}\right)-y_{T_n}\right)\right] \leq E\left[e^{-\rho T_n} n^{h(\gamma-1)} x_{T_n}^{-\gamma}\left(F\left(y_{T_n}\right)-y_{T_n}\right)\right].$$

Since the analysis will be obvious if $x^{-\gamma}(F(y)-y)$ is uniformly bounded (notice here $x=1+l-\rho y$), it is sufficient to consider $x_{T_n}=\frac{1}{n}$. Because $F\left(y^b\right)=y^b$ and $F'\left(y^b\right)=1$, by Taylor expansion we know that $F\left(y^b-\frac{1}{n\rho}\right)-\left(y^b-\frac{1}{n\rho}\right)$ can be written as $\psi(n)\frac{1}{n}$ when n is sufficiently large, and $\psi(n)\to 0$ as $n\to\infty$. Therefore we have to show that, as $n\to\infty$,

$$E\left[e^{-\rho T_n} n^{(\gamma-1)(1+h)}\right]\psi(n)\to 0$$

and a sufficient condition is that,

$$E\left[e^{-\rho T_n}\right] n^{(\gamma-1)(1+h)}\to K$$

where K is bounded.

We apply existing analytical results in the literature to show our claim. To do so, we have to separate our two state variables. We define

$$T_n^D=\inf\left\{t: D_t=\frac{1}{n^h}\right\}, T_n^x=\inf\left\{t: x_t=\frac{1}{n}\right\}.$$

We want to bound $E\left[e^{-\rho T_n}\right]$ by the sum of $E\left[e^{-\rho T_n^D}\right]$ and $E\left[e^{-\rho T_n^x}\right]$; note that they are Laplace transforms of the first-hitting time distribution of a GBM and Bessel processes, respectively. The Laplace transform of T_n is simply

$$E\left[e^{-\rho T_n}\right]=\int_0^\infty e^{-\rho T} d\mathbf{F}(T)=\rho\int_0^\infty e^{-\rho T}\mathbf{F}(T)\,dT,$$

where the bold $\mathbf{F}$ denotes the distribution function of T_n. The similar relation also holds for T_n^D or T_n^x. Denote $\mathbf{F}^D(\cdot)$ (or $\mathbf{F}^x(\cdot)$) as the distribution function for T_n^D (or T_n^x), and notice that

$$\begin{aligned} 1-\mathbf{F}(T) &= \Pr(T_n>T)=\Pr\left(T_n^D>T, T_n^x>T\right)>\Pr\left(T_n^D>T\right)\Pr\left(T_n^x>T\right) \\ &= 1-\mathbf{F}^D(T)-\mathbf{F}^x(T)+\mathbf{F}^D(T)\mathbf{F}^x(T), \end{aligned}$$

because $\mathbf{1}_{\left\{T_n^D>T\right\}}$ and $\mathbf{1}_{\left\{T_n^x>T\right\}}$ are positively correlated (both take the value 1 when Z is high).[21] Therefore $\mathbf{F}(T)<\mathbf{F}^D(T)+\mathbf{F}^x(T)$, or

$$E\left[e^{-\rho T_n}\right] n^{(\gamma-1)(1+h)}<E\left[e^{-\rho T_n^D}\right] n^{(\gamma-1)(1+h)}+E\left[e^{-\rho T_n^x}\right] n^{(\gamma-1)(1+h)}$$

Using the standard result of the Laplace transform of the first-hitting time distribution for a GBM process, we can easily verify that as $n\to\infty$, the first term $E\left[e^{-\rho T_n^D}\right] n^{(\gamma-1)(1+h)}$ vanishes under our parameters when $h=0.9$ (in fact, this relates to the parameter restriction for a standard GBM/CRRA economy).

Step 3: Regulated Bessel Process The challenging task is the second term. Notice that our economy (i.e., evolution of x) differs from the evolution of a Bessel process when x is far away from 0; therefore an extra care needs to be taken. We consider a regulated Bessel process which is reflected at some positive constant $\overline{x}$. Intuitively, by doing so, we are putting an upper bound for $E\left[e^{-\rho T_n^x}\right]$, as the reflection makes x_t to hit $\frac{1}{n}$ more likely (therefore, a larger $\mathbf{F}^x$). Also, for a sufficiently small $\overline{x}>0$, when $x\in(0,\overline{x}]$, x can be approximated by a Bessel process with a dimension $\gamma+2-\varepsilon$. Therefore, $\mathbf{F}^x$ must be bounded by the first-hitting time distribution of a Bessel process with a dimension δ, where δ takes value from $\gamma+2-\epsilon$ to $\gamma+2$, where ϵ is sufficiently small. Finally, note that by considering a Bessel process we are neglecting certain drift for x. However, one can easily check that when x is close to 0, the adjustment term for μ_y is $-\frac{1+l}{\rho}\gamma\sigma^2<0$. This implies that we are neglecting a positive drift for x—which potentially makes hitting less likely—thereby yielding an upper-bound estimate.

We have the following Lemma from the Bessel process.

[21] Technically, using the technique of Malliavian derivatives, we can show that both x_s and D_s have positive diffusions in the martingale representations for all s. Then, the running minimum $\underline{x}_T=\min\{x_t: 0<t<T\}$ and $\underline{D}_T=\min\{D_t: 0<t<T\}$ have positive loadings always on the martingale representations (using the technique in *Methods of Mathematical Finance*, Karatzas and Shreve (1998), Page 367). The same technique can be applied to $\mathbf{1}_{\left\{T_n^x>T\right\}}=\mathbf{1}_{\left\{\underline{x}_T>T\right\}}$ and $\mathbf{1}_{\left\{T_n^D>T\right\}}=\mathbf{1}_{\left\{\underline{D}_T>T\right\}}$, as an indicator function can be approximated by a sequence of differentiable increasing functions.

Lemma 1 *Consider a Bessel process x with $\delta > 2$ which is reflected at $\overline{x} > 0$. Let $\nu = \frac{\delta}{2} - 1$. Starting from $x_0 \leq \overline{x}$, we consider the hitting time $T_n^x = \inf\left\{t : x_t = \frac{1}{n}\right\}$. Then we have*

$$E\left[e^{-\rho T_n^x}\right] \propto n^{-2\nu} \text{ as } n \to \infty$$

Proof. Due to the standard results in Bessel process and the Laplace transform of the hitting time (e.g., see Borodin and Salminen (1996), Chapter 2), we have

$$E\left[e^{-\rho T_n^x}\right] = \frac{\varphi(x_0)}{\varphi\left(\frac{1}{n}\right)},$$

where

$$\varphi(z) = c_1 z^{-\nu} I_v\left(\sqrt{2\rho} z\right) + c_2 z^{-\nu} K_v\left(\sqrt{2\rho} z\right),$$

and $I_v(\cdot)$ (and $K_v(\cdot)$) is modified Bessel function of the first (and second) kind of order v. Because R is a reflecting barrier, the boundary condition is

$$\varphi'(\overline{x}) = 0,$$

which pins down the constants c_1 and c_2 (up to a constant multiplication; notice that this does not affect the value of $E\left[e^{-\rho T_n^x}\right]$). Therefore the growth rate of $E\left[e^{-\rho T_n^x}\right]$ is determined by $n^{\nu} K_v\left(\sqrt{2\rho} n^{-1}\right)$ as K_v dominates I_v near 0. Since $K_v(x)$ has a growth rate $x^{-\nu}$ when $x \to 0$, the result is established. ■

For any y_0, redefine starting point as $x_0 = \min(1 + l - y_0, \overline{x})$; clearly this leads to an upper-bound estimate for $E\left[e^{-\rho T_n^x}\right]$. However, since for all $\delta \in [\gamma + 2 - \epsilon, \gamma + 2]$, the above Lemma tells us that for any $\varepsilon \in [0, \epsilon]$, when $n \to \infty$,

$$n^{(\gamma-1)(1+h)} E\left[e^{-\rho T_n^x}\right] \propto n^{(\gamma-1)(1+h)} n^{-2\nu} = n^{(\gamma-1)(1+h)-\gamma+\varepsilon} \to 0$$

uniformly if $\gamma = 2$ and $h = 0.9$ (and for some sufficiently small $\epsilon > 0$). Therefore we obtain our desirable result.

Finally $c_t \xi_t > 0$ implies that $\int_0^\infty c_t \xi_t dt$ converges monotonically, and therefore the specialist's budget equation $\lim_{T\to\infty} E \int_0^T \xi_t c_t dt = \xi_0 D_0 (F_0 - y_0)$ holds for all stopping times that converge to infinity. Q.E.D.

C Appendix for Section 6

C.1 Borrowing Subsidy

We have the same ODE as in Appendix A. The only difference is that

$$\mu_y = \frac{1}{1 - \theta_s F'} \left(\theta_s + l + (r + \sigma^2 - g)\hat{\theta}_b - \hat{\theta}_b \Delta r - \rho y + \frac{1}{2} \theta_s F'' \sigma_y^2\right).$$

C.2 Direct Asset Purchase

In this case, the intermediary holds $1 - s$ of the risky asset (where s is a function of (y, D)). In the unconstrained region, $\alpha^h = 1$, and

$$\frac{\alpha^I\left(w + \alpha^h (1 - \lambda) w^h\right)}{P} = 1 - s$$

which implies that $\alpha^I = \frac{(1-s)F}{F-\lambda y}$. Therefore the households' holding of the risky asset through intermediaries is

$$\theta_s^I = \frac{(1 - s)(1 - \lambda) y}{F - \lambda y},$$

and the total holding is $\theta_s = \theta_s^I + s = \frac{(1-s)(1-\lambda)y}{F-\lambda y} + s(y, D)$.

In the constrained region, $\alpha^h = \frac{m(F-y)}{(1-\lambda)y}$ and $\alpha^I = \frac{1}{1+m} \frac{(1-s)F}{F-y}$. So

$$\theta_s^I = \frac{m(F - y)}{(1 - \lambda) y} \frac{1}{1 + m} \frac{(1 - s) F}{F - y} (1 - \lambda) \frac{y}{F} = \frac{m}{1 + m} (1 - s)$$

and the total holding is

$$\theta_s = \frac{m}{1 + m} (1 - s) + s = \frac{m + s}{1 + m}.$$

The same constraint cutoff applies $y^c = \frac{m}{1-\lambda+m} F^c$.

C.3 Government Debt

We only characterize the increasing government debt case, i.e, government debt outstanding at t is $B_t = \max\left(0, bD_t\left(y_t - y^*\right)\right)$. In the region of $y_t > y^*$, $B_t = bw_t^h - bD_t y^*$, and the tax paid by households is

$$\tau_t = bw_t^h r_t dt - bD_t y^* r_t dt - bd\left(w_t^h\right) + by^* dD_t.$$

Substituting into the household's budget equation, $dw^h = \theta_s dP + D\theta_s dt + r_t\theta_b dt + lDdt - \rho w_t^h dt - \tau_t$, we find:

$$\begin{aligned}
\sigma_y &= -\frac{\widehat{\theta}_b - b\left(y - y^*\right)}{1 - b - \theta_s F'}\sigma \\
\mu_y &= \frac{1}{1 - b - \theta_s F'}\left(\theta_s + l + \left(r + \sigma^2 - g\right)\left(\widehat{\theta}_b - b\left(y - y^*\right)\right) - \rho y + \frac{1}{2}\theta_s F'' \sigma_y^2\right).
\end{aligned}$$

The ODE remains the same as before (equation (17)); after substituting for μ_y and σ_y, and collecting the terms, we arrive at the final ODE:

$$\begin{aligned}
& G' \frac{(\hat{\theta}_b - b\left(y - y^*\right))^2\sigma^2}{2}\frac{G}{\theta_s F}\left(\frac{(1+l)(1-b) + \rho y(\gamma - 1)(1-b) + \rho\gamma b y^*}{1 + l - \rho y + \rho\gamma G\left(\hat{\theta}_b - b\left(y - y^*\right)\right)}\right) \\
= \;& \rho + g(\gamma - 1) - \frac{1}{F} + \\
& \frac{\gamma(1-\gamma)\sigma^2}{2}\left(1 + \frac{\rho G\left(\hat{\theta}_b - b\left(y - y^*\right)\right)}{1 + l - \rho y}\right)\frac{(1-b)y - (1-b)G\left(\hat{\theta}_b - b\left(y - y^*\right)\right) + by^*}{\theta_s F}\left[\frac{1 + l - \rho y - \rho G\left(\hat{\theta}_b - b\left(y - y^*\right)\right)}{1 + l - \rho y + \rho\gamma G\left(\hat{\theta}_b - b\left(y - y^*\right)\right)}\right] \\
& - \left(\frac{(1 + l - \rho y)\left((1-b)G - 1\right)}{\theta_s F} + \gamma\rho G\right)\frac{\theta_s + l + \left(\hat{\theta}_b - b\left(y - y^*\right)\right)\left(g(\gamma - 1) + \rho\right) - \rho y}{1 + l - \rho y + \rho\gamma G\left(\hat{\theta}_b - b\left(y - y^*\right)\right)},
\end{aligned}$$

where $G(y) \equiv \frac{1}{1 - b - \theta_s F'}$ again. Clearly this is for $y > y^*$; when $y < y^*$, we can simply set $b = 0$.

Now we derive the boundary conditions. For $y = 0$, the same boundary condition (20) as without government debt applies. When $y = \frac{1+l}{\rho}$, the specialist's wealth is zero, and we must have

$$\frac{1+l}{\rho} = F\left(\frac{1+l}{\rho}\right) + b\left(\frac{1+l}{\rho} - y^*\right),$$

where the LHS is the total wealth in this economy. This implies that $F\left(\frac{1+l}{\rho}\right) = \frac{1+l}{\rho}(1-b) + by^*$.

Finally, in the numerical solutions we focus on the case where $y^c > y^*$, i.e., the government issue some public debt before the capital constraint is binding. Therefore, the capital constraint binds when $(1-\lambda)w^h = mw = m\left(P - w^h + b\left(w^h - Dy^*\right)\right)$, and this implies,

$$y^c = \frac{m\left(F^c - by^*\right)}{1 - \lambda + m(1-b)}.$$