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Fiscal Policy over the Real Business Cycle: A Positive Theory*

Abstract

This paper presents a political economy theory of the behavior of fiscal policy over the business cycle. The theory predicts that, in both booms and recessions, fiscal policies are set so that the marginal cost of public funds obeys a submartingale. In the short run, fiscal policy can be pro-cyclical with government debt spiking up upon entering a boom. However, in the long run, fiscal policy is counter-cyclical with debt increasing in recessions and decreasing in booms. Government spending increases in booms and decreases during recessions, while tax rates decrease during booms and increase in recessions. Data on tax rates from the G7 countries supports the submartingale prediction, and the correlations between fiscal policy variables and national income implied by the theory are consistent with much of the existing evidence from the U.S. and other countries.

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1 Introduction

Real business cycle theory develops the idea that business cycles can be generated by random fluctuations in productivity. At the core of this research program, the fundamental issues are how individuals react to productivity shocks and how these reactions affect the macro economy. While the issue of reaction to shocks is typically studied at the *individual level*, it can also be raised at the *societal level*. How do individuals, through their political institutions, collectively decide to adjust fiscal policies in response to changes in productivity? Moreover, what is the role of changes in fiscal policy in amplifying or dampening shocks? Though understanding individual responses to shocks can be addressed with the tools of basic microeconomics, understanding societal responses requires a study of how collective choices are made in complex dynamic environments.

In the last two decades, political economy has made important progress, both theoretically and empirically, in understanding how governments function and the type of distortions that the political process generates in an economy. This *first generation* of research, however, has largely focused on static or two period models that are not well suited to answer the questions raised by real business cycle theory. When longer time horizons are considered, other important elements of the environment (such as shocks, rational forward looking agents, etc) are muted. Thus, the basic question as to how governments react to business cycles is not well understood. Because of this, empirical analysis on the cyclical behavior of fiscal policy remains largely guided by normative models of policy making.

With the aim of filling this void, this paper presents a positive theory of the behavior of fiscal policy over the business cycle. The theory integrates a dynamic political economy model of policy-making of the form used in Battaglini and Coate (2007, 2008) with a neoclassical real business cycle framework with serially correlated productivity shocks. The theory delivers a number of interesting predictions concerning the cyclical behavior of fiscal policy variables. Moreover, these predictions appear consistent with much of the evidence from the U.S. and other countries.

The economic model underlying the theory is a dynamic stochastic general equilibrium model in which a single good is produced using labor. This good can be consumed or used to produce a public good. Labor productivity follows a two state, serially-correlated Markov process. When productivity is high, the economy is in a "boom" and, when it is low, a "recession". The political economy component of the model assumes that policy choices in each period are made by a legislature comprised of representatives elected by single-member, geographically-defined districts. The legislature can raise revenues in two ways: via a proportional tax on labor income and by issuing one period risk-free bonds. The legislature can also purchase bonds and use the interest earnings to help finance future public spending if it so chooses. Public revenues are used to finance the provision of the public good and to provide targeted district-specific transfers, which are interpreted as pork-barrel spending. The legislature makes policy decisions by majority (or super-majority) rule and legislative policy-making is modelled as non-cooperative bargaining. The level of public debt and the persistent level of productivity are the state variables, creating a dynamic linkage across policy-making periods.

The most striking prediction of the theory concerns the dynamic evolution of the so-called *marginal cost of public funds* (MCPF). The MCPF, a basic concept in public finance, is the social marginal cost of raising an additional unit of tax revenue. It takes into account the distortionary costs of taxation for the economy. In our model, it depends upon the tax rate and the elasticity of labor supply. Our theory implies that, at each point in time and over all phases of the cycle, the equilibrium choice of fiscal policies is such that the MCPF obeys a submartingale.[1] This means the expected MCPF next period is always at least as large as the current MCPF and is sometimes strictly larger. This prediction contrasts with that emerging from a planning model which implies that the MCPF obeys a martingale. Political distortions therefore create a wedge between the current MCPF and the future MCPF. Moreover, this wedge is likely to be greater the lower is the current MCPF, the lower is the level of government debt and the higher is the productivity of the economy.

The theory also implies that fiscal policy will converge to a stochastic steady state in which policy varies predictably over the business cycle. Upon entering a boom, public spending will increase, tax rates will fall, but the primary surplus will increase. Over the course of the boom, public spending will continue to increase until it reaches a ceiling level, and tax rates and the primary surplus will decrease until they reach floor levels. When the economy enters a recession, public spending will decrease, tax rates will increase, but the primary surplus will fall. As the recession progresses, public spending will continue to decrease, tax rates will continue to increase, and the primary surplus will increase. The overall fiscal stance as measured by the long run pattern

[1] In our model the assumptions of the standard submartingale convergence theorem are not satisfied, so the MCPF does not converge to a constant as $t \to \infty$. Indeed, we show that in the long run the MCPF will have a non degenerate stationary distribution.

of debt is counter-cyclical: government debt decreases in booms and increases in recessions.[2]

Perhaps the most interesting feature of the long run cyclical behavior of fiscal policy is that debt *falls* when the economy enters a boom. Intuitively, one might have guessed just the opposite. After all, a boom increases both current and expected future productivity, which reduces the expected marginal cost of borrowing. This reduction in cost might be expected to lead legislators to increase debt and use the proceeds to provide pork to their districts. This intuition is correct, but ignores the fact that any increase in debt will have permanent effects. Thus, such a pro-cyclical, debt-financed pork-fest can occur only in the short run, the first time the economy moves from recession to boom. After it occurs, the level of debt is too high in recessions for it to ever occur again.

The submartingale prediction of the theory can be tested with time series data on tax rates and an estimate of the elasticity of labor supply and we find supporting evidence for the prediction in data from the U.S. and the other G7 countries. The cyclical predictions of the theory imply that debt should be negatively correlated with changes in GDP, while spending should be positively correlated. The implication concerning debt is consistent with evidence from the U.S. and that concerning spending is consistent with evidence from the U.S. states and many other countries. The theory implies that the relationship between the primary surplus and changes in GDP depends on the phase of the cycle and thus is theoretically ambiguous. This may help explain the varied correlations that are found in the data. The theory also offers new predictions on the cyclical behavior of the primary surplus, tax revenues, and pork-barrel spending that await testing.

The organization of the remainder of the paper is as follows. Section 2 explains how our paper relates to prior work on the theory of fiscal policy. Section 3 outlines the model and Section 4 establishes a benchmark by describing socially optimal fiscal policies. Section 5 defines political equilibrium, develops a useful characterization of equilibrium, and establishes existence. Section

[2] There are a number of definitions of "counter-cyclical" fiscal policy in the literature. Consistent with a Keynesian perspective, Kaminsky, Reinhart and Vegh (2004) and Talvi and Vegh (2005) define fiscal policy to be counter-cyclical if government spending rises in recessions and tax rates fall. Adopting a neoclassical perspective, Alesina, Campante, and Tabellini (2007) define as counter-cyclical "a policy that follows the tax smoothing principle of holding constant tax rates and discretionary spending as a fraction of GDP over the cycle". Our definition is that fiscal policy is counter-cyclical if debt falls in booms and rises in recessions. Like Alesina, Campante, and Tabellini, our definition is motivated by tax smoothing principles. However, it recognizes the fact that in a world with incomplete markets and unanticipated productivity shocks, these principles do not imply constant tax rates or government spending over the cycle. While reflecting a neoclassical perspective, our definition does not discriminate between a neoclassical and Keynesian view of optimal fiscal policy over the cycle: in both cases, government debt will rise in recessions and fall in booms. As suggested by Kaminsky, Reinhart and Vegh (2004), the way to discriminate between these views is to look at the behavior of tax rates and public spending. We will discuss this point in greater detail in Section 6.

6 derives the submartingale result on the marginal cost of public funds and Section 7 explores the cyclical properties of fiscal policy. Section 8 evaluates the empirical implications of the theory and Section 9 concludes.

2 Related literature

The bulk of theoretical work on the cyclical behavior of fiscal policy has been normative. The theoretical framework that has guided empirical work is the tax smoothing theory of fiscal policy with perfect foresight (Barro (1979)). This theory implies that the government should perfectly smooth both tax rates and government spending by borrowing in recessions and repaying in booms (see, for example, Talvi and Vegh (2005)). The empirical literature on cyclicality sees the evidence from developed countries as broadly in line with these predictions, while that from developing countries is not. In particular, government spending is strongly pro-cyclical in developing countries.[3] This has led the literature to regard the perfect foresight tax smoothing model as an adequate positive model for developed countries but not for developing countries.

A variety of theories have been advanced to explain the stronger pro-cyclical behavior of government spending in developing countries. In an early attempt to explain the phenomenon, Gavin and Perotti (1997) note that pro-cyclical policies may be induced by tighter debt constraints in recessions. Borrowing limits in recessions would force contractionary policies; as the limits are relaxed in booms, we would observe expansionary policies. Other authors point to the dysfunctional political systems that pervade developing countries. In a dynamic common pool framework in which multiple groups compete for a share of the national pie, Lane and Tornell (1998) and Tornell and Lane (1999) suggest that group competition can increase following a positive income shock which may lead spending to increase more than proportionally to the increase in income (the so-called *voracity effect*). In the context of a perfect foresight tax smoothing model, Talvi and Vegh (2005) show that if spending pressures increase with the size of the primary surplus, then optimal fiscal policy will imply a pro-cyclical pattern of spending. In a political agency framework, Alesina, Campante and Tabellini (2007) show that when faced with corrupt governments whose debt and consumption choices are hard to observe, citizens may rationally demand higher public spending in a boom.

[3] The empirical literature is reviewed in more detail in Section 8 below.

We take issue with the literature's view that the perfect foresight tax smoothing model is adequate to explain the cyclical behavior of fiscal policy in developed economies. First, the empirical evidence shows that government spending tends to be pro-cyclical even in developed economies. Second, under the more palatable assumption that cyclical variations are not perfectly foreseen, the tax smoothing approach has trouble explaining cyclical fiscal policy in the long run. Specifically, in environments with incomplete markets, the approach often implies that the government should self-insure, eventually accumulating sufficient assets to finance government spending out of the interest earnings from these assets (Aiyagari et al (2002)).[4] Thus, in the long run, this model predicts no cyclical pattern in government spending or the primary surplus. Third, while political systems are admittedly less dysfunctional in developed countries, policies are determined by the voting decisions of elected representatives and these representatives are interested in redistributing to their constituents. These political forces will lead policy to depart from the normative ideal and it is important to understand how.

We see our theory as complementary to the political economy theories of Lane and Tornell and Alesina, Campante and Tabellini. They are interested in modelling different, and much more dysfunctional, political systems than us. As noted in the introduction, in the short run there may be episodes of procyclical fiscal policy that may resemble the voracity effect identified by Lane and Tornell. However, our analysis differs from their work in that our economy is subject to recurrent cyclical shocks rather than a one time permanent shock. This accounts for our conclusions that the voracity effect can not survive in the long run.

More generally, the theory presented here is part of a *second generation* of research in political economy attempting to develop models in more general dynamic environments of interest to macroeconomists. Examples of this type of work include Acemoglu, Golosov and Tsyvinski (2006), Azzimonti (2007), Battaglini and Coate (2008), Hassler et al (2003), Hassler et al (2005), Krussel and Rios-Rull (1999), Song, Zilibotti and Storesletten (2007) and Yared (2007). The particular model presented here builds on the model developed in Battaglini and Coate (2008). It differs in assuming, first, that labor productivity is stochastic rather than constant, and, second, that citizens' valuation of the public good is constant rather than subject to i.i.d. shocks. Thus, revenue

[4] Different conclusions arise when there are complete markets and the government can issue state-contingent debt. We focus on the incomplete markets assumption here because we feel that it is the most appropriate for a positive analysis. We refer the reader to Chari, Christiano and Kehoe (1994) for a comprehensive analysis of optimal fiscal policy in a real business cycle model with complete markets.

shocks replace public spending shocks as the driver of fiscal policy. More importantly, the revenue shocks are persistent because labor productivity follows a serially correlated Markov Process. This allows us to explore the cyclical behavior of fiscal policy. This work also advances our previous research by developing the implications of political decision-making for the martingale properties of tax rates and the marginal cost of public funds.

Finally, we note that our theory is related to, but distinct from, the literature on the *political business cycle*.[5] This literature focuses on cyclical effects of expansionary fiscal policies generated by the attempts of incumbent politicians to win elections. These effects arise when voters are myopic, or when there is asymmetric information about politicians' abilities and incumbents use spending as a signalling device. We assume rational forward-looking voters and complete information, so the phenomena underlying political business cycles are not present in our model. Our goal is to study how politicians *react* to shocks to the real economy rather than to present a theory of how the political system generates cycles around elections.

3 The model

3.1 The economic environment

A continuum of infinitely-lived citizens live in n identical districts indexed by $i = 1, ..., n$. The size of the population in each district is normalized to be one. There is a single (nonstorable) consumption good, denoted by z, that is produced using a single factor, labor, denoted by l, with the linear technology $z = wl$. There is also a public good, denoted by g, that can be produced from the consumption good according to the linear technology $g = z/p$.

Citizens consume the consumption good, benefit from the public good, and supply labor. Each citizen's per period utility function is

$$z + Ag^{\alpha} - \frac{l^{(1+1/\varepsilon)}}{\varepsilon + 1}, \tag{1}$$

where $\alpha \in (0, 1)$ and $\varepsilon > 0$. The parameter A measures the value of the public good to the citizens. Citizens discount future per period utilities at rate δ.

The productivity of labor w varies across periods in a random way, reflecting the business cycle. Specifically, the economy can either be in a *boom* or a *recession*. Labor productivity is w_H

[5] See Drazen (2000) and Persson and Tabellini (2000) for excellent reviews of the political business cycle literature.

in a boom and w_L in a recession, where $w_L < w_H$. The state of the economy follows a first order Markov process, with transition matrix

$$\begin{bmatrix} \alpha_{LL} & \alpha_{LH} \\ \alpha_{HL} & \alpha_{HH} \end{bmatrix}.$$

Thus, conditional on the economy being in a recession, the probability of remaining in a recession is α_{LL} and the probability of transitioning to a boom is α_{LH}. Similarly, conditional on being in a boom, the probability of remaining in a boom is α_{HH} and the probability of transitioning to a recession is α_{HL}. Though in many environments it is natural to assume that states are persistent, this assumption is not necessary for our results. However, we do require that α_{HH} exceeds α_{LH}, so that the economy is more likely to be in a boom if it was in a boom the previous period.[6]

There is a competitive labor market and competitive production of the public good. Thus, the wage rate is equal to w_H in a boom and w_L in a recession and the price of the public good is p. There is also a market in risk-free one period bonds. The assumption of a constant marginal utility of consumption implies that the equilibrium interest rate on these bonds must be $\rho = 1/\delta - 1$. At this interest rate, citizens will be indifferent as to their allocation of consumption across time.

3.2 Government policies

The public good is provided by the government. The government can raise revenue by levying a proportional tax on labor income. It can also borrow and lend by selling and buying bonds. Revenues can not only be used to finance the provision of the public good but can also be diverted to finance targeted district-specific transfers which are interpreted as (non-distortionary) pork-barrel spending.

Government policy in any period is described by an $n+3$-tuple $\{r, g, x, s_1, ..., s_n\}$, where r is the income tax rate, g is the amount of the public good provided, x is the amount of bonds sold, and s_i is the proposed transfer to district i's residents. When x is negative, the government is buying bonds. In each period, the government must also repay any bonds that it sold in the

[6] Our basic model assumes that in the "up-part" of the business cycle there is a single productivity level w_H, and in the "down-part" a single productivity level w_L. Thus, within booms and recessions, there is no variation in productivity. While this is a rather spartan conception of a business cycle, the model can be extended to incorporate within state productivity shocks by assuming that productivity in state θ is given by $w_\theta + \omega$ where ω is an i.i.d "shock" with mean zero, range $[-\overline{\omega}, \overline{\omega}]$. Though the introduction of i.i.d shocks makes the distinction between booms and recessions less clear-cut, the equilibrium of the extended model has the same structure as the equilibrium of the simpler model described in the text and produces the same predictions of the key correlation between macro variables. A more complete analysis of this extension is available from the authors.

previous period. Thus, if it sold b bonds in the previous period, it must repay $(1+\rho)b$ in the current period. The government's initial debt level in period 1 is given exogenously and is denoted by b_0.

In a period in which government policy is $\{r, g, x, s_1, ..., s_n\}$ and the state of the economy (i.e., boom or recession) is $\theta \in \{L, H\}$, each citizen will supply an amount of labor

$$l_\theta^*(r) = \arg\max_l \{w_\theta(1-r)l - \frac{l^{(1+1/\varepsilon)}}{\varepsilon+1}\}. \tag{2}$$

It is straightforward to show that $l_\theta^*(r) = (\varepsilon w_\theta(1-r))^\varepsilon$, so that ε is the elasticity of labor supply. A citizen in district i who simply consumes his net of tax earnings and his transfer will obtain a per period utility of $u_\theta(r, g) + s_i$, where

$$u_\theta(r, g) = \frac{\varepsilon^\varepsilon (w_\theta(1-r))^{\varepsilon+1}}{\varepsilon+1} + Ag^\alpha. \tag{3}$$

Since citizens are indifferent as to their allocation of consumption across time, their lifetime expected utility will equal the value of their initial bond holdings plus the payoff they would obtain if they simply consumed their net earnings and transfers in each period.

Government policies must satisfy three feasibility constraints. The first is that revenues must be sufficient to cover expenditures. To see what this implies, consider a period in which the initial level of government debt is b, the policy choice is $\{r, g, x, s_1, ..., s_n\}$, and the state of the economy is θ. Expenditure on public goods and debt repayment is $pg + (1+\rho)b$, tax revenue is

$$R_\theta(r) = nrw_\theta l_\theta^*(r) = nrw_\theta(\varepsilon w_\theta(1-r))^\varepsilon, \tag{4}$$

and revenue from bond sales is x. Letting the *net of transfer surplus* (i.e., the difference between revenues and spending on public goods and debt repayment) be denoted by

$$B_\theta(r, g, x; b) = R_\theta(r) - pg + x - (1+\rho)b, \tag{5}$$

the constraint requires that $B_\theta(r, g, x; b) \geq \sum_i s_i$.

The second constraint is that the district-specific transfers must be non-negative (i.e., $s_i \geq 0$ for all i). This rules out financing public spending via district-specific lump sum taxes. With lump sum taxes, there would be no need to impose the distortionary labor tax and hence no tax smoothing problem.

The third and final constraint is that the amount of government borrowing must be feasible. In particular, there is an upper limit $\overline{x}$ on the amount of bonds the government can sell. This limit is motivated by the unwillingness of borrowers to hold bonds that they know will not be repaid. If the government were borrowing an amount x such that the interest payments exceeded the maximum possible tax revenues in a recession; i.e., $\rho x > \max_r R_L(r)$, then, if the economy were in recession, it would be unable to repay the debt *even if it provided no public goods or transfers.* Thus, the maximum level of debt is $\overline{x} = \max_r R_L(r)/\rho$.

We avoid assuming that there is any "ad hoc" limit on the amount of bonds that the government can purchase (see Aiyagari et al (2002)). In particular, the government is allowed to hold sufficient bonds to permit it to always finance the Samuelson level of the public good from the interest earnings. This level of bonds is given by $\underline{x} = -pg_S/\rho$, where g_S is the level of the public good that satisfies the *Samuelson Rule.*[7] Since the government will never want to hold more bonds than this, there is no loss of generality in constraining the choice of debt to the interval $[\underline{x}, \overline{x}]$ and we will do this below.[8] We also assume that the initial level of government debt, b_0, belongs to the interval $(\underline{x}, \overline{x})$.

3.3 The political process

Government policy decisions are made by a legislature consisting of representatives from each of the n districts. One citizen from each district is selected to be that district's representative. Since all citizens have the same policy preferences, the identity of the representative is immaterial and hence the selection process can be ignored.[9] The legislature meets at the beginning of each period. These meetings take only an insignificant amount of time, and representatives undertake private sector work in the rest of the period just like everybody else. The affirmative votes of $q < n$ representatives are required to enact any legislation.

To describe how legislative decision-making works, suppose the legislature is meeting at the beginning of a period in which the current level of public debt is b and the state of the economy is θ. One of the legislators is randomly selected to make the first proposal, with each representative having an equal chance of being recognized. A proposal is a policy $\{r, g, x, s_1, ..., s_n\}$ that satisfies

[7] The Samuelson Rule is that the sum of marginal benefits equal the marginal cost, which means that g_S satisfies the first order condition that $n\alpha A g^{\alpha-1} = p$.

[8] By assuming that the government can choose to borrow any amount in the interval $[\underline{x}, \overline{x}]$, we are implicitly assuming that labor productivity is sufficiently high that the amount spent on public goods is never higher than national income. A sufficient condition for this is that $nw_L(\varepsilon w_L(\frac{\varepsilon}{1+\varepsilon}))^{\varepsilon} > pg_S$ (see Battaglini and Coate (2008) for details).

[9] While citizens may differ in their bond holdings, this has no impact on their policy preferences.

the feasibility constraints. If the first proposal is accepted by q legislators, then it is implemented and the legislature adjourns until the beginning of the next period. At that time, the legislature meets again with the difference being that the initial level of public debt is x and that the state of the economy may have changed. If, on the other hand, the first proposal is not accepted, another legislator is chosen to make a proposal. There are $T \geq 2$ such proposal rounds, each of which takes a negligible amount of time. If the process continues until proposal round T, and the proposal made at that stage is rejected, then a legislator is appointed to choose a default policy. The only restrictions on the choice of a default policy are that it be feasible and that it involve a uniform district-specific transfer (i.e., $s_i = s_j$ for all i, j).

4 The social planner's solution

To create a normative benchmark with which to compare the political equilibrium, we begin by describing what fiscal policy would look like if policies were chosen by a social planner who wished to maximize aggregate utility. The planner's problem can be formulated recursively. In a period in which the current level of public debt is b and the state of the economy is θ, the problem is to choose a policy $\{r, g, x, s_1, ..., s_n\}$ to solve:

$$\begin{array}{c} \max \ u_\theta(r,g) + \frac{\sum_i s_i}{n} + \delta[\alpha_{\theta H} v_H^\circ(x) + \alpha_{\theta L} v_L^\circ(x)] \\ s.t. \ \ s_i \geq 0 \ \text{ for all } i, \ \sum_i s_i \leq B_\theta(r,g,x;b), \ \& \ x \in [\underline{x}, \overline{x}], \end{array} \tag{6}$$

where $v_\theta^\circ(x)$ denotes the representative citizen's value function in state θ (net of bond holdings).

Surplus revenues will optimally be rebated back to citizens and hence $\sum_i s_i = B_\theta(r,g,x;b)$. Thus, we can reformulate the problem as choosing a tax-public good-debt triple (r,g,x) to solve:

$$\begin{array}{c} \max \ u_\theta(r,g) + \frac{B_\theta(r,g,x;b)}{n} + \delta[\alpha_{\theta H} v_H^\circ(x) + \alpha_{\theta L} v_L^\circ(x)] \\ s.t. \ \ B_\theta(r,g,x;b) \geq 0 \ \& \ x \in [\underline{x}, \overline{x}]. \end{array} \tag{7}$$

The problem in this form is fairly standard. The citizen's value functions v_L° and v_H° solve the pair of functional equations

$$v_\theta^\circ(b) = \max_{(r,g,x)} \left\{ \begin{array}{c} u_\theta(r,g) + \frac{B_\theta(r,g,x;b)}{n} + \delta[\alpha_{\theta H} v_H^\circ(x) + \alpha_{\theta L} v_L^\circ(x)] \\ \\ s.t. \ \ B_\theta(r,g,x;b) \geq 0 \ \& \ x \in [\underline{x}, \overline{x}] \end{array} \right\} \quad \theta \in \{L, H\} \tag{8}$$

and the planner's policies in state θ, $\{r^{\circ}_{\theta}(b), g^{\circ}_{\theta}(b), x^{\circ}_{\theta}(b)\}$, are the optimal policy functions for this program.

In any given state (b, θ) the planner's optimal policies $\{r^{\circ}_{\theta}(b), g^{\circ}_{\theta}(b), x^{\circ}_{\theta}(b)\}$ are implicitly defined by three conditions. The first is that the social marginal benefit of the public good is equal to the social marginal cost of financing it; that is,

$$n\alpha A g^{\alpha-1} = p(\frac{1-r}{1-r\,(1+\varepsilon)}). \tag{9}$$

To interpret this, note that $(1-r)/(1-r(1+\varepsilon))$ measures the *marginal cost of public funds* (MCPF) - the social cost of raising an additional unit of revenue via a tax increase. The term on the right hand side therefore represents the cost of financing an additional unit of the public good. The condition is just the *Samuelson Rule* modified to take account of the fact that taxation is distortionary and it determines the optimal public good level for any given tax rate. The second condition is that the marginal cost of public funds today equals the expected marginal cost of debt tomorrow; that is,[10]

$$\frac{1-r}{1-r\,(1+\varepsilon)} = -n\delta[\alpha_{\theta H} v^{\circ\prime}_{H}(x) + \alpha_{\theta L} v^{\circ\prime}_{L}(x)]. \tag{10}$$

This ensures that, on the margin, the cost of financing public goods via taxation equals that of financing them by issuing debt. The final condition is that the net of transfer surplus be zero; that is,

$$B_{\theta}(r, g, x; b) = 0. \tag{11}$$

This implies that the planner raises no more revenues than are necessary to finance public good spending.

Using these conditions, it is possible to show that for each state θ the optimal tax rate and debt level are increasing in b and the optimal public good level is decreasing in b. Using the *Envelope Theorem*, it is also straightforward to show that the marginal cost of debt tomorrow in state θ is just the marginal cost of public funds tomorrow in state θ; that is,

$$-n\delta v^{\circ\prime}_{\theta}(x) = (\frac{1-r^{\circ}_{\theta}(x)}{1-r^{\circ}_{\theta}(x)\,(1+\varepsilon)}). \tag{12}$$

Substituting this into (10), yields the Euler equation for the planner's problem:

$$\frac{1-r^{\circ}_{\theta}(b)}{1-r^{\circ}_{\theta}(b)\,(1+\varepsilon)} = \alpha_{\theta H}(\frac{1-r^{\circ}_{H}(x^{\circ}_{\theta}(b))}{1-r^{\circ}_{H}(x^{\circ}_{\theta}(b))\,(1+\varepsilon)}) + \alpha_{\theta L}(\frac{1-r^{\circ}_{L}(x^{\circ}_{\theta}(b))}{1-r^{\circ}_{L}(x^{\circ}_{\theta}(b))\,(1+\varepsilon)}). \tag{13}$$

[10] Note that in deriving (10) we are ignoring the upperbound $x \leq \overline{x}$. We show in the Appendix (Section 10.6) that this is without loss of generality.

This equation tells us that the optimal debt level equalizes the current MCPF with the corresponding expected MCPF and implies that *the MCPF obeys a martingale.*[11] The condition illustrates the planner's desire to smooth taxation between periods. Since the MCPF is a convex function of the tax rate r, the martingale property implies that the current tax rate exceeds the expected tax rate; that is, $r_{\theta}^{\circ}(b) > \alpha_{\theta H} r_{H}^{\circ}(x_{\theta}^{\circ}(b)) + \alpha_{\theta L} r_{L}^{\circ}(x_{\theta}^{\circ}(b))$. Thus, the tax rate behaves as a supermartingale.

The Euler equation (13) is the key to understanding the dynamic evolution of the system. It implies that the planner raises debt in a recession and lowers it in a boom. He raises debt in a recession because he anticipates that the economic environment can only improve in the future. If it does improve, the MCPF will be lower since tax rates are lower in booms than in recessions.[12] Thus, debt must increase to maintain equation (13). Likewise, when the economy is in a boom, the planner anticipates that the economic environment can only get worse in the future and thus increases debt. The upshot is that debt behaves counter-cyclically. On the other hand, public good spending behaves pro-cyclically with spending increasing in booms and falling in recessions.

What happens in the long run? Consider what would happen if the economy were in a boom for a very long period of time. Then, tax rates would fall, public good provision would increase, and debt levels would fall. Eventually, the debt level would reach the lower bound $\underline{x}$. At this point, the government would have accumulated sufficient assets to finance a first best level of the public good without taxation. From this point on, whatever the state of the economy, the government could set the tax rate equal to zero, the public good to the Samuelson level, and balance its budget. The planner would have no incentive to either accumulate further assets or to reduce assets because tax rates and public good levels would be totally smooth. A steady state would be reached. In the convergence to the steady state, the MCPF continues to be a martingale, but it becomes degenerate since both the current and expected MCPF converge to 1. Moreover, the tax rate converges to zero.

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[11] A formal derivation of (13) is available in the Appendix (Section 10.6). Aiyagari et al [2002] show a similar result for the planner's solution under similar assumptions. To ease the comparison, however, note that our MCPF corresponds to the negative of their Lagrangian multiplier ψ. It should also be noted that in their model the planner's MCPF follows a supermartingale because the upper bound on debt will bind with positive probability. This however depends on the fact that g_t is an exogenous process. As we show in the appendix, this can not happen in our framework because g_t is endogenous.

[12] While tax rates being lower in booms than in recessions (i.e., $r_{H}^{\circ}(b) < r_{L}^{\circ}(b)$) seems natural, it may not be immediate how to prove it. Since the planner's solution is a special case of the political equilibrium when $q = n$, the result will follow from Lemma 2 in Section 7.

We conclude therefore that if the economy were in a boom for a sufficiently long period of time, the debt level would fall to $\underline{x}$ and a steady state would be reached. Now observe that with probability one there must eventually arise a boom period sufficiently long to allow the planner to reach the debt level $\underline{x}$. Thus, we have that:[13]

Proposition 1. *The social planner's solution converges to a steady state in which the debt level is $\underline{x}$, the tax rate is* 0*, and the public good level is g_S.*

The key point to note is that, while in the short run debt displays the counter-cyclical pattern usually associated with the tax smoothing approach, this disappears in the long run. Moreover, all other fiscal policy variables are also constant. Proposition 1 thus underscores the point made in Section 2: when cyclical variations are not perfectly anticipated, the tax smoothing approach has difficulty explaining cyclical fiscal policy in the long run.

5 Political equilibrium

5.1 Definition

To characterize behavior when policies are chosen by a legislature, we look for a symmetric Markov-perfect equilibrium. In this type of equilibrium any representative selected to propose at round $\tau \in \{1, ..., T\}$ of the meeting at some time t makes essentially the same proposal and this depends only on the current level of public debt (b) and the state of the economy (θ). Similarly, at the voting stage of a round τ, the probability a legislator votes for a proposal depends only on the proposal itself and the state (b, θ). As is standard in the theory of legislative voting, we focus on weakly stage undominated strategies, which implies that legislators vote for a proposal if they prefer it (weakly) to continuing on to the next proposal round.

An equilibrium can be described by a collection of proposal functions $\{r_\theta^\tau(b),\ g_\theta^\tau(b),\ x_\theta^\tau(b),$ $s_\theta^\tau(b)\}_{\tau=1}^T$ which specify the proposal made by the proposer in round τ of the meeting in a period in which the state is (b, θ). Here $r_\theta^\tau(b)$ is the proposed tax rate, $g_\theta^\tau(b)$ is the public good level, $x_\theta^\tau(b)$ is the new level of public debt, and $s_\theta^\tau(b)$ is a transfer offered to the districts of $q-1$ randomly selected representatives. The proposer's district receives the surplus revenues $B_\theta(r_\theta^\tau(b), g_\theta^\tau(b), x_\theta^\tau(b); b) - (q-1)s_\theta^\tau(b)$. Associated with any equilibrium are a collection of value functions $\{v_\theta^\tau(b)\}_{\tau=1}^{T+1}$ which

[13] A formal proof of the following result is presented in the Appendix (Section 10.13). A similar conclusion holds when public spending shocks rather than revenue shocks are the driver of fiscal policy (see Aiyagari et al (2002) and Battaglini and Coate (2008)).

specify the expected future payoff of a legislator at the beginning of proposal round τ in a period in which the state is (b, θ).

We focus, without loss of generality, on equilibria in which at each round τ, proposals are immediately accepted by at least q legislators, so that on the equilibrium path, no meeting lasts more than one proposal round. Accordingly, the policies that are actually implemented in equilibrium are those proposed in the first round. In what follows, we will drop the superscript and refer to the round 1 value function as $v_\theta(b)$ and the round 1 policy proposal as $\{r_\theta(b),\, g_\theta(b),\, x_\theta(b),\, s_\theta(b)\}$.

In equilibrium, there is a reciprocal feedback between the policy proposals $\{r^\tau_\theta(b),\, g^\tau_\theta(b),\, x^\tau_\theta(b),\, s^\tau_\theta(b)\}_{\tau=1}^{T}$ and the associated value functions $\{v^\tau_\theta(b)\}_{\tau=1}^{T+1}$. On the one hand, given that future payoffs are described by the value functions, the prescribed policy proposals must maximize the proposer's payoff subject to the incentive constraint of getting the required number of affirmative votes and the appropriate feasibility constraints. Formally, given $\{v^\tau_\theta(b)\}_{\tau=1}^{T+1}$, for each proposal round τ and state (b, θ), the proposal $\{r^\tau_\theta(b),\, g^\tau_\theta(b),\, x^\tau_\theta(b),\, s^\tau_\theta(b)\}$ must solve the problem:

$$\begin{aligned} &\max_{(r,g,x,s)} u_\theta(r,g) + B_\theta(r,g,x;b) - (q-1)\,s + \delta[\alpha_{\theta H} v_H(x) + \alpha_{\theta L} v_L(x)] \\ &\quad s.t. \quad u_\theta(r,g) + s + \delta[\alpha_{\theta H} v_H(x) + \alpha_{\theta L} v_L(x)] \geq v^{\tau+1}_\theta(b), \\ &\qquad\qquad B_\theta(r,g,x;b) \geq (q-1)s, \quad s \geq 0 \;\&\; x \in [\underline{x}, \overline{x}]. \end{aligned}$$

The first constraint is the incentive constraint and the remainder are feasibility constraints. The formulation reflects the assumption that on the equilibrium path, the proposal made in round 1 is accepted.

On the other hand, the value functions $\{v^\tau_\theta(b)\}_{\tau=1}^{T+1}$ are themselves determined by the equilibrium policy proposals. The legislators' round 1 value functions $v_L(b)$ and $v_H(b)$ are determined recursively using $\{r_\theta(b),\, g_\theta(b),\, x_\theta(b),\, s_\theta(b)\}$ by the system:

$$v_\theta(b) = u_\theta(r_\theta(b), g_\theta(b)) + \frac{B_\theta(r_\theta(b), g_\theta(b), x_\theta(b); b)}{n} + \delta[\alpha_{\theta H} v_H(x_\theta(b)) + \alpha_{\theta L} v_L(x_\theta(b))] \quad \theta \in \{L, H\}. \tag{14}$$

To understand this recall that a legislator is chosen to propose in round 1 with probability $1/n$. If chosen to propose, he obtains a payoff in that period of

$$u_\theta(r_\theta(b), g_\theta(b)) + B_\theta(r_\theta(b), g_\theta(b), x_\theta(b); b) - (q-1)s_\theta(b).$$

If he is not chosen to propose, but is included in the coalition of legislators whose districts receive a transfer, he obtains $u_\theta(r_\theta(b), g_\theta(b)) + s_\theta(b)$, and, if he is not included, he obtains just

$u_\theta(r_\theta(b), g_\theta(b))$. The probability that his district will receive a transfer, conditional on not being chosen to propose, is $(q-1)/(n-1)$. Taking expectations, the pork barrel transfers $s_\theta(b)$ cancel and the period payoff is as described in (14).

The value functions for rounds 2 and beyond are determined by the associated policy proposals and the round 1 value functions. For all proposal rounds $\tau = 1,..,T-1$ the expected future payoff of a legislator if the round τ proposal is rejected is

$$v_\theta^{\tau+1}(b) = u_\theta\left(r_\theta^{\tau+1}(b), g_\theta^{\tau+1}(b)\right) + \frac{B_\theta(r_\theta^{\tau+1}(b), g_\theta^{\tau+1}(b), x_\theta^{\tau+1}(b); b)}{n} + \delta[\alpha_{\theta H} v_H(x_\theta^{\tau+1}(b)) + \alpha_{\theta L} v_L(x_\theta^{\tau+1}(b))].$$

This reflects the assumption that, in the out-of-equilibrium event that play reaches proposal round $\tau+1$, the proposal made at that point will be immediately accepted. Recall that if the round T proposal is rejected, the assumption is that a legislator is appointed to choose a default tax rate, public good level, level of debt and a uniform transfer. Thus,

$$v_\theta^{T+1}(b) = \max_{(r,g,x)} \left\{ u_\theta(r,g) + \frac{B_\theta(r,g,x;b)}{n} + \delta[\alpha_{\theta H} v_H(x) + \alpha_{\theta L} v_L(x)] : B_\theta(r,g,x;b) \geq 0 \ \& \ x \in [\underline{x}, \overline{x}] \right\}.$$

We say that an equilibrium is *well-behaved* if the associated round 1 legislators' value functions v_L and v_H are continuous and concave on $[\underline{x}, \overline{x}]$. In what follows, we will first characterize a well-behaved equilibrium and then establish the existence of such an equilibrium. Henceforth, when we refer to an "equilibrium", it is to be understood that it is well-behaved.

5.2 Characterization

To understand equilibrium behavior, note that to get support for his proposal the proposer must obtain the votes of $q-1$ other representatives. Accordingly, given that utility is transferable, he is effectively making decisions to maximize the utility of q legislators. It is therefore *as if* a randomly chosen *minimum winning coalition* (mwc) of q representatives is selected in each period and this coalition chooses a policy choice to maximize its aggregate utility. Formally, this means that, when the state is (b, θ), the tax-public good-debt triple (r, g, x) solves the problem

$$\begin{aligned} &\max u_\theta(r,g) + \tfrac{B_\theta(r,g,x;b)}{q} + \delta[\alpha_{\theta H} v_H(x) + \alpha_{\theta L} v_L(x)] \\ &s.t. \quad B_\theta(r,g,x;b) \geq 0 \ \& \ x \in [\underline{x}, \overline{x}]\,. \end{aligned} \tag{15}$$

In any given state (b, θ), there are two possibilities: either the mwc will provide pork to the districts of its members or it will not. Providing pork requires reducing public good spending or increasing taxation in the present or the future (if financed by issuing additional debt). When b is high and/or the economy is in a recession, the opportunity cost of revenues may be too high to make this attractive. In this case, the mwc will not provide pork, so $B_\theta(r, g, x; b) = 0$. From (15), it is clear that the outcome will then be *as if* the mwc is maximizing the utility of the legislature as a whole. Indeed, the policy choice will be identical to that a benevolent planner would choose in the same state and with the same value function.

When b is low and/or the economy is in a boom, the opportunity cost of revenues is lower. Less tax revenues need to be devoted to debt repayment when b is low and both current and expected future tax revenues are more plentiful when the economy is in a boom. As a result, the mwc will allocate revenues to pork and policies will diverge from those that would be chosen by a planner. Interestingly, it turns out that this diversion of resources toward pork, effectively creates lower bounds on how low the tax rate and debt level can go, and an upper bound on how high the level of the public good can be.

To show this, we must first characterize the policy choices that the mwc selects when it provides pork. Consider again problem (15) and suppose that the constraint $B_\theta(r, g, x; b) \geq 0$ is not binding. Using the first-order conditions for this problem, we find that the optimal tax rate r^* satisfies the condition that

$$\frac{1}{q} = \frac{\left[\frac{1-r^*}{1-r^*(1+\varepsilon)}\right]}{n}. \tag{16}$$

The condition says that the benefit of raising taxes in terms of increasing the per-coalition member transfer $(1/q)$ must equal the per-capita MPCF. Similarly, the optimal public good level g^* satisfies the condition that

$$\alpha A g^{*\alpha-1} = \frac{p}{q}. \tag{17}$$

This says that the per-capita benefit of increasing the public good must equal the per-coalition member reduction in transfers that providing the additional unit necessitates. The optimal public debt level x_θ^* satisfies the condition that

$$x_\theta^* = \arg\max\{\frac{x}{q} + \delta[\alpha_{\theta H} v_H(x) + \alpha_{\theta L} v_L(x)] : x \in [\underline{x}, \overline{x}]\}. \tag{18}$$

The optimal level balances the benefit of increasing debt in terms of increasing the per-coalition member transfer with the expected per-capita cost of an increase in the debt level.

We can now make precise how the legislature's ability to divert resources toward pork-barrel spending effectively creates endogenous bounds on the policy choices.

Proposition 2. *The equilibrium value functions $v_H(b)$ and $v_L(b)$ solve the system of functional equations*

$$v_\theta(b) = \max_{(r,g,x)} \left\{ \begin{array}{c} u_\theta(r,g) + \frac{B_\theta(r,g,x;b)}{n} + \delta[\alpha_{\theta L} v_L(x) + \alpha_{\theta H} v_H(x)] \\ s.t. \;\; B_\theta(r,g,x;b) \geq 0, \; r \geq r^*, \; g \leq g^* \; \& \; x \in [x_\theta^*, \overline{x}] \end{array} \right\} \;\; \theta \in \{L, H\} \tag{19}$$

and the equilibrium policies $\{r_\theta(b),\, g_\theta(b),\, x_\theta(b)\}$ are the optimal policy functions for this program.

Thus, the equilibrium policy choices solve a *constrained planner's problem* in which the tax rate can not fall below r^*, the public good level can not exceed g^*, and debt can not fall below the state contingent threshold x_θ^*.[14] However, there is a fundamental difference with the planner's problem (8). The thresholds that constrain the policies are endogenous because they depend on the economic fundamentals and, in the case of x_L^* and x_H^*, on the equilibrium: so rather than being constraints that *affect* the value function, they are determined simultaneously *with* the value function.

Given Proposition 2, the nature of the equilibrium policies in a given state θ is clear. For any equilibrium, define b_θ^* to be the value of debt such that the triple (r^*, g^*, x_θ^*) satisfies the constraint that $B_\theta(r^*, g^*, x_\theta^*; b) = 0$. This is given by:

$$b_\theta^* = \frac{R_\theta(r^*) + x_\theta^* - pg^*}{1+\rho}. \tag{20}$$

Then, if the debt level b is such that $b \leq b_\theta^*$ the tax-public good-debt triple is (r^*, g^*, x_θ^*) and the net of transfer surplus $B_\theta(r^*, g^*, x_\theta^*; b)$ is used to finance transfers. If $b > b_\theta^*$ the budget constraint binds so that no transfers are given. The tax rate and public debt level strictly exceed (r^*, x_θ^*) and the public good level is strictly less than g^*. In this case, therefore, the solution can be characterized by obtaining the first order conditions for problem (19) with only the budget constraint binding. These are conditions (9), (10), and (11) except with the equilibrium value functions. It is easy to show that the tax rate and debt level are increasing in b, while the public good level is decreasing in b.[15]

[14] This result extends Proposition 4 of Battaglini and Coate (2008) by showing that when shocks are persistent the lower bound on debt in the constrained planning problem will be state-contingent.

[15] Details are available from the authors upon request.

5.3 Existence

To prove the existence of an equilibrium, we first establish that the conditions of Proposition 2 are not only necessary but also sufficient.

Proposition 3: *Suppose that the value functions $v_H(b)$ and $v_L(b)$ solve the system of functional equations (19) where x_L^* and x_H^* satisfy (18). Then, there exists an equilibrium in which the round 1 value functions are $v_H(b)$ and $v_L(b)$ and the round 1 policy choices $\{r_\theta(b), g_\theta(b), x_\theta(b)\}$ are the optimal policy functions for program (19).*

Together with Proposition 2, this result might be seen as rationalizing the practice of capturing political economy considerations in complex macroeconomic models by adding exogenous constraints on the planner's set of policy instruments (see, for example, Ayagari et al. (2002)). Propositions 2 and 3, however, qualify this practice by making clear that not only must the constraints be endogenous, but also they should apply to all policy variables (debt, taxes, and public good provision) and depend on the state of the economy.

Using Proposition 3 we can now establish the existence of an equilibrium by showing that there must exist a pair of value functions $v_H(b)$ and $v_L(b)$ and a pair of debt thresholds x_L^* and x_H^* such that: (i) $v_H(b)$ and $v_L(b)$ solve (19) given x_L^* and x_H^*, and, (ii) x_L^* and x_H^* solve (18) given $v_H(b)$ and $v_L(b)$. In this way, we obtain:

Proposition 4. *There exists an equilibrium.*

6 Tax smoothing in political equilibrium

As discussed in Section 4, the social planner smooths taxation over time by equalizing the current MCPF with the expected MCPF next period. This implies that the MCPF behaves as a martingale and the tax rate as a supermartingale. In this section, we explain how political decision making distorts tax smoothing.

Note first that in political equilibrium, whether the mwc is providing pork or not, the debt level must be such that the MCPF today equals the expected marginal cost of debt tomorrow; that is,[16]

$$\frac{1-r_\theta(b)}{1-r_\theta(b)\,(1+\varepsilon)} = -n\delta[\alpha_{\theta H}v_H'(x_\theta(b)) + \alpha_{\theta L}v_L'(x_\theta(b))]. \tag{21}$$

[16] Again, in deriving (21) we are ignoring the upperbound $x \leq \overline{x}$. In the Appendix (Section 10.6) we prove that this is without loss of generality.

If, for example, the MCPF exceeded the expected marginal cost of debt, the mwc could shift the financing of its spending program from taxation to debt and make each coalition member better off.

To develop the implications of equation (21), the next step is to develop an expression for the marginal cost of debt in each state.

Lemma 1. *For each state of the economy* $\theta \in \{L, H\}$*, the equilibrium value function* $v_\theta(\cdot)$ *is differentiable for all* b *such that* $b \neq b^*_\theta$*. Moreover:*

$$-v'_\theta(b) = \begin{cases} \left(\frac{1-r_\theta(b)}{1-r_\theta(b)(1+\varepsilon)}\right)\left(\frac{1+\rho}{n}\right) & \text{if } b > b^*_\theta \\ \left(\frac{1+\rho}{n}\right) & \text{if } b < b^*_\theta \end{cases} . \tag{22}$$

To understand this, recall that when the initial debt level b exceeds b^*_θ, there is no pork, so to pay back an additional unit of debt requires an increase in taxes. This means that the cost of an additional unit of debt is equal to the repayment amount $1+\rho$ multiplied by the per capita MCPF. By contrast, when b is less than b^*_θ, pork will be reduced to pay back additional debt since that is the marginal use of resources. The cost of an additional unit of debt is thus equal to $1+\rho$ multiplied by the expected per capita reduction in pork which is $1/n$. Notice that the value function is not differentiable at $b = b^*_\theta$. The left hand derivative at $b = b^*_\theta$ is equal to $(1+\rho)/n$ and the right hand derivative is equal to $(1+\rho)/q$ (since the tax rate $r_\theta(x)$ equals r^* at $b = b^*_\theta$).[17] This discontinuity reflects the fact that increasing taxes is more costly than reducing pork because the marginal cost of taxation exceeds 1.

Using Lemma 1, we can rewrite equation (21) as follows:

$$\frac{1-r_\theta(b)}{1-r_\theta(b)\,(1+\varepsilon)} = \Pr\left(\theta' \text{ s.t. } x_\theta(b) \leq b^*_{\theta'} \,|\theta\right) + \sum_{\theta' \text{ s.t. } x_\theta(b) \leq b^*_{\theta'}} \alpha_{\theta\theta'} \frac{1-r_{\theta'}(x_\theta(b))}{1-r_{\theta'}(x_\theta(b))\,(1+\varepsilon)}. \tag{23}$$

Now recall from the characterization that when $x_\theta(b)$ is less than or equal to $b^*_{\theta'}$, the tax rate $r_{\theta'}(x_\theta(b))$ will equal r^*. Thus, equation (23) can be rewritten as:[18]

$$\frac{1-r_\theta(b)}{1-r_\theta(b)\,(1+\varepsilon)} = E\left[\frac{1-r_{\theta'}(x_\theta(b))}{1-r_{\theta'}(x_\theta(b))\,(1+\varepsilon)} \,|\theta\right] - \Pr\left(\theta' \text{ s.t. } x_\theta(b) \leq b^*_{\theta'} \,|\theta\right)\left[\frac{\varepsilon r^*}{1-r^*\,(1+\varepsilon)}\right]. \tag{24}$$

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[17] The set of sub-gradients of the value function v_θ at $x = b^*_\theta$ is $[-(\frac{1+\rho}{q}), -(\frac{1+\rho}{n})]$.

[18] Equation (24) is obtained by adding and subtracting $\Pr\left(\theta' \text{ s.t. } x_\theta(b) \leq b^*_{\theta'} \,|\theta\right)\left[\frac{1-r^*}{1-r^*(1+\varepsilon)}\right]$ from the right hand side of (23).

The current MCPF is therefore less than or equal to the expected future MCPF. This yields:

Proposition 5. *In any equilibrium, the marginal cost of public funds is a submartingale; that is,*

$$\frac{1-r_\theta(b)}{1-r_\theta(b)\,(1+\varepsilon)} \leq \alpha_{\theta H}[\frac{1-r_H(x_\theta(b))}{1-r_H(x_\theta(b))\,(1+\varepsilon)}] + \alpha_{\theta L}[\frac{1-r_L(x_\theta(b))}{1-r_L(x_\theta(b))\,(1+\varepsilon)}], \tag{25}$$

with the inequality strict when b is sufficiently low.

Why when the inequality in equation (25) is strict does the mwc not find it optimal to raise taxes and reduce debt in order to equalize the current MCPF with the expected future MCPF? The answer is that if next period's mwc is providing pork, the correspondent increase in revenues will simply be diverted toward pork. This creates a wedge between the current MCPF and the expected future MCPF.[19]

What can we say about the evolution of the tax rate? As noted in the discussion of the planner's solution, when the MCPF obeys a martingale, the tax rate behaves as a supermartingale. In states (b, θ) such that $x_\theta(b)$ is less than or equal to $b^*_{\theta'}$ for some θ' however, two forces push the difference between the current and expected tax rate in opposite directions: the convexity of the MCPF pushes the difference up, and the submartingale property of the MCPF pushes it down. As we prove in the following proposition, there are states in which the first force dominates, implying that the current tax rate is strictly less than the expected future tax rate. This yields:

Proposition 6. *The tax rate is not a martingale of any type; that is, there exist states such that next period's expected tax rate exceeds the current tax rate and states such that the opposite is true.*

By the same logic, it is easy to prove that debt and the public good level will not be martingales of any type as well. In Section 7.2 we will show that the distribution of the MCPF, the tax rate, the public good level and debt all converge to a unique stationary distribution.

7 Cyclical behavior of fiscal policies

From the characterization in Section 5.2, we understand the nature of the equilibrium policies in a given state of the economy θ. This section first explains how policies compare across booms and recessions. It then use this understanding to explore the behavior of fiscal policies over the business cycle.

[19] While it was not noted, Proposition 5 is also true in the model of Battaglini and Coate (2008).

7.1 Comparing policies in booms and recessions

To compare policies across states, the key step is to understand how the political constraints change over the cycle, i.e. the relationship between x_H^* and x_L^*. The determination of the debt levels x_H^* and x_L^* can be illustrated in a simple diagram. However, some preliminary work is necessary to pave the way for the graphical analysis. First note from (18) that if the expected value function $\alpha_{\theta H}v_H + \alpha_{\theta L}v_L$ is differentiable at x_θ^*, the optimal public debt level x_θ^* satisfies the condition that:

$$\frac{1}{q} = -\delta[\alpha_{\theta H}v_H'(x_\theta^*) + \alpha_{\theta L}v_L'(x_\theta^*)]. \tag{26}$$

This tells us that the benefit of increasing debt in terms of increasing the per-coalition member transfer must equal the expected per-capita marginal cost of an increase in the debt level.[20]

The expected marginal cost of debt depends on the marginal cost of debt in both states and thus the next step is to understand how the marginal cost differs across states. Lemma 1 implies that the marginal cost of debt in a recession lies above that in a boom if two conditions are satisfied. First, the threshold debt level in a boom exceeds that in a recession (i.e., $b_H^* > b_L^*$), and, second, the tax rate in a recession is larger than that in a boom when pork is not provided (i.e., $r_H(b) < r_L(b)$ for all $b \geq b_H^*$). Intuitively, we would expect that both these conditions would be satisfied. After all, in a recession, not only is the tax base smaller but also low wages are expected to persist over time, so the expected cost of borrowing will be higher. The following result confirms these intuitions.

Lemma 2. *In any equilibrium: (i) $b_H^* > b_L^*$, and, (ii) $r_H(b) < r_L(b)$ for all $b \geq b_H^*$.*

We can now illustrate the determination of x_θ^* in an equilibrium. The horizontal axis of Figure 1 measures the level of debt and the vertical the marginal cost of debt. On the Figure, we have graphed the discounted marginal cost of debt in the two states $-\delta v_L'(b)$ and $-\delta v_H'(b)$. Following Lemma 2, the marginal cost of debt in a recession lies above that in a boom for $b \geq b_L^*$. We have also combined these two curves to form the expected discounted marginal cost of debt $-\delta[\alpha_{\theta H}v_H' + \alpha_{\theta L}v_L']$, which lies between the other two curves. The debt level x_θ^* occurs where the expected marginal cost of debt intersects the horizontal line emanating from $1/q$.

It is clear from the Figure that x_θ^* must be greater than b_L^* and can be no larger than b_H^*.

[20] If the expected value function is not differentiable at x_θ^*, then there must exist subgradients ξ_H and ξ_L of the functions v_H and v_L at x_θ^* such that $1/q = -\delta[\alpha_{\theta H}\xi_H + \alpha_{\theta L}\xi_L]$.

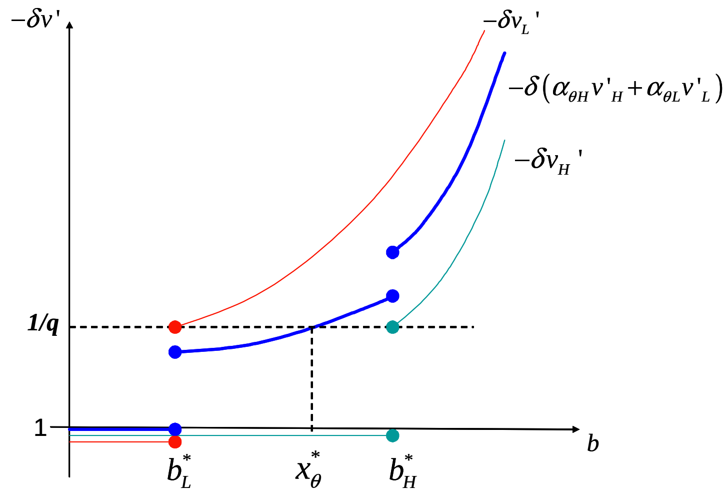

Figure 1: The determination of x_θ^*: the interior solution.

However, it is possible that x_θ^* equals b_H^*. This case, which is illustrated in Figure 2, arises when the expected marginal cost curve lies everywhere below $1/q$ on the interval (b_L^*, b_H^*). It will necessarily arise when $\alpha_{\theta H}$ is sufficiently close to 1 in which case legislators anticipate being in a boom in the next period and have no incentive to restrain their pork consumption in anticipation of a recession. In this case, the expected marginal cost function $\alpha_{\theta H} v_H + \alpha_{\theta L} v_L$ will not be differentiable at x_θ^*.

Since α_{HH} is strictly larger than α_{LH}, the expected marginal cost of debt in a boom lies to the right of that in a recession. Accordingly, if $x_L^* < b_H^*$ as in Figure 1, the amount that the mwc borrows when providing pork in a boom (x_H^*) will be bigger than the amount they borrow in a recession (x_L^*). On the other hand, if $x_L^* = b_H^*$, then $x_H^* = x_L^* = b_H^*$. This case arises only when α_{LH} is sufficiently close to α_{HH} to make a recession barely persistent. Under these circumstances, legislators would not find it optimal to borrow less when providing pork during a recession than during a boom because the recession is sufficiently likely to revert to a boom. From here on, we will assume that the transition probabilities are such that $x_L^* < x_H^*$ which we see as the most interesting case.[21]

We can now compare fiscal policies in booms and recessions. In addition to public spending,

[21] A sufficient condition for this to be true is that recessions are sufficiently likely, that is α_{LL} is sufficiently high.

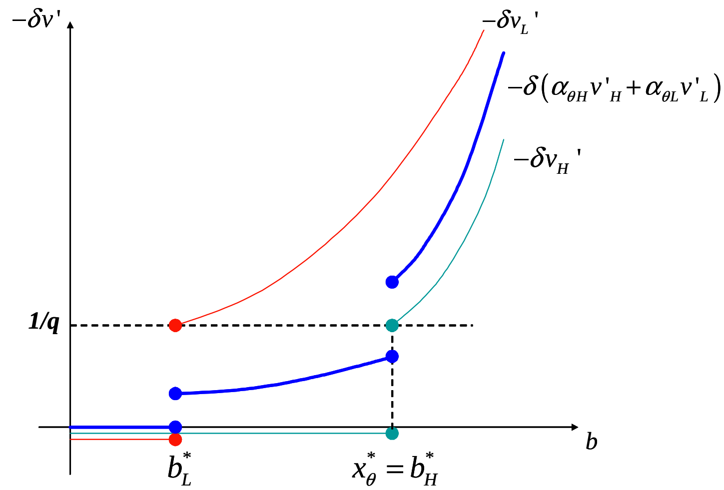

Figure 2: The determination of x_θ^*: the corner solution.

taxes and debt, we will also be interested in the primary surplus which is the difference between tax revenues and public spending other than interest payments. In our model, it is the difference between tax revenues and spending on the public good and pork. Using the budget constraint, we may write the primary surplus when the state of the economy is θ and the current debt level is b as $PS_\theta(b) = (1+\rho)b - x_\theta(b)$.

The comparison of policies will depend on the initial level of debt b. When b is less than b_L^* the mwc provides pork in both booms and recessions (since $b_L^* < b_H^*$ by Lemma 2). In this case, the tax rate and public good provision are constant across states, respectively at r^* and g^*, while debt will be higher in a boom than in a recession (respectively, x_H^* versus x_L^*). Tax revenues will be higher in a boom and these extra revenues, together with the extra borrowing, will be used to finance higher levels of pork-barrel spending. The primary surplus will be lower in a boom because borrowing is higher.

When b is between b_L^* and b_H^* the mwc provides pork in a boom but not in a recession. In this case, taxes will be higher in a recession and public good provision will be lower. Over this interval of initial debt levels, the new level of debt will be constant in a boom, but increasing in a recession. We show in the Appendix that there will be a threshold initial debt level $\widehat{b} \in (b_L^*, b_H^*)$ such that new debt will be higher in a recession if and only if $b > \widehat{b}$. Tax revenues will be higher in a boom when $b \geq \widehat{b}$, while the primary surplus will be higher in a boom if and only if $b \geq \widehat{b}$.

Finally, when b exceeds b_H^* the mwc does not provide pork in either state. In this range, public good levels will be lower in a recession ($g_L(b) < g_H(b)$), tax rates will be higher ($r_L(b) > r_H(b)$), and public borrowing will be higher ($x_L(b) > x_H(b)$). Tax revenues and the primary surplus will be higher in a boom.

7.2 Policy dynamics

With this understanding of how policies compare across the two states of the economy, we can now turn to the dynamic evolution of policy. Clearly, the key to understanding the dynamics is to understand how debt behaves. The cyclical behavior of all the remaining fiscal policies will follow from the behavior of debt given the results we already have.

The fundamental result concerning the dynamic evolution of debt is the following:

Lemma 3. *In any equilibrium: (i)* $x_L(b) > b$ *for all* $b \in [\underline{x}, \overline{x})$*, and, (ii)* $x_H(b) > b$ *for all* $b \in (\underline{x}, x_H^*)$ *and* $x_H(b) < b$ *for all* $b \in (x_H^*, \overline{x}]$*.*

Part (i) implies that the debt level always increases in a recession. Intuitively, if we are in a recession today, the economic environment can only improve in the future. This makes it worthwhile for the legislature to increase debt. Part (ii) implies that the debt level decreases in a boom if the initial debt level exceeds x_H^* and increases otherwise. Figure 3 graphs the functions $x_L(b)$ and $x_H(b)$.

We can now infer the cyclical behavior of debt. Note first that, in the short run, it is possible for debt to behave pro-cyclically - jumping up when the economy enters a boom. To see this, suppose that the economy's initial level of debt (b_0) is less than x_H^* and the economy starts out in a recession. Then, once the first boom arrives, if the level of accumulated debt remains less than x_H^*, debt will increase to x_H^* upon entering the boom. The boom increases both current and expected future productivity, which reduces the expected marginal cost of debt. Debt-financed pork instantaneously becomes more attractive for the mwc because of the downward shift in the expected marginal cost of borrowing. Debt jumps to a level at which equality between the marginal benefit of pork and the expected marginal cost of borrowing is reestablished and, during this process, a "pork-fest" occurs. This is very similar to the logic underlying Lane and Tornell's voracity effect.

In the long run, however, debt must behave counter-cyclically - decreasing when the economy

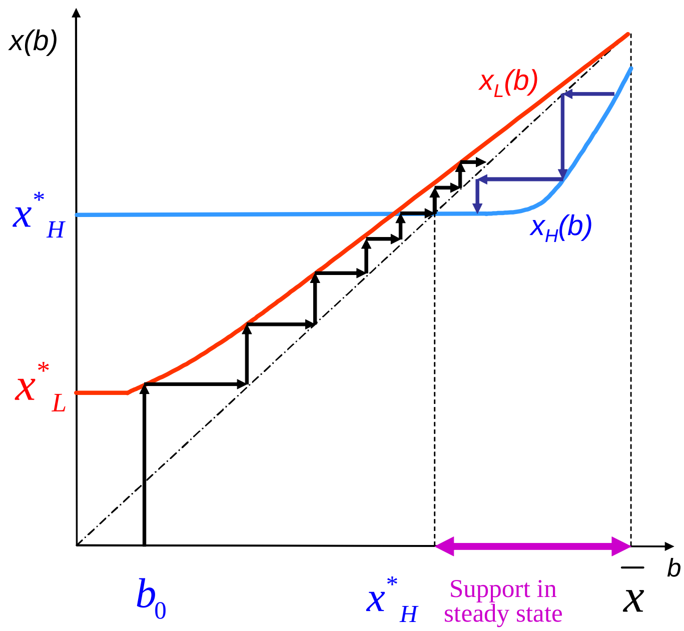

Figure 3: Equilibrium dynamics

enters a boom and increasing when it enters a recession.[22] For once such a pro-cyclical debt expansion has occurred it can never happen again. The damage of the pork-fest to public finances is permanent. This is clear from Figure 3. The debt level is bounded below by x^*_H in a boom and, as demonstrated in Lemma 3, it is increasing in a recession. In the long-run, therefore, once the first boom has occurred and debt has jumped up to x^*_H, fiscal policy will behave counter-cyclically: in a recession, debt will increase and, in a boom, debt will decrease down to x^*_H and then remain constant. Moreover, we can show that no matter what the economy's initial debt level, the same distribution of debt emerges in the long run. To summarize:

Proposition 7. *In any equilibrium, the debt distribution strongly converges to a unique, non degenerate, invariant distribution with support on $[x^*_H, \overline{x}]$. The dynamic pattern of debt is counter-cyclical. When the economy enters a recession, debt will increase and will continue to increase as long as the recession persists. When the economy enters a boom, debt decreases and, during the boom, continues to decline until it reaches x^*_H.*

[22] As noted earlier, the voracity effect papers just consider the implications of a one time positive income shock.

It is important to be clear why our theory does not predict recurrent episodes of pro-cyclical fiscal policy in the long run. As we said, these episodes occur only after the arrival of an unexpected increase in productivity that increases politicians' expectation of future revenues and triggers a permanent increase in debt. In our economy there is no permanent growth, so there is a limit to these positive productivity "surprises". Specifically, such a surprise only occurs the first time the economy moves from a recession to a boom. Once this has happened, the level of debt already incorporates the effects of potential productivity growth. In an economy with permanent growth, positive technological surprises may lead to constant (though stochastic) increases in productivity. We conjecture that in such an economy pro-cyclical "pork fests" will occur even in the long run whenever the upperbound on productivity is increased.[23] The result in Proposition 7 is therefore best interpreted as applying to a mature economy in which these growth effects are not a dominant force.

Since the remaining fiscal policies are all functions of debt, Proposition 7 implies that the distribution of these policies will also be invariant in the long term. Combining Proposition 7 with our understanding of equilibrium policies from the previous section, allows us to predict their long-run cyclical behavior. We deal first with taxes and public good spending.

Proposition 8. *In any equilibrium, in the long run, when the economy enters a recession, the tax rate increases and public good provision decreases. Moreover, the tax rate will continue to increase and public good provision will continue to decrease as long as the recession persists. When the economy enters a boom, the tax rate decreases and public good provision increases. During the boom, the tax rate continues to decline and public good provision continues to increase until they reach, respectively,* r^* *and* g^**.*

The cyclical behavior of the tax rate determines the dynamics of the MCPF. Proposition 8 implies that the MCPF will increase when the economy enters a recession and continue to increase as long as the recession persists. At any point in time, the MCPF is finite because the tax rate is always lower than the revenue maximizing level (which is $1/(1+\varepsilon)$). In a sufficiently long recession, however, the tax rate may become arbitrarily close to $1/(1+\varepsilon)$, and so the MCPF may become arbitrarily large.[24] When the MCPF is large, however, it must behave as a martingale. For,

[23] The behavior of fiscal policy in such an economy is a very interesting subject for further research.

[24] It is perhaps worthwhile to point out that the fact the MCPF is arbitrarily large when r is close to the peak of the Laffer curve only means that at that point tax revenues can not be further increased by increasing r. Moreover,

by then, b will be bigger than b_H^*. Proposition 8 also implies that the MCPF will decrease when the economy enters a boom and continue to decline until it reaches its floor level (which is n/q). Along this decreasing path, the MCPF will eventually start to behave as a strict submartingale. After a sufficiently long recession, however, the MCPF will temporarily continue to behave as a martingale even when the economy returns to a boom because it will take time for debt to reduce to a level such that the probability of the event $\{(b, \theta) \,|x_\theta(b) < b_H^*\}$ is positive.

We turn next to the cyclical pattern of pork-barrel spending.

Proposition 9. *In any equilibrium, in the long run, pork-barrel spending will not occur during a recession. Moreover, it will only occur during a boom once the debt accumulated during prior recessions has been paid off and debt has reached x_H^*.*

The only circumstance in which pork-barrel spending begins immediately when the economy moves into a boom is when the accumulated debt level is less than b_H^*. In all other cases, debt is paid down before pork-barrel spending starts up.

When combined with the dynamic pattern of public good spending described in Proposition 8, an important implication of Proposition 9 is that total public expenditure (which includes pork-barrel spending and public good provision) is pro-cyclical. The equilibrium changes in public spending and taxes therefore serve to amplify the business cycle. These predictions are distinctive and serve to nicely differentiate the predictions of our neoclassical theory from what would be expected if government were following a Keynesian counter-cyclical fiscal policy. For, in a recession, a Keynesian government would reduce taxes and increase public spending to bolster aggregate demand.

Our final fiscal policy of interest is the primary surplus.

Proposition 10. *In any equilibrium, in the long run, when the economy enters a recession, the primary surplus jumps down and then starts gradually increasing. When the economy enters a boom, the primary surplus jumps up and then starts gradually declining until it reaches a minimal level of ρx_H^*.*

This long run behavior is illustrated in Figure 4, which is drawn under the assumption that x_H^* is positive. Notice that because in long run equilibrium debt always exceeds x_H^*, the primary surplus

since it can be shown that $\sup_t E(MCPF_t)$ is unbounded, the standard submartingale convergence theorem does not apply and so the MCPF does not converge to a constant (see for example Shiryayev (1996)); and this does not imply that the MCPF converges to an arbitrarily large number. Indeed it is clear that the MCPF must recurrently drop to its floor level $\frac{n}{q}$ in the long run.

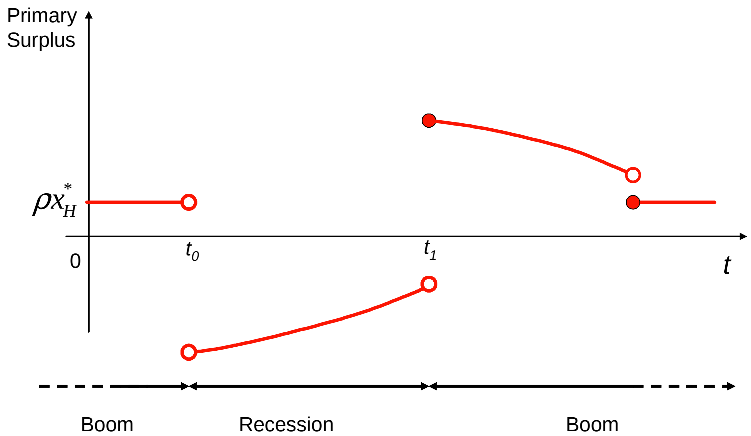

Figure 4: The long run behavior of primary surplus.

will be larger in a boom than a recession for any given level of observed debt.[25]

8 Evaluating the theory

In this section, we develop and evaluate the empirical implications of our theory. Our results contribute to two different strands of empirical research: the literature studying the martingale properties of tax rates and the literature on the cyclicality of fiscal policies. We discuss each in turn.

8.1 Martingale properties of tax rates

Stimulating a significant empirical literature, Barro (1979, 1981) conjectured that the marginal tax rate on income should follow a martingale in an optimal tax plan.[26] The discussion of the planner' problem in Section 4 has qualified this observation, showing that it is the MCPF that should evolve as a martingale in an optimal plan and the tax rate should obey a supermartingale.[27] In a political equilibrium, however, Proposition 5 shows that the MCPF follows a submartingale and Proposition 6 shows that the tax rate does not follow a martingale of any type.

[25] This follows from the results in section 7.1 once it is observed from Figure 3 that $\widehat{b}$ is smaller than x_H^*.

[26] See, for example, Bizer and Durlauf (1990), Hess (1993), and Sahasakul (1986).

[27] As noted in footnote 11, Aiyagari et al (2002) demonstrate a similar result concerning the MCPF for the planner's solution of their model. In fact, they generalize the result by showing that the MCPF is a "risk adjusted" martingale under less restrictive assumptions on the utility function.

The empirical predictions of the equilibrium concerning the dynamic evolution of the MCPF can be further refined with the results of Section 7. Proposition 7 tells us that, in the long run, debt is contained in the interval $[x_H^*, \overline{x}]$ and we know that x_H^* exceeds b_L^*. Thus, equation (24) tells us that when the probability of the event $\{(b,\theta) \,|x_\theta(b) < b_H^*\}$ is positive, the MCPF will be strictly less than the expected MCPF, but when it is zero the MCPF will obey a martingale. Since $x_\theta(b)$ is increasing in b, we may conclude that the MCPF is more likely to behave as a martingale the higher is the debt level. Moreover, since $x_L(b)$ exceeds $x_H(b)$ in the long run, the MCPF is more likely to behave as a martingale in recessions. Since the MCPF is increasing in b and higher in recessions, it follows that the MCPF will behave as a strict submartingale when it is sufficiently low.

How can we test these predictions concerning the dynamics of the MCPF? In our model, the $MCPF$ in any period t just depends on the elasticity of labor supply and the income tax rate in period t; that is,

$$MCPF_t = \frac{1 - r_t}{1 - r_t(1+\varepsilon)}. \tag{27}$$

Under the assumptions of our model, therefore, the predictions can be tested with time series data on tax rates and an estimate of the elasticity of labor supply. While we recognize that under more general assumptions the $MCPF$ will likely depend on other variables, we will explore the dynamics of the $MCPF$ empirically assuming that (27) holds. Our efforts should therefore be interpreted as an initial exploration.

To test our predictions on the dynamic evolution of the MCPF using (27), we define $\Delta MCPF_\theta(b)$ to be the difference between the current MCPF in state (b,θ) and the expected future MCPF; that is,

$$\Delta MCPF_\theta(b) = \frac{1 - r_\theta(b)}{1 - r_\theta(b)\,(1+\varepsilon)} - E\left[\frac{1 - r_{\theta'}(x_\theta(b))}{1 - r_{\theta'}(x_\theta(b))\,(1+\varepsilon)} \,|\theta\right].$$

Figure 5 shows the time series of $MCPF_t$ and $\Delta MCPF_t$ from 1951 to 2003 for the U.S.. The value of $MCPF_t$ is computed from r_t using (27) and an estimate of the elasticity of labor supply ε. We assume that ε equals one and take our data on labor income tax rates from McDaniel (2007).[28] To compute $\Delta MCPF_t$, we needed an estimate of the expectation of $MPCF_{t+1}$ conditional on the information available at t. To this end, we assumed that $MCPF_t$ follows

[28] McDaniel's work extends the methodology of Mendoza et al. (1994) using data up to 2003 and preceeding 1967. Results similar to those reported in Figure 5 and 6 are obtained with alternative assumptions on the elasticity of labor supply (ε=0.5, 1.5 or 2).

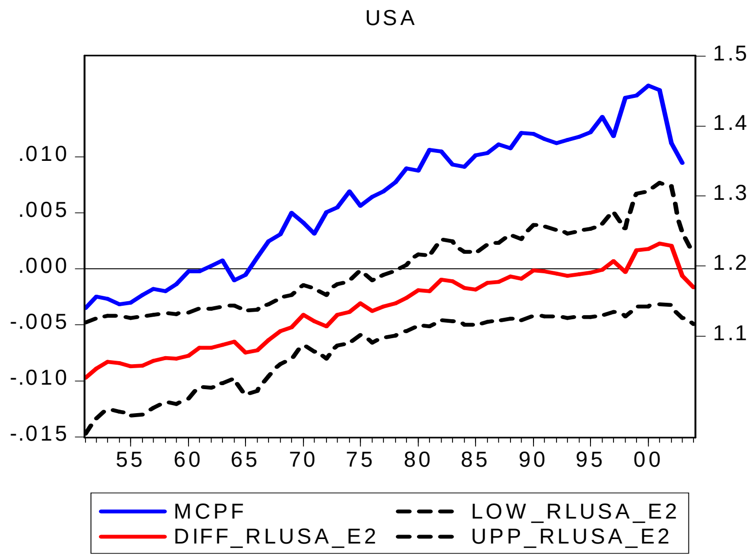

Figure 5: Time series of $MCPF_t$ and $\Delta MCPF_t$ from 1951 to 2003 for the USA. The higher solid line (in blue) represents $MCPF_t$, the lower solid line (in red) represents $\Delta MCPF_t$. The dotted lines represent a +1/-1 standard deviation error band on $\Delta MCPF_t$.

an $AR(1)$ process and estimated the time series $E\left[MCPF_{t+1}\left|MCPF_t\right.\right]$ by Bayesian methods.[29]

The error bands in Figure 5 are then constructed by simulating the posterior distribution of $MCPF_t - E\left[MCPF_{t+1}\left|MCPF_t\right.\right].$

Figure 5 illustrates two striking facts. First, the difference $\Delta MCPF_t$ is negative for the vast majority of time periods.[30] Second, it approaches zero only in the region in which $MCPF_t$ is high. The pattern therefore seems consistent with the prediction that the MCPF behaves as a strict submartingale when small enough, and as a martingale for higher values.

Remarkably, the same pattern emerges for the remaining G7 countries. Indeed, the evidence is even more compelling. Figure 6 shows that for four of our six countries the time series $\Delta MCPF_t$ is negative for all t. The only country in which $\Delta MCPF_t$ appears to be positive for a significant

[29] For the estimation we adopt a flat prior with a standard correction to avoid overweighting the initial observations (Dummy Observation Prior). The results, however, do not change if we assume a Natural Conjugate Prior. Similar results are also found using classical estimation methods.

[30] Indeed, we can reject the hypothesis that the long term expected value of $\Delta MCPF_\theta(b)$ is positive at practically any level of significance. The process $\Delta MCPF_t$ is covariance stationary but autocorrelated. The Central Limit Theorem applies to its sample mean after correcting the variance of the statistic to account for the autocorrelation of $\Delta MCPF_t$.

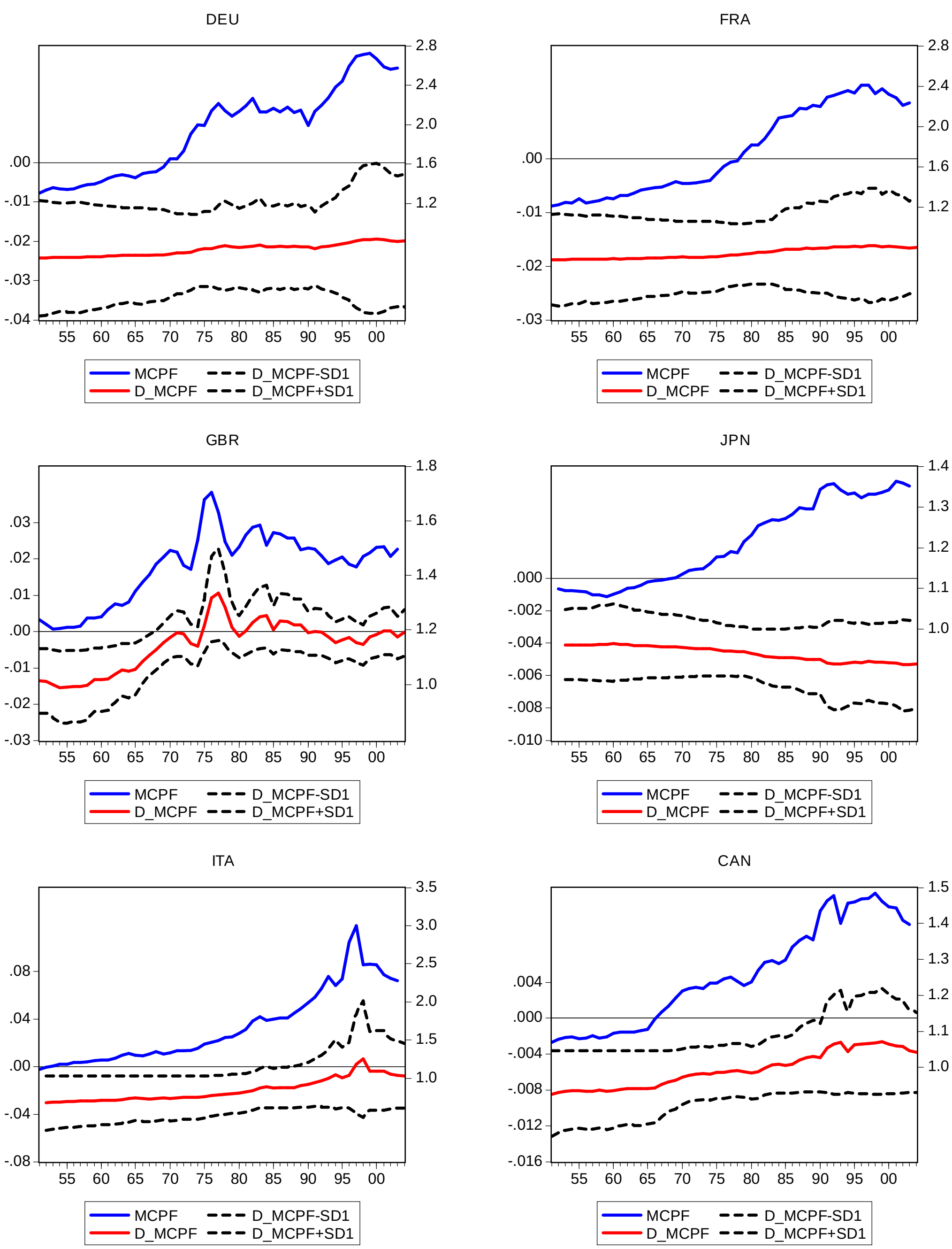

Figure 6: Time series of $MCPF_t$ and $\Delta MCPF_t$ from 1950 to 2003 for the G7 countries excluding the US.

fraction of time (1975-1985) is the U.K., but even in this case we can reject the hypothesis that the long term expected value of $\Delta MCPF_t$ is positive at practically any level of significance. Indeed, the deviation can probably be explained as an effect of the discovery of the North Sea Oil which, though discovered in the 60s, started being extracted in the early 70s. Moreover, the Figure also shows that for all countries except Japan $MCPF_t$ tends to be high when $\Delta MCPF_t$ is closer to zero.[31]

8.2 The cyclicality of fiscal policy

The empirical literature on the cyclicality of fiscal policy focuses on the correlation of debt, government spending, and the primary surplus, with changes in GDP. We first develop the implications of our theory for these correlations. We then place these implications in context by comparing them with those of the perfect foresight tax smoothing model that has guided past empirical work. Finally, we discuss the consistency of our theory's implications with the existing evidence and identify other cyclicality predictions that might be studied in future work.

The empirical implications of the theory Consider first the correlation of debt and GDP. Proposition 7 implies that debt levels go down upon entering a boom and continue to decline over the course of a boom. By contrast, debt levels are increasing over the course of a recession. Since GDP levels are increasing over the course of a boom and decreasing over the course of a recession, debt and GDP are always moving in the opposite direction.[32] Thus, *the theory predicts that debt and GDP should be negatively correlated.*

Next consider the correlation of spending and GDP. Propositions 8 and 9 imply that public spending levels go up upon entering a boom and continue to increase over the course of a boom until they reach a ceiling level. By contrast, spending levels are decreasing over the course of a recession. Since GDP levels are increasing over the course of a boom and decreasing over the course of a recession, *the theory predicts that spending and GDP should be positively correlated.* Notice, however, that the theory provides no predictions on the correlation between spending *as a proportion of* GDP and GDP. This is because when GDP increases both the numerator and the

[31] The $MCPF_t$ and $\Delta MCPF_t$ would be positively correlated even for Japan if we started the time series after 1960.

[32] While productivity levels are constant, GDP levels are increasing (decreasing) during booms (recessions) because tax rates are decreasing (increasing) (Proposition 8).

denominator of the ratio increase and which increases more will depend on how the parameters of the model are calibrated.

Turning to the correlation of the primary surplus and GDP, Proposition 10 implies that the primary surplus, after jumping up upon entering a boom, then declines over the course of a boom. By contrast, the primary surplus is increasing over the course of a recession. Since GDP levels are increasing over the course of a boom and decreasing over the course of a recession, the primary surplus and GDP move in the same direction when the economy transitions to a boom and in the opposite direction over the course of a boom or a recession. Accordingly, the theory provides no clear prediction concerning the correlation of primary surplus and GDP.[33]

While the correlation of tax rates and tax revenues with changes in GDP has not been a focus of this strand of empirical work, it is worth noting what our theory has to say on this. From Proposition 8, it is straightforward to see that the tax rate is negatively correlated with GDP. This immediately implies that *tax revenues as a proportion of GDP will be negatively correlated with GDP*. The theory provides no clear prediction concerning the correlation of tax revenues with GDP. At the onset of a boom, these variables move in the same direction: both GDP and tax revenues increase. However, during a boom, the decreasing tax rate moves tax revenues and GDP in opposite directions.

The empirical implications of the perfect foresight tax smoothing model The perfect foresight tax smoothing model implies that the government will smooth both tax rates and government spending by borrowing in recessions and repaying in booms. Thus, debt will be increasing in booms and decreasing in recessions, implying that debt is positively correlated with changes in GDP. Government spending will be uncorrelated with changes in GDP, but government spending as a proportion of GDP will be negatively correlated. The primary surplus will be positively correlated with changes in GDP, as will primary surplus as a proportion of GDP. Tax rates and tax revenues as a proportion of GDP will be uncorrelated with changes in GDP, while tax revenues will be positively correlated.

The prediction concerning the correlation of debt with GDP is identical to that of our theory,

[33] It is tempting to wonder if this ambiguity could be resolved by further theoretical analysis, but we suspect that this will not be the case. The offsetting forces generating the ambiguity seem perfectly natural and there seems no good reason why one should outweigh the other. Understanding precisely the circumstances under which one force dominates the other will require numerical simulation of the model, a task we leave for future work.

while the implications concerning spending diverge. The theories also diverge on the correlation of tax revenue as a proportion of GDP with changes in GDP. The tax smoothing model yields sharper implications concerning the correlation of GDP changes with primary surplus, primary surplus as a proportion of GDP, spending as a proportion of GDP and tax revenues, than our model. These implications follow from the theoretical prediction of perfect smoothing of taxes and spending which in turn stems from the assumption that cyclical variations are perfectly anticipated. They make the perfect foresight tax smoothing model something of a *straw man*.[34]

The existing evidence The correlation between debt and income changes has been studied by Barro (1986) for the U.S. federal government. Using data from the period 1916-1982, he found a negative correlation between changes in debt and changes in GNP. This evidence is consistent with both the perfect foresight tax smoothing model and with our theory.[35]

The correlation between government consumption (which excludes transfers and debt interest payments) and changes in GDP has been studied extensively for the U.S. both at the federal and state level, and for different groups of countries aggregated according to geographical location and stage of economic development.[36] The basic findings are that government consumption tends to be slightly pro-cyclical for developed economies, and much more pro-cyclical for developing countries.[37] As noted in section 2, these findings have been interpreted as suggesting that fiscal policy is consistent with the perfect foresight tax smoothing model in developed countries and inconsistent in developing countries. Strictly speaking, of course, neither finding is consistent with the tax smoothing model. However, both are consistent with our theory. It is also important to note that these findings suggest that the governments of most countries are not following a

[34] Relaxing the assumption of perfectly anticipated shocks, yields less stark results in the short run, but, as Proposition 1 shows, untenable results in the long run.

[35] Barro runs regressions of the form $(b_t - b_{t-1})/y_t = \alpha \cdot X_t + \beta yvar_t + \varepsilon_t$,where b_t is debt, y_t is GNP, X_t is a vector of control variables, $yvar_t$ is a business cycle indicator, and ε_t is a shock. The business cycle indicator takes on negative values during a boom and positive values during a recession. He finds that the coefficient β is positive, suggesting that debt behaves counter-cyclically. On the other hand, he also finds that the coefficient β is greater than 1 suggesting that debt falls more than proportionally to GNP in a recession. His interpretation is that, in a recession, not only do tax revenue falls but also tax rates are reduced. The latter implication is not consistent with our model, which predicts that tax rates will be higher in a recession. It is more consistent with the idea that the U.S. government is following a Keynesian counter-cyclical fiscal policy.

[36] Gavin and Perotti (1997) compare a sample of Latin American countries with a sample of industrialized countries. Sorensen, Wu and Yosha (2001) study the U.S. states. Lane (2003) looks at all the OECD countries. Alesina, Campante, and Tabellini (2007), Kaminsky, Reinhart and Vegh (2004), Talvi and Vegh (2005), and Woo (2006) look at data sets containing a broad sample of developed and developing countries.

[37] See, in particular, Alesina, Campante, and Tabellini (2007), Gavin and Perotti (1997), Kaminsky, Reinhart and Vegh (2004), Talvi and Vegh (2005), and Woo (2006).

Keynesian counter-cyclical fiscal policy.

The correlation between the primary surplus and changes in GDP has also been studied extensively.[38] These studies show that there is a positive correlation between the primary surplus and changes in GDP. However, again there is a difference between developed and developing counties. Consistent with the findings on spending, the cyclical increase in the primary surplus is much greater for developed countries. In particular, the primary surplus as a proportion of GDP is strongly positively correlated with changes in GDP in developed countries and only weakly positively or even negatively correlated in developing countries.[39] Again, while the findings from the developing countries appear inconsistent with the perfect foresight tax smoothing model, they are, in principle, consistent with our theory.

Suggestions for further empirical research Future empirical research in this vein might usefully explore the predictions of the theory developed here more systematically. One implication of interest that would seem straightforward to test is the negative correlation of tax revenues as a proportion of GDP with GDP. Of more interest, but much harder to test because of measurement issues, is the prediction stated in Proposition 9 that pork is pro-cyclical. Another interesting possibility would be to study the behavior of the primary surplus in more detail. The theory implies that, in long run equilibrium, for any given debt level that might be observed, the primary surplus will be higher in a boom than in a recession. This is a testable proposition. Also of interest is the fact that the primary surplus is decreasing in booms and increasing in recessions. Finally, it would be useful to numerically simulate the model to generate a deeper understanding of the relationships implied by the model and the factors that determine the degree of cyclicality of the fiscal variables. It may be that the differences between developed and developing countries that are observed in the data can be traced to some underlying difference in the fundamentals.[40]

[38] Bohn (1998) studies the U.S. federal government, while Sorensen, Wu and Yosha (2001) look at the U.S. states. Lane (2003) considers the OECD countries and Alesina, Campante, and Tabellini (2007) look at a large sample of developed and developing countries.

[39] See, in particular, Alesina, Campante, and Tabellini (2007), Bohn (1998) and Gavin and Perotti (1997).

[40] For example, Talvi and Vegh (2005) document that tax base fluctuations are much greater in developing countries.

9 Conclusion

The literature on real business cycles studies how competitive markets react to random fluctuations in productivity or other economic fundamentals. At the core of this literature there is the question of how agents *individually* respond to these shocks by adjusting their consumption and saving levels. In this paper we have studied the complementary question of how agents, through their political institutions, *collectively* react to these same shocks by adjusting fiscal policy. Given the importance of the public sector in contemporary market economies, answering this question is clearly a necessary condition for a satisfactory positive theory of business cycles.

Our theory assumes that society delegates the choice of fiscal policy to a legislature comprised of representatives elected by single-member, geographically-defined districts. While representatives are perfect agents of their constituents, the theory incorporates a realistic distributional conflict by assuming that they can target revenues back to their districts via pork-barrel spending. This distributional conflict means that the legislature's policy choices solve a particular "constrained" planning problem. The constraints consist of an upper bound on public good provision, a lower bound on the tax rate, and state contingent lower bounds on debt. Paradoxically, the addition of these constraints to the planning problem, produces a long run cyclical pattern of fiscal policy that is much more consistent with traditional tax smoothing principles than that emerging from the unconstrained planner's solution.[41]

Our theory yields three central predictions. The first is that equilibrium fiscal policies are such that the marginal cost of public funds obeys a submartingale. This is a sharp prediction that has the great merit of being straightforward to test. Given that in the planner's solution the MCPF obeys a martingale, it also provides a simple and intuitive way of understanding how political decision-making distorts fiscal policy.

The second prediction is that, in the long run, debt displays a counter-cyclical pattern, increasing in recessions and decreasing in booms. This is contrary to the intuitions emerging from the literature on the voracity effect which suggest that the distributional conflict created by pork-barrel spending would result in debt increasing in booms. However, this literature considers a one time only positive shock, whereas in our model, the economy is subject to recurrent cyclical shocks. While a "voracity effect"-style debt expansion can arise when the economy first moves

[41] Similar conclusions arise when public spending shocks are the driver of fiscal policy (Battaglini and Coate (2008)).

from recession to boom, after it occurs, the level of debt is too high in recessions for it to ever occur again.

The third prediction is that, in the long run, public spending and tax rates display a procyclical pattern, with spending increasing in booms and decreasing in recessions, and tax rates decreasing in booms and increasing in recessions. The equilibrium changes in public spending and taxes therefore serve to amplify the business cycle. This prediction serves to nicely differentiate our theory both from a perfect foresight tax smoothing model and from what would be expected if government were following a Keynesian counter-cyclical fiscal policy.

We hope that our theory will provide a new benchmark for empirical research on the cyclical behavior of fiscal policy. When compared with the current benchmark theory - the perfect foresight tax smoothing model - it both rests on more satisfactory assumptions and delivers a richer set of predictions. The theory's implications concerning the correlation of debt and government spending with changes in GDP are consistent with evidence from the U.S. and the other G7 countries. In addition, the theory's predictions concerning the dynamic evolution of the MCPF find some support in data from the U.S. and eight other countries. These, and the theory's other novel implications, warrant further empirical investigation.

The ultimate payoff of having a more satisfactory theoretical account of the behavior of fiscal policy is not only to improve our predictive ability, but also to be able to evaluate policy proposals that seek to change fiscal and political constitutions. Policies of this form include balanced-budget rules, debt limits, and super-majority budget approval requirements. The firm micro-foundations of our theory make it particularly suitable for welfare analysis of such policies and this is also an important topic for future research.[42]

[42] See Azzimonti, Battaglini and Coate (2008) for a quantitative application of the Battaglini and Coate (2008) model along these lines.

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10 Appendix

10.1 Proof of Proposition 1

The social planner's solution arises as a special case of the political equilibrium when the legislature operates by unanimity rule; that is, in which $q = n$. We will therefore delay proof of this proposition until after we have understood the behavior of the political equilibrium.

10.2 Proof of Proposition 2

Let $\{r_\theta^\tau(b),\ g_\theta^\tau(b),\ x_\theta^\tau(b),\ s_\theta^\tau(b)\}_{\tau=1}^T$ be an equilibrium with associated value functions $v_L(b)$ and $v_H(b)$. It is enough to show that for $\theta \in \{L, H\}$, $\{r_\theta(b),\ g_\theta(b),\ x_\theta(b)\}$ solves the problem

$$\begin{array}{c} \max_{(r,g,x)} u_\theta(r,g) + \frac{B_\theta(r,g,x;b)}{n} + \delta[\alpha_{\theta L} v_L(x) + \alpha_{\theta H} v_H(x)] \\ s.t.\ \ B_\theta(r,g,x;b) \geq 0,\ r \geq r^*,\ g \leq g^* \ \&\ x \in [x_\theta^*, \overline{x}], \end{array} \tag{28}$$

where x_H^* and x_L^* satisfy (18). For then it would follow immediately from (14) that the value functions $v_L(b)$ and $v_H(b)$ have the required properties.

We begin by making precise the claim made in Section 5 that, given transferable utility, the proposer is effectively making decisions to maximize the collective utility of q legislators under the assumption that they get to divide any surplus revenues among their districts.

Lemma A.1: *Let $\{r_\theta^\tau(b),\ g_\theta^\tau(b),\ x_\theta^\tau(b),\ s_\theta^\tau(b)\}_{\tau=1}^T$ be an equilibrium with associated value functions $v_L(b)$ and $v_H(b)$. Then, for all states (b,θ), the tax rate-public good-public debt triple $(r_\theta^\tau(b), g_\theta^\tau(b), x_\theta^\tau(b))$ proposed in any round τ solves the problem*

$$\begin{array}{c} \max_{(r,g,x)} u_\theta(r,g) + \frac{B_\theta(r,g,x;b)}{q} + \delta[\alpha_{\theta L} v_L(x) + \alpha_{\theta H} v_H(x)] \\ s.t. \quad B_\theta(r,g,x;b) \geq 0 \ \&\ x \in [\underline{x}, \overline{x}] \,. \end{array} \tag{29}$$

Moreover, the transfer to coalition members is given by

$$s_\theta^\tau(b) = v_\theta^{\tau+1}(b) - u_\theta(r_\theta^\tau(b), g_\theta^\tau(b)) - \delta E v_{\theta'}(x_\theta^\tau(b)).$$

Proof: The proof of this result is similar to the proof of an analogous result in Battaglini and Coate (2008) and thus is omitted. A proof is available from the authors upon request.

As we argued in the text, if the constraint $B_\theta(r,g,x;b) \geq 0$, is not binding, then the solution to problem (29) is (r^*, g^*, x_θ^*). On the other hand, if the constraint is binding, then the solution to this problem solves the problem

$$\begin{array}{c} \max_{(r,g,x)} u_\theta(r,g) + \frac{B_\theta(r,g,x;b)}{n} + \delta[\alpha_{\theta L} v_L(x) + \alpha_{\theta H} v_H(x)] \\ s.t. \quad B_\theta(r,g,x;b) \geq 0 \ \& \ x \in [\underline{x}, \overline{x}] \,. \end{array} \tag{30}$$

Letting b^*_θ be as defined in (20), we conclude that $\{r_\theta(b),\, g_\theta(b),\, x_\theta(b)\} = (r^*, g^*, x^*_\theta)$ when $b \leq b^*_\theta$ and solves (30) when $b > b^*_\theta$. Thus, we need to show (i) that when $b \leq b^*_\theta$ the solution to problem (28) is (r^*, g^*, x^*_θ), and, (ii) that when $b > b^*_\theta$ the constraints $r \geq r^*$, $g \leq g^*$ and $x \geq x^*_\theta$ will not be binding in problem (28). For (ii), note first that the solution to (30) when $b = b^*_\theta$ is (r^*, g^*, x^*_θ) and second that the optimal tax rate and debt level for problem (30) are non decreasing in b and the public good is non increasing in b. For (i), note that when $b \leq b^*_\theta$ the budget constraint cannot be binding in problem (28) and, if the budget constraint is not binding, the individual constraints $r \geq r^*$, $g \leq g^*$ and $x \geq x^*_\theta$ must all bind. ■

10.3 Proof of Proposition 3

Let $\widetilde{v}_H$ and $\widetilde{v}_L$ be a pair of value functions and $\widetilde{x}_H$ and $\widetilde{x}_L$ a pair of debt levels such that (i) $\widetilde{v}_H$ and $\widetilde{v}_L$ solve (19) given $\widetilde{x}_H$ and $\widetilde{x}_L$, and, (ii) $\widetilde{x}_H$ and $\widetilde{x}_L$ solve (18) given $\widetilde{v}_H$ and $\widetilde{v}_L$. Let $(\widetilde{r}_\theta(b), \widetilde{g}_\theta(b), \widetilde{x}_\theta(b))$ be the corresponding optimal policies that solve the program in (19). For each proposal round τ and state of the economy $\theta = H, L$ define the following strategies:

$$(r^\tau_\theta(b), g^\tau_\theta(b), x^\tau_\theta(b)) = (\widetilde{r}_\theta(b), \widetilde{g}_\theta(b), \widetilde{x}_\theta(b));$$

for proposal rounds $\tau = 1, ..., T-1$

$$s^\tau_\theta(b) = B_\theta(\widetilde{r}_\theta(b), \widetilde{g}_\theta(b), \widetilde{x}_\theta(b); b)/n;$$

and for proposal round T

$$s^T_\theta(b) = v^{T+1}_\theta(b) - u_\theta(\widetilde{r}_\theta(b), \widetilde{g}_\theta(b)) - \delta\left[\alpha_{\theta H} \widetilde{v}_H(\widetilde{x}_\theta(b)) + \alpha_{\theta L} \widetilde{v}_L(\widetilde{x}_\theta(b))\right];$$

where

$$v^{T+1}_\theta(b) = \max_{(r,g,x)} \left\{ \begin{array}{c} u_\theta(r,g) + \frac{B_\theta(r,x,g;b)}{n} + \delta\left[\alpha_{\theta H} \widetilde{v}_H(x) + \alpha_{\theta L} \widetilde{v}_L(x)\right] \\ \text{s.t. } B_\theta(r,x,g;b) \geq 0 \ \& \ x \in [\underline{x}, \overline{x}] \end{array} \right\}.$$

Given these proposals, the legislators' round one value functions are given by $\widetilde{v}_H$ and $\widetilde{v}_L$. This follows from the fact that

$$v^1_\theta(b) \quad = \quad u_\theta(\widetilde{r}_\theta(b), \widetilde{g}_\theta(b)) + \frac{B_\theta(\widetilde{r}_\theta(b), \widetilde{g}_\theta(b), \widetilde{x}_\theta(b); b)}{n} +$$

$$\delta[\alpha_{\theta H}\widetilde{v}_H(\widetilde{x}_\theta(b)) + \alpha_{\theta L}\widetilde{v}_L(\widetilde{x}_\theta(b))] = \widetilde{v}_\theta(b).$$

Similarly, the round $\tau = 2, ..., T$ legislators' value functions are given by $\widetilde{v}_H$ and $\widetilde{v}_L$.

To show that $\{r_\theta^\tau(b), g_\theta^\tau(b), x_\theta^\tau(b), s_\theta^\tau(b)\}_{\tau=1}^T$ together with the associated value functions $\{v_\theta^\tau(b)\}_{\tau=1}^{T+1}$ describe an equilibrium, we need only show that for proposal rounds $\tau = 1, .., T$ the proposal $(r_\theta^\tau(b), g_\theta^\tau(b), x_\theta^\tau(b), s_\theta^\tau(b))$ solves the problem

$$\begin{aligned} &\max_{(r,g,x,s)} u_\theta(r,g) + B_\theta(r,g,x;b) - (q-1)\,s + \delta[\alpha_{\theta H}\widetilde{v}_H(x) + \alpha_{\theta L}\widetilde{v}_L(x)] \\ &s.t. \quad u_\theta(r,g) + s + \delta[\alpha_{\theta H}\widetilde{v}_H(x) + \alpha_{\theta L}\widetilde{v}_L(x)] \geq \Upsilon_\theta^{\tau+1}(b) \\ &B_\theta(r,g,x;b) \geq (q-1)s, \quad s \geq 0 \;\&\; x \in [\underline{x}, \overline{x}], \end{aligned}$$

where $\Upsilon_\theta^{\tau+1}(b) = \widetilde{v}_\theta(b)$ for $\tau = 1..., T-1$, and $\Upsilon_\theta^{T+1}(b) = v_\theta^{T+1}(b)$. We show this result only for $\tau = 1, ..., T-1$ – the argument for $\tau = T$ being analogous.

Consider some proposal round $\tau = 1, ..., T-1$. Let (b, θ) be given. To simplify notation, let

$$(\widehat{r}, \widehat{g}, \widehat{x}, \widehat{s}) = (\widetilde{r}_\theta(b), \widetilde{g}_\theta(b), \widetilde{x}_\theta(b), \frac{B_\theta(\widetilde{r}_\theta(b), \widetilde{g}_\theta(b), \widetilde{x}_\theta(b); b)}{n}).$$

Since $\widetilde{x}_\theta$ solves (18), it follows from the discussion in Section 5.2 (and it can easily be formally verified) that $(\widehat{r}, \widehat{g}, \widehat{x})$ solves the problem:

$$\begin{aligned} &\max_{(r,g,x)} u_\theta(r,g) + \tfrac{B_\theta(r,g,x;b)}{q} + \delta[\alpha_{\theta H} v_H(x) + \alpha_{\theta L} v_L(x)] \\ &s.t. \quad B_\theta(r,g,x;b) \geq 0 \;\&\; x \in [\underline{x}, \overline{x}], \end{aligned}$$

and that

$$\widehat{s} = \widetilde{v}_\theta(b) - u_\theta(\widehat{r}, \widehat{g}) - \delta[\alpha_{\theta H}\widetilde{v}_H(\widehat{x}) + \alpha_{\theta L}\widetilde{v}_L(\widehat{x})].$$

Suppose that $(\widehat{r}, \widehat{g}, \widehat{x}, \widehat{s})$ does not solve the round τ proposer's problem. Then there exist some (r', g', x', s') which achieves a higher value of the proposer's objective function. We know that $s' \geq \widetilde{v}_\theta(b) - u_\theta(r', g') - \delta[\alpha_{\theta H}\widetilde{v}_H(x') + \alpha_{\theta L}\widetilde{v}_L(x')]$. Thus, we have that the value of the proposer's objective function satisfies

$$\begin{aligned} &u_\theta(r', g') + B_\theta(r', g', x'; b) - (q-1)\,s' + \delta[\alpha_{\theta H}\widetilde{v}_H(x') + \alpha_{\theta L}\widetilde{v}_L(x')] \\ \leq\; &q\{u_\theta(r', g') + \delta[\alpha_{\theta H}\widetilde{v}_H(x') + \alpha_{\theta L}\widetilde{v}_L(x')]\} + B_\theta(r', g', x'; b). \end{aligned}$$

But since $B_\theta(r', g', x'; b) \geq 0$, we know that

$$\begin{aligned} &q\{u_\theta(r', g') + \delta[\alpha_{\theta H}\widetilde{v}_H(x') + \alpha_{\theta L}\widetilde{v}_L(x')]\} + B_\theta(r', g', x'; b) \\ \leq\; &q\{u_\theta(\widehat{r}, \widehat{g}) + \delta[\alpha_{\theta H}\widetilde{v}_H(\widehat{x}) + \alpha_{\theta L}\widetilde{v}_L(\widehat{x})]\} + B_\theta(\widehat{r}, \widehat{g}, \widehat{x}; b). \end{aligned}$$

But the right hand side of the inequality is the value of the proposer's objective function under the proposal $(\widehat{r}, \widehat{g}, \widehat{x}, \widehat{s})$. This therefore contradicts the assumption that (r', g', x', s') achieves a higher value for the proposer's problem. $\blacksquare$

10.4 Proof of Proposition 4

By Proposition 3, we can establish the existence of an equilibrium by showing that we can find a pair of value functions $v_H(b)$ and $v_L(b)$ and a pair of debt thresholds x_L^* and x_H^* such that (i) $v_H(b)$ and $v_L(b)$ solve (19) given x_L^* and x_H^*, and, (ii) x_L^* and x_H^* solve (18) given $v_H(b)$ and $v_L(b)$. We simply sketch how to do this here, the details are available on request.

Let F denote the set of all real valued functions $v(\cdot)$ defined over the set $[\underline{x}, \overline{x}]$ that are continuous and concave. For each $\theta \in \{H, L\}$ and any $z_\theta \in [(R_L(r^*) - pg^*)/\rho, \overline{x}]$, define the operator $T^\theta_{z_\theta} : F \times F \to F$ as follows:

$$T^\theta_{z_\theta}(v_H, v_L)(b) = \left\{ \begin{array}{c} \max_{(r,g,x)} u_\theta(r,g) + \frac{B_\theta(r,g,x;b)}{n} + \delta\left[\alpha_{\theta H} v_H(x) + \alpha_{\theta L} v_L(x)\right] \\ s.t. \quad B_\theta(r,g,x;b) \geq 0,\ r \geq r^*,\ g \leq g^* \ \&\ x \in [z_\theta, \overline{x}] \end{array} \right\}.$$

Let $\mathbf{z} = (z_H, z_L)$ and let $T_\mathbf{z}(v_H, v_L)(b) = (T^H_{z_H}(v_H, v_L)(b), T^L_{z_L}(v_H, v_L)(b))$ be the corresponding function from $F \times F$ to itself. For any $\mathbf{z} \in [(R_L(r^*) - pg^*)/\rho, \overline{x}]^2$, it can be verified that $T_\mathbf{z}$ is a contraction and admits a unique fixed point $v_\mathbf{z}$ (where we use the subscript $\mathbf{z}$ to indicate that this fixed point depends on $\mathbf{z}$). Given $v_\mathbf{z}$, let

$$M_\theta(\mathbf{z}) = \arg\max\{\frac{x}{q} + \delta\left[\alpha_{\theta H} v_{H\mathbf{z}}(x) + \alpha_{\theta L} v_{L\mathbf{z}}(x)\right] : x \in [z_\theta, \overline{x}]\}$$

and let $M(\mathbf{z}) = M_H(\mathbf{z}) \times M_L(\mathbf{z})$. Then, we have an equilibrium if we can find a fixed point of this correspondence, $\mathbf{z} \in M(\mathbf{z})$. This can be proven by showing that M satisfies the conditions of *Kakutani's Fixed Point Theorem.* $\blacksquare$

10.5 Proof of Lemma 1

From Proposition 2, we know that

$$v_\theta(b) = \max_{(r,g,x)} \left\{ \begin{array}{c} u_\theta(r,g) + \frac{B_\theta(r,g,x;b)}{n} + \delta[\alpha_{\theta L} v_L(x) + \alpha_{\theta H} v_H(x)] \\ s.t. \ \ B_\theta(r,g,x;b) \geq 0,\ r \geq r^*,\ g \leq g^* \ \&\ x \in [x^*_\theta, \overline{x}] \end{array} \right\}.$$

Moreover, from the discussion in the text, we know that if $b \leq b^*_\theta$ the optimal policies are (r^*, g^*, x^*_θ), and, if $b > b^*_\theta$ the constraints $r \geq r^*$, $g \leq g^*$ and $x \geq x^*_\theta$ in the maximization problem will not be binding, but the budget constraint will be binding.

Suppose first that $b_o < b^*_\theta$. Then, we know that in a neighborhood of b_o it must be the case that

$$v_\theta(b) = u_\theta(r^*, g^*) + \frac{B_\theta(r^*, g^*, x^*_\theta; b)}{n} + \delta[\alpha_{\theta H} v_H(x^*_\theta) + \alpha_{\theta L} v_L(x^*_\theta)].$$

Thus, it is immediate that the value function $v_\theta(b)$ is differentiable at b_o and that

$$v'_\theta(b_o) = -(\frac{1+\rho}{n}).$$

Now suppose that $b_o > b^*_\theta$. Then, we know that in a neighborhood of b_o it must be the case that

$$v_\theta(b) = \max_{(r,g,x)} \left\{ \begin{array}{c} u_\theta(r,g) + \frac{B_\theta(r,g,x;b)}{n} + \delta\left[\alpha_{\theta H} v_H(x) + \alpha_{\theta L} v_L(x)\right] \\ \\ B_\theta(r,g,x;b) \geq 0 \ \&\ x \in [x^*_\theta, \overline{x}] \end{array} \right\}.$$

Define the function

$$g(b) = \frac{R_\theta(r_\theta(b_o)) + x_\theta(b_o) - (1+\rho)b}{p}$$

and let

$$\eta(b) = u_\theta(r_\theta(b_o), g(b)) + \frac{B_\theta(r_\theta(b_o), g(b), x_\theta(b_o); b)}{n} + \delta\left[\alpha_{\theta H} v_H(x_\theta(b_o)) + \alpha_{\theta L} v_L(x_\theta(b_o))\right].$$

Notice that $(r_\theta(b_o), g(b), x_\theta(b_o))$ is a feasible policy when the initial debt level is b so that in a neighborhood of b_o we must have that $v_\theta(b) \geq \eta(b)$. Moreover, $\eta(b)$ is twice continuously differentiable with derivatives

$$\eta'(b) = -\alpha A g(b)^{\alpha-1}(\tfrac{1+\rho}{p})$$
$$\eta''(b) = -(1-\alpha)\alpha A g(b)^{\alpha-2}(\tfrac{1+\rho}{p})^2 < 0$$

The second derivative property implies that $\eta(b)$ is strictly concave. It follows from Theorem 4.10 of Stokey, Lucas and Prescott (1989) that $v_\theta(b)$ is differentiable at b_o with derivative $v'_\theta(b) = \eta'(b_o) = -\alpha A g_\theta(b_o)^{\alpha-1}(\frac{1+\rho}{p})$. To complete the proof note that $(r_\theta(b_o), g_\theta(b_o))$ must solve the problem:

$$\max_{(r,g)} \left\{ \begin{array}{c} u_\theta(r,g) + \frac{B_\theta(r,g,x_\theta(b_o);b_o)}{n} \\ \\ B_\theta(r,g,x_\theta(b_o);b_o) \geq 0 \end{array} \right\},$$

which implies that $\alpha n A g_\theta(b_o)^{\alpha-1} = p[\frac{1-r_\theta(b_o)}{1-r_\theta(b_o)(1+\varepsilon)}]$. Thus, we have that

$$v_\theta'(b) = -[\frac{1-r_\theta(b_o)}{1-r_\theta(b_o)(1+\varepsilon)}](\frac{1+\rho}{n}).$$

■

10.6 Proof of Proposition 5

We proceed in two steps. First we prove that both in a political equilibrium and in the planner's solution, the upperbound on debt $x \leq \overline{x}$ does not bind for any $b < \overline{x}$. This establishes equation (10) in Section 4 and equation (21) in Section 6. Then we prove the statement of Proposition 5 and equation (13) of Section 4

Step 1. Consider a particular political equilibrium and let $\{r_\theta(b), g_\theta(b), x_\theta(b)\}$ be the associated equilibrium policies. We wish to prove that for any state (b,θ) there is an $\epsilon(b,\theta) > 0$ such that $x_\theta(b) < \overline{x} - \epsilon(b,\theta)$. Since the planner's solution arises as a special case of the political equilibrium when $q = n$, this would imply that the same property holds for the planner's solution. Assume that there is a state (b,θ) such that $x_\theta(b)$ is arbitrarily close to $\overline{x}$; that is, $x_\theta(b) = \overline{x} - \eta$, where η is arbitrarily small. We can write $g_\theta(b) = \phi(r_\theta(b))$ where $\phi(r)$ is a continuous function implicitly defined by the solution of the equation $n\alpha A g^{\alpha-1} = [\frac{1-r}{1-r(1+\varepsilon)}]p$. Since $x_\theta(b) > x_\theta^*$, we must have

$$B_\theta(r_\theta(b), \phi(r_\theta(b)), x_\theta(b); b) = 0. \tag{31}$$

Thus, we can express all the policy choices as a function of η, where $x_\theta(b) = \overline{x} - \eta = x(\eta)$, $r_\theta(b) = r(\eta)$ solves (31) and $g_\theta(b) = \phi(r(\eta)) = g(\eta)$. Note that as $\eta \to 0$, we have $r(\eta) \to \widetilde{r} < 1/(1+\varepsilon)$. For if $r(\eta) \to 1/(1+\varepsilon)$, then $g(\eta) \to 0$ and (31) would not be satisfied since $b < \overline{x}$. Moreover, $r(\eta) \to \widetilde{r}$ implies $g(\eta) \to \widetilde{g} > 0$.

From the first order condition on debt, we have that:

$$\begin{aligned}(\frac{1-r(\eta)}{1-r(\eta)(1+\varepsilon)}) &\geq -\delta[\alpha_{\theta H} v_H'(x(\eta)) + \alpha_{\theta L} v_L'(x(\eta))] \\ &\geq -\delta\alpha_{\theta L} v_L'(x(\eta)) = -\delta\alpha_{\theta L}\left(\frac{1-r_L(x(\eta))}{1-r_L(x(\eta))(1+\varepsilon)}\right).\end{aligned} \tag{32}$$

It is easy to see that $r_L(x(\eta)) \to 1/(1+\varepsilon)$ as $\eta \to 0$. This implies that the right hand side of (32) diverges to infinity, while the left hand side converges to a finite value: a contradiction. ■

Step 2. We now prove that the deadweight loss of taxation is a submartingale when $q < n$, with strict inequality for some states (b,θ). The argument in Section 6 establishes that the MCPF is a submartingale (equation (24)). To complete the statement of the proposition, note that if $q < n$, then $r^* > 0$, and $b^*_\theta > \underline{x}$. It is also easy to show that if $q < n$, there is a $b' > x^*_H$ such that for any $b \leq b'$ we have $\Pr\left(\theta' \text{ s.t. } x_\theta(b) \leq b^*_{\theta'} \,|\theta\right) > 0$ for any θ. For these states (25) holds as a strict inequality. To prove (13), note that if $q = n$, then $r^* = 0$. In this case, $\Pr\left(\theta' \text{ s.t. } x_\theta(b) \leq b^*_{\theta'} \,|\theta\right) \frac{r^*\varepsilon}{1-r^*(1+\varepsilon)} = 0$: which implies that both in an equilibrium with unanimity and in the planner's solution, the MCPF is a martingale. ■

10.7 Proof of Proposition 6

Clearly for all states (b, θ) such that $r^*_\theta(b) = r^*$, we have $r^* < E\left(r_{\theta'}\left(x_\theta(b)\right)|\theta\right)$. We now prove that there are states in which $r_\theta(b) > E\left(r_{\theta'}(x_\theta(b))\,|\theta\right)$. We know from Lemma 2 and 3 below that $r_H(x_\theta(b)) < r_\theta(b)$ for all states (b, θ) with $b \in [\underline{x}, \overline{x}]$ and that both $\lim_{b\to\overline{x}} r_L(x_L(b)) = 1/(1+\varepsilon)$ and $\lim_{b\to\overline{x}} r_L(b) = 1/(1+\varepsilon)$. So there is a η such that $r_H(x_L(b)) < r_L(b) - \eta$ for all $b \in [\underline{x}, \overline{x}]$, and there is a b' such that $r_L(x_L(b'')) < r_L(b'') + \frac{\eta}{2}$ for $b'' > b'$. This implies that for $b > b'$ we have

$$\begin{aligned} E\left(r_{\theta'}\left(x_L(b)\right)|\theta = L\right) &= \alpha_{LL} r_L(x_L(b)) + \alpha_{LH} r_H(x_L(b)) \\ &= \alpha_{LL} r_L(b) + \alpha_{LH} r_L(b) - \frac{\eta}{2} \\ &< r_L(b). \end{aligned}$$

This shows that for all states $(\widetilde{b}, \widetilde{\theta})$ such that $\widetilde{b} > b'$ and $\widetilde{\theta} = L$, we have $E\left(r_{\theta'}\left(x_{\widetilde{\theta}}(\widetilde{b})\right)\middle|\widetilde{\theta}\right) < r_{\widetilde{\theta}}(\widetilde{b})$. While for states $(\widetilde{b}, \widetilde{\theta})$ such that $\widetilde{b} \leq b^*_{\widetilde{\theta}}$, we have $E\left(r_{\theta'}\left(x_{\widetilde{\theta}}(\widetilde{b})\right)\middle|\widetilde{\theta}\right) > r_{\widetilde{\theta}}(\widetilde{b})$. So $r_\theta\left(b\right)$ is not a martingale of any type. ■

10.8 Proof of Lemma 2

(i) We will establish that $x^*_H \geq x^*_L$ which will imply the result. Suppose that, to the contrary, that $x^*_H < x^*_L$. There are two possibilities. The first is that $b^*_L < b^*_H$. In this case, it follows from (26) and Lemma 1 that $b^*_L < x^*_H < x^*_L \leq b^*_H$ and that x^*_H and x^*_L satisfy the following two first order conditions:

$$\alpha_{HL}\left(\frac{1 - r_L(x^*_H)}{1 - r_L(x^*_H)(1+\varepsilon)}\right) + \alpha_{HH} = \frac{n}{q},$$

and

$$\alpha_{LL}\left(\frac{1 - r_L(x^*_L)}{1 - r_L(x^*_L)(1+\varepsilon)}\right) + \alpha_{LH} \leq \frac{n}{q} \ \left(= \text{if } x^*_L < b^*_H\right).$$

But since $x_H^* < x_L^*$ we know that

$$\frac{1 - r_L(x_H^*)}{1 - r_L(x_H^*)(1+\varepsilon)} < \frac{1 - r_L(x_L^*)}{1 - r_L(x_L^*)(1+\varepsilon)}.$$

In addition, $\alpha_{HL} \leq \alpha_{LL}$ and hence the above two first order conditions are clearly inconsistent.

The second possibility is that $b_L^* > b_H^*$. In this case, it follows from (26) and Lemma 1 that $b_H^* < x_H^* < x_L^* \leq b_L^*$. Since $x_H^* > b_H^*$, it must be that in a boom with debt level $b = x_H^*$ the policy is such that $r_H(x_H^*) > r^*$, $g_H(x_H^*) < g^*$, and $x_H(x_H^*) > x_H^*$. This implies that

$$\begin{aligned} 0 &= B_H(r_H(x_H^*), g_H(x_H^*), x_H(x_H^*); x_H^*) \\ &> B_H(r^*, g^*, x_H^*; x_H^*) = R_H(r^*) - pg^* - \rho x_H^* > R_H(r^*) - pg^* - \rho x_L^*. \end{aligned} \tag{33}$$

Since $x_L^* < b_L^*$, it must be that in a recession with debt level $b = x_L^*$, the policy is such that $r_L(x_L^*) = r^*$, $g_L(x_L^*) = g^*$, and $x_L(x_L^*) = x_L^*$. This implies:

$$0 \leq B_L(r^*, g^*, x_L^*; x_L^*) = R_L(r^*) - pg^* - \rho x_L^* < R_H(r^*) - pg^* - \rho x_L^*,$$

which is in contradiction with (33).

(ii) When $b \geq b_H^*$, we know from the discussion following Proposition 2 that $\{r_\theta(b),\ g_\theta(b),\ x_\theta(b)\}$ satisfies the following three equations:

$$n\alpha A g^{\alpha-1} = p[\frac{1-r}{1-r(1+\varepsilon)}],$$

$$[\frac{1-r}{1-r(1+\varepsilon)}] = -\delta n[\alpha_{\theta H} v_H'(x) + \alpha_{\theta L} v_L'(x)],$$

and

$$B_\theta(r, g, x; b) = 0.$$

Suppose, contrary to the claim in the Lemma, that $r_L(b) \leq r_H(b)$. Then it follows immediately that $g_L(b) \geq g_H(b)$. In addition, we have that

$$-\delta n[\alpha_{HH} v_H'(x_H(b)) + \alpha_{HL} v_L'(x_H(b))] \geq -\delta n[\alpha_{LH} v_H'(x_L(b)) + \alpha_{LL} v_L'(x_L(b))].$$

Suppose that it were the case that $-v_H'(x_H(b)) \leq -v_L'(x_H(b))$. Then, since $\alpha_{HH} > \alpha_{LH}$, we would have that

$$-\delta n[\alpha_{HH} v_H'(x_H(b)) + \alpha_{HL} v_L'(x_H(b))] \leq -\delta n[\alpha_{LH} v_H'(x_H(b)) + \alpha_{LL} v_L'(x_H(b))].$$

Combining these two inequalities we could conclude that $x_H(b) \geq x_L(b)$. But then we would have

$$0 = B_H(r_H(b), g_H(b), x_H(b); b) > B_L(r_L(b), g_L(b), x_L(b); b) = 0$$

a contradiction. Thus, we would have shown that $r_L(b) > r_H(b)$.

It follows that we can prove the result by showing the following result:

Lemma A.2: *If v_H and v_L are differentiable at $b \in [\underline{x}, \overline{x}]$, then*

$$-v_H'(b) \leq -v_L'(b).$$

Proof: As in the proof of Proposition 4, let F denote the set of all real valued functions $v(\cdot)$ defined over the set $[\underline{x}, \overline{x}]$ that are continuous and concave. For $\theta \in \{H, L\}$, define the operator $T^\theta : F \times F \to F$ as follows:

$$T^\theta(v_H, v_L)(b) = \left\{ \begin{array}{c} \max_{(r,g,x)} u_\theta(r,g) + \frac{B_\theta(r,g,x;b)}{n} + \delta\left[\alpha_{\theta H} v_H(x) + \alpha_{\theta L} v_L(x)\right] \\ \\ s.t. \quad B_\theta(r,g,x;b) \geq 0,\ r \geq r^*,\ g \leq g^* \ \&\ x \in [x_\theta^*, \overline{x}] \end{array} \right\}.$$

Let $T(v_H, v_L)(b) = (T^H(v_H, v_L)(b), T^L(v_H, v_L)(b))$ be the corresponding function from $F \times F$ to itself. From Proposition 2, we know that $(v_H, v_L) = T(v_H, v_L)$. Moreover, T is a contraction.

Now let $\widetilde{v}_H$ and $\widetilde{v}_L$ belong to F and assume that for any b if ξ_L and ξ_H are sub-gradients of $\widetilde{v}_L$ and $\widetilde{v}_H$ at b, then we have that: $-\xi_L \geq -\xi_H$. Define $\mathbf{v}_0 = (\widetilde{v}_H, \widetilde{v}_H)$ and consider the sequence of functions $\langle v_{\theta k}(b) \rangle_{k=1}^\infty$ for $\theta = H, L$, defined inductively as follows: $v_{\theta 1} = T^\theta(\mathbf{v}_0)$, and $v_{\theta k+1} = T^\theta(v_{Hk}, v_{Lk})$. Let $\mathbf{v}_k = (v_{Hk}, v_{Lk})$ and note that, since T is a contraction, $\langle \mathbf{v}_k \rangle_{k=1}^\infty$ must converge to (v_H, v_L).

Finally, for all $\mu > 0$, let

$$X_\theta^\mu(\mathbf{v}_k) = \arg\max_x \{\frac{x}{\mu} + \delta\left[\alpha_{\theta H} v_{Hk}(x) + \alpha_{\theta L} v_{Lk}(x)\right] : x \in [x_\theta^*, \overline{x}]\}$$

and let $x_\theta^\mu(\mathbf{v}_k)$ be the largest element of the compact set $X_\theta^\mu(\mathbf{v}_k)$. Notice that $x_\theta^\mu(\mathbf{v}_k)$ is non-increasing in μ. Also let

$$b_\theta^*(x) = \frac{R_\theta(r^*) + x - pg^*}{1+\rho}.$$

Then we have:

Claim: *For all k, for any $b \in [\underline{x}, \overline{x}]$ if ξ_L^k and ξ_H^k are sub-gradients of v_{Lk} and v_{Hk} at b then we have that: $-\xi_L^k \geq -\xi_H^k$. In addition, if $b \in (b_H^*(x_H^q(\mathbf{v}_{k-1})), \overline{x}]$, then v_{Hk} and v_{Lk} are differentiable at b and $-v_{Lk}'(b) > -v_{Hk}'(b)$.*

Proof: The proof proceeds via induction. Consider the claim for $k = 1$. In state θ if (r, g, x) is a solution to the problem

$$\begin{array}{c} \max u_\theta(r,g) + \frac{B_\theta(r,g,x;b)}{n} + \delta[\alpha_{\theta H}\widetilde{v}_H(x) + \alpha_{\theta L}\widetilde{v}_L(x)] \\ B_\theta(r,g,x;b) \geq 0,\ g \leq g^*,\ r \geq r^* \ \&\ x \in [x^*_\theta, \overline{x}] \end{array},$$

then: (i) if $b \leq b^*_\theta(x^n_\theta(\mathbf{v}_0))$, $(r,g) = (r^*, g^*)$ and $x \in X^n_\theta(\mathbf{v}_0) \cap \{x : B_\theta(r^*, g^*, x; b) \geq 0\}$; (ii) if $b \in (b^*_\theta(x^n_\theta(\mathbf{v}_0)), b^*_\theta(x^q_\theta(\mathbf{v}_0))]$, $(r,g) = (r^*, g^*)$ and $B_\theta(r^*, g^*, x; b) = 0$; and, (iii) if $b > b^*_\theta(x^q_\theta(\mathbf{v}_0))$, (r, g, x) is uniquely defined and the budget constraint is binding. Moreover, $r > r^*$ and $g < g^*$. Denote the solution in case (iii) as $(r_\theta(b; \mathbf{v}_0), g_\theta(b; \mathbf{v}_0), x_\theta(b; \mathbf{v}_0))$.

It follows from this that, if $b \leq b^*_\theta(x^n_\theta(\mathbf{v}_0))$

$$T^\theta(\mathbf{v}_0)(b) = u_\theta(r^*, g^*) + \frac{B_\theta(r^*, g^*, x^n_\theta(\mathbf{v}_0); b)}{n} + \delta[\alpha_{\theta H}\widetilde{v}_H(x^n_\theta(\mathbf{v}_0)) + \alpha_{\theta L}\widetilde{v}_L(x^n_\theta(\mathbf{v}_0))].$$

Thus, $T^\theta(\mathbf{v}_0)$ is differentiable and its derivative is

$$-\frac{dT^\theta(\mathbf{v}_0)(b)}{db} = \frac{1+\rho}{n}.$$

If $b \in (b^*_\theta(x^n_\theta(\mathbf{v}_0)), b^*_\theta(x^q_\theta(\mathbf{v}_0))]$, then

$$T^\theta(\mathbf{v}_0)(b) = u_\theta(r^*, g^*) + \delta[\alpha_{\theta H}\widetilde{v}_H(pg^* + (1+\rho)b - R_\theta(r^*)) + \alpha_{\theta L}\widetilde{v}_L(pg^* + (1+\rho)b - R_\theta(r^*))].$$

It follows that if μ_θ is a sub-gradient of $T^\theta(\mathbf{v}_0)$ at b there exist sub-gradients ξ_H and ξ_L of $\widetilde{v}_H$ and $\widetilde{v}_L$ at $pg^* + (1+\rho)b - R_\theta(r^*)$ such that $\mu_\theta = \alpha_{\theta H}\xi_H + \alpha_{\theta L}\xi_L$. Notice that in this range, $b \in (b^*_\theta(x^n_\theta(\mathbf{v}_0)), b^*_\theta(x^q_\theta(\mathbf{v}_0))]$ and hence if μ_θ is a sub-gradient of $T^\theta(\mathbf{v}_0)$ at b

$$-\delta\mu_\theta(1+\rho) \in (\frac{1+\rho}{n}, \frac{1+\rho}{q}].$$

If $b > b^*_\theta(x^q_\theta(\mathbf{v}_0))$ then

$$T^\theta(\mathbf{v}_0)(b) = \max_{(r,g,x)} \left\{ \begin{array}{c} u_\theta(r,g) + \frac{B_\theta(r,g,x;b)}{n} + \delta[\alpha_{\theta H}\widetilde{v}_H(x) + \alpha_{\theta L}\widetilde{v}_L(x)] \\ B_\theta(r,g,x;b) \geq 0 \ \&\ x \in [x^*_\theta, \overline{x}] \end{array} \right\}.$$

By the same argument used to prove Lemma 1, $T^\theta(\mathbf{v}_0)$ is differentiable and its derivative is

$$-\frac{dT^\theta(\mathbf{v}_0)(b)}{db} = \frac{1 - r_\theta(b; \mathbf{v}_0)}{n(1 - r_\theta(b; \mathbf{v}_0)(1+\varepsilon))}(1+\rho).$$

Since $r_\theta(b; \mathbf{v}_0) > r^*$, in this range we have that

$$-\frac{dT^\theta(\mathbf{v}_0)(b)}{db} > \frac{(1+\rho)}{q}.$$

Given the expressions for the derivatives and subgradients derived above, the result would follow for $k = 1$ if: (i) $b_L^*(x_L^n(\mathbf{v}_0)) \leq b_H^*(x_H^n(\mathbf{v}_0))$; (ii) $b_L^*(x_L^q(\mathbf{v}_0)) \leq b_H^*(x_H^q(\mathbf{v}_0))$; (iii) for all $b \in (b_H^*(x_H^n(\mathbf{v}_0)), b_L^*(x_L^q(\mathbf{v}_0)))$ if ξ_H and ξ_L are subgradients of $\widetilde{v}_H$ and $\widetilde{v}_L$ at $pg^* + (1+\rho)b - R_H(r^*)$ and ξ_H' and ξ_L' are subgradients of $\widetilde{v}_H$ and $\widetilde{v}_L$ at $pg^* + (1+\rho)b - R_L(r^*)$, then

$$-\delta[\alpha_{HH}\xi_H + \alpha_{HL}\xi_L](1+\rho) \leq -\delta[\alpha_{LH}\xi_H' + \alpha_{LL}\xi_L'](1+\rho);$$

and, (iv) for all $b > b_H^*(x_H^q(\mathbf{v}_0))$

$$\frac{1 - r_L(b;\mathbf{v}_0)}{(1 - r_L(b;\mathbf{v}_0)(1+\varepsilon))} > \frac{1 - r_H(b;\mathbf{v}_0)}{(1 - r_H(b;\mathbf{v}_0)(1+\varepsilon))}.$$

We will now establish that these four conditions are satisfied. For the first, we will show that $x_H^n(\mathbf{v}_0) \geq x_L^n(\mathbf{v}_0)$. Recall that by definition $x_\theta^n(\mathbf{v}_0)$ is the largest element in the compact set

$$X_\theta^n(\mathbf{v}_0) = \arg\max_x \{\frac{x}{n} + \delta\left[\alpha_{\theta H}\widetilde{v}_H(x) + \alpha_{\theta L}\widetilde{v}_L(x)\right] : x \in [x_\theta^*, \overline{x}]\}.$$

As shown in part (i) of this Lemma, we have that $x_H^* \geq x_L^*$. We can assume wlog that $x_L^n(\mathbf{v}_0) > x_L^*$. Thus, there exists sub-gradients ξ_H and ξ_L of $\widetilde{v}_H$ and $\widetilde{v}_L$ at $x_L^n(\mathbf{v}_0)$ such that

$$\frac{1}{n} = -\delta\left[\alpha_{LH}\xi_H + \alpha_{LL}\xi_L\right].$$

Suppose that $x \leq x_L^n(\mathbf{v}_0)$. Then, if ξ_H' and ξ_L' of $\widetilde{v}_H$ and $\widetilde{v}_L$ at x then since $-\xi_H \leq -\xi_L$, $\alpha_{HH} \geq \alpha_{LH}$, and $-\xi_\theta' \leq -\xi_\theta$, we know that:

$$-\delta\left[\alpha_{HH}\xi_H' + \alpha_{HL}\xi_L'\right] \leq -\delta\left[\alpha_{LH}\xi_H + \alpha_{LL}\xi_L\right] = \frac{1}{n}.$$

This implies that $x_H^n(\mathbf{v}_0) \geq x_L^n(\mathbf{v}_0)$. A similar argument establishes the second condition.

For the third condition, let $b \in (b_H^*(x_H^n(\mathbf{v}_0)), b_L^*(x_L^q(\mathbf{v}_0)))$, let ξ_H and ξ_L be subgradients of $\widetilde{v}_H$ and $\widetilde{v}_L$ at $pg^* + (1+\rho)b - R_H(r^*)$, and let ξ_H' and ξ_L' be subgradients of $\widetilde{v}_H$ and $\widetilde{v}_L$ at $pg^* + (1+\rho)b - R_L(r^*)$. Then we have

$$\begin{aligned} -\delta[\alpha_{HH}\xi_H + \alpha_{HL}\xi_L](1+\rho) &\leq -\delta[\alpha_{LH}\xi_H + \alpha_{LL}\xi_L](1+\rho) \\ &\leq -\delta[\alpha_{LH}\xi_H' + \alpha_{LL}\xi_L'](1+\rho), \end{aligned}$$

where the first inequality follows from the facts that $\alpha_{HH} \geq \alpha_{LH}$ and $-\xi_H \leq -\xi_L$, and the second inequality follows from the facts that $\widetilde{v}_H$ and $\widetilde{v}_L$ are concave and that $R_H(r^*) > R_L(r^*)$.

For the fourth condition, note that $(r_\theta(b;\mathbf{v}_0), g_\theta(b;\mathbf{v}_0), x_\theta(b;\mathbf{v}_0))$ is defined by the following three conditions:

$$n\alpha A g_\theta(b;\mathbf{v}_0)^{\alpha-1} = p[\frac{1-r_\theta(b;\mathbf{v}_0)}{1-r_\theta(b;\mathbf{v}_0)(1+\varepsilon)}],$$

there exist subgradients ξ_H and ξ_L be subgradients of $\widetilde{v}_H$ and $\widetilde{v}_L$ at $x_\theta(b;\mathbf{v}_0)$ such that

$$[\frac{1-r_\theta(b;\mathbf{v}_0)}{1-r_\theta(b;\mathbf{v}_0)(1+\varepsilon)}] = -\delta n[\alpha_{\theta H}\xi_H + \alpha_{\theta L}\xi_L],$$

and

$$B_\theta(r_\theta(b;\mathbf{v}_0), g_\theta(b;\mathbf{v}_0), x_\theta(b;\mathbf{v}_0); b) = 0.$$

Suppose to the contrary that $r_H(b;\mathbf{v}_0) \geq r_L(b;\mathbf{v}_0)$. Then, $g_H(b;\mathbf{v}_0) \leq g_L(b;\mathbf{v}_0)$ and $x_H(b;\mathbf{v}_0) \geq x_L(b;\mathbf{v}_0)$. It follows that

$$\begin{aligned} 0 &= B_H(r_H(b;\mathbf{v}_0), g_H(b;\mathbf{v}_0), x_H(b;\mathbf{v}_0); b) \geq B_H(r_L(b;\mathbf{v}_0), g_L(b;\mathbf{v}_0), x_L(b;\mathbf{v}_0); b) \\ &> B_L(r_L(b;\mathbf{v}_0), g_L(b;\mathbf{v}_0), x_L(b;\mathbf{v}_0); b). \end{aligned}$$

This is a contradiction.

Now assume that the claim is true for $t = 1, ...k$ and consider it for $k+1$. By the induction step, for any $b \in [\underline{x}, \overline{x}]$ if ξ_{Lk} is a sub-gradient of v_{Lk} at b and ξ_{Hk} is a sub-gradient of v_{Hk} at b then we have that: $-\xi_{Lk} \geq -\xi_{Hk}$. It follows that v_{Hk} and v_{Lk} have the same properties as the functions $\widetilde{v}_H$ and $\widetilde{v}_L$ and the same argument as above applies to step $k+1$. ■

We can now prove Lemma A.2. Given Lemma 1, all we need to do is to establish that if $b \in (b_H^*, \overline{x}]$ then

$$-v_L'(b) \geq -v_H'(b).$$

Suppose, to the contrary, that there exists some $b' \in (b_H^*, \overline{x}]$ such that $-v_L'(b') < -v_H'(b')$. Let $\varepsilon > 0$ be such that $b' - \varepsilon > b_H^*$. Given that $\mathbf{v}_k$ converges to (v_H, v_L), it must be the case that $x_H^q(\mathbf{v}_k)$ converges to $x_H^q(v_H, v_L) = x_H^*$ as $k \to \infty$. Thus, for sufficiently large k, $b_H^*(x_H^q(\mathbf{v}_k)) < b' - \varepsilon$. For any k sufficiently large, therefore, the Claim implies that v_{Hk} and v_{Lk} are differentiable on $(b'-\varepsilon, \overline{x}]$ and $-v_{Lk}'(b) > -v_{Hk}'(b)$ for any $b \in (b'-\varepsilon, \overline{x}]$. Thus, by Theorem 25.7 of Rockafellar (1970), we know that $\lim_{k\to\infty} v_{\theta k}'(b) = v_\theta'(b)$ for any $b \in (b'-\varepsilon, \overline{x}]$, which includes b': a contradiction. ■

10.9 Proof of the results of Section 7.1

In Section 7.1 we claim that (i) if $b \geq b_H^*$ then $r_L(b) > r_H(b)$, $g_L(b) < g_H(b)$ and $x_L(b) > x_H(b)$; and (ii) there is a $\widehat{b} \in (b_L^*, b_H^*)$ such that new debt will be higher in a recession than a boom if and only if $b > \widehat{b}$. We begin with part (i). We have already shown in Lemma 2 that when $b \geq b_H^*$, $r_L(b) > r_H(b)$. The first order conditions tells us that $\{r_\theta(b),g_\theta(b)\}$ must satisfy the following equality:

$$n\alpha A g^{\alpha-1} = p[\frac{1-r}{1-r(1+\varepsilon)}],$$

which implies that $g_L(b) < g_H(b)$. In addition, since $x_H^* \leq b_H^* \leq b$, by Lemma 3 below we have that

$$x_H(b) \leq b < x_L(b).$$

Part (ii) follows from the facts that $x_L(b)$ is increasing in b on the interval $(b_L^*, b_H^*]$, $x_H(b)$ is constant on the interval $(b_L^*, b_H^*]$, $x_H(b_L^*) > x_L(b_L^*)$, and $x_H(b_H^*) < x_L(b_H^*)$ (by part (i)).

We also claim that tax revenues will be higher in a boom when $b \geq \widehat{b}$. To see this note first that

$$R_H(r_H(b)) \geq p g_H(b) - x_H(b) + (1+\rho)b$$

and that

$$R_L(r_L(b)) = p g_L(b) - x_L(b) + (1+\rho)b.$$

Now note that $g_H(b) > g_L(b)$ and, for $b \geq \widehat{b}$, $x_L(b) \geq x_H(b)$. ■

10.10 Proof of Lemma 3

(i) If $b \leq b_L^*$, we have that $x_L(b) = x_L^* > b_L^* \geq b$. Assume then that $b > b_L^*$. Suppose, contrary to the claim, that $x_L(b) \leq b$. By Lemma 1, we have that

$$-\delta n v_L'(b) = \frac{1-r_L(b)}{1-r_L(b)(1+\varepsilon)}.$$

Since $x_L(b) < \overline{x}$ the first order conditions for $(r_L(b), g_L(b), x_L(b))$ imply that there must exist sub-gradients ξ_L and ξ_H of v_L and v_H at $x_L(b)$ such that

$$\frac{1-r_L(b)}{1-r_L(b)(1+\varepsilon)} = -\delta n \left[\alpha_{LH}\xi_H + \alpha_{LL}\xi_L\right].$$

Since $r_L(b) > r^*$, for this equation to hold we must have that $x_L(b) > b_L^*$ and hence we know by Lemma 1 that

$$\xi_L = -\frac{1-r_L(x_L(b))}{1-r_L(x_L(b))(1+\varepsilon)}(\frac{1+\rho}{n}).$$

In addition, it must be the case that

$$-\delta\xi_H < -\delta\xi_L.$$

Clearly, this is case if $x_L(b) \leq b_H^*$. If $x_L(b) > b_H^*$, the inequality follows from the fact that $r_L(x_L(b)) > r_H(x_L(b))$. Thus, we have that

$$\begin{aligned}
\frac{1-r_L(b)}{1-r_L(b)(1+\varepsilon)} &= -\delta n\left[\alpha_{LH}\xi_H + \alpha_{LL}\xi_L\right] \\
&< -\delta n \xi_L \\
&= \frac{1-r_L(x_L(b))}{1-r_L(x_L(b))(1+\varepsilon)}.
\end{aligned}$$

But this is a contradiction because the facts that $r_L(\cdot)$ is increasing and that $b \geq x_L(b)$, imply that $r_L(b) \geq r_L(x_L(b))$.

(ii) If $b \leq b_H^*$, we have that $x_H(b) = x_H^*$. Thus, $x_H(b) > b$ if $b < x_H^*$ and $x_H(b) < b$ if $b \in (x_H^*, b_H^*]$. Assume then that $b > b_H^*$. Suppose, contrary to the claim, that $x_H(b) \geq b$. By Lemma 1, we have that

$$-\delta n v_H'(b) = \frac{1-r_H(b)}{1-r_H(b)(1+\varepsilon)}.$$

Since $x_H(b) \geq b > b_H^* > b_L^*$, then we know from the first order condition for $x_H(b)$ and Lemma 1 that

$$\frac{1-r_H(b)}{1-r_H(b)(1+\varepsilon)} \geq \alpha_{LH}\left(\frac{1-r_H(x_H(b))}{1-r_H(x_H(b))(1+\varepsilon)}\right) + \alpha_{LL}\left(\frac{1-r_L(x_H(b))}{1-r_L(x_H(b))(1+\varepsilon)}\right) \ (= \text{ if } x_H(b) < \overline{x}),$$

Since $r_H(x_H(b)) < r_L(x_H(b))$, this equation implies that

$$\frac{1-r_H(b)}{1-r_H(b)(1+\varepsilon)} > \left(\frac{1-r_H(x_H(b))}{1-r_H(x_H(b))(1+\varepsilon)}\right).$$

But this is a contradiction because the facts that $r_H(\cdot)$ is increasing and $b \leq x_H(b)$, imply that $r_H(b) \leq r_H(x_H(b))$. ■

10.11 Proof of Proposition 7

The dynamic pattern of debt described in the proposition follows immediately from Lemma 3. Thus, to prove the proposition we must show that the debt distribution converges strongly to a unique invariant distribution. To this end, define the state space $S = [\underline{x}, \overline{x}] \times \{L, H\}$ with associated σ-algebra $\mathcal{F} = \mathcal{B} \times \mathcal{H}$, where $\mathcal{B}$ is the family of Borel sets that are subsets of $[\underline{x}, \overline{x}]$, and $\mathcal{H}$ is the family of subsets of $\{L, H\}$. For any subset $A \in \mathcal{F}$, let $\mu_t(A)$ denote the probability that the state lies in A in period t. The probability measure μ_t describes the debt distribution in period t; for example, the probability that in period t the debt level lies between x_a and x_b in a boom is given by $\mu_t(([x_a, x_b], H))/\mu_t(([\underline{x}, \overline{x}], H))$. We are thus interested in the long run behavior of μ_t.

The probability distribution μ_1 depends on the initial level of debt b_0 and the initial state of the economy. To describe the probability distribution in periods $t \geq 2$ we must first describe the transition function implied by the equilibrium. This transition function is given by:

$$Q(A\,|b,\theta\,) = \begin{cases} \sum_{\{\theta' :\ \text{s.t.}(x_{\theta'}(b),\theta')\in A\}} \alpha_{\theta\theta'} & \text{if } \exists\ \theta'\ \ s.t.\ \ (x_{\theta'}(b),\theta') \in A \\ 0 & otherwise \end{cases}.$$

Intuitively, $Q(A\,|b,\theta\,)$ is the probability that a set A is reached in one step if the initial state is (b,θ). Using this notation, the probability distribution in period $t \geq 2$ is defined inductively as:

$$\mu_t(A) = \sum_{\theta} \int_b Q(A\,|b,\theta\,)\mu_{t-1}(db,\theta).$$

The probability distribution μ^* is an invariant distribution if

$$\mu^*(A) = \sum_{\theta} \int_b Q(A\,|b,\theta\,)\mu^*(db,\theta).$$

We now show that the sequence of distributions $\langle \mu_t \rangle_{t=1}^{\infty}$ converges strongly to a unique invariant distribution.

By Theorem 11.12 in Stokey, Lucas and Prescott (1989), it is enough to show that the transition function Q satisfies the M *condition* (see the definition in Stokey, Lucas and Prescott (1989)). To this end, let $Q^1(A\,|b,\theta\,) = Q(A\,|b,\theta\,)$ and define recursively:

$$Q^n(A\,|b,\theta\,) = \sum_{\theta'} \int_{b'} Q(A\,|b',\theta'\,)Q^{n-1}(db',\theta'\,|b,\theta\,).$$

Thus, $Q^n(A\,|b,\theta\,)$ is the probability that a set A is reached in n steps if the initial state is (b,θ). To establish that Q satisfies the M condition, it is sufficient to prove that there exists a state (x^*,θ^*), an integer $N \geq 1$ and a number $\varepsilon > 0$, such that for any initial state (b,θ), $Q^N((x^*,\theta^*)\,|b,\theta\,) > \varepsilon$ (See Exercises 11.5 and 11.4 in Stokey, Lucas and Prescott (1989)).

Consider the state (x_H^*, H). Define $\eta = \min_{b\in[b_H^*,\overline{x}]}[b - x_H(b)]$. Since, by Lemma 3, $x_H(b) < b$ for any $b \in [b_H^*, \overline{x}]$, we have that $\eta > 0$. Let N be the smallest integer larger than $\frac{\overline{x}-b_H^*}{\eta} + 1$. Then, we claim that for any initial state (b,θ), we have that:

$$Q^N((x_H^*, H)\,|b,\theta\,) \geq \alpha_{LH}\,(\alpha_{HH})^{N-1} > 0.$$

If this claim is true, then by choosing $\varepsilon \in \left(0, \alpha_{LH} \left(\alpha_{HH}\right)^{N-1}\right)$, we have the desired condition.

To see that the claim is true, suppose first that the initial state (b, θ) is such that $b \leq b_H^*$. With probability of at least α_{LH} the state will be (x_H^*, θ_H) in the next period and it will remain there for as long as the economy remains in a boom (which happens with probability α_{HH}). Next suppose that the initial state (b, θ) is such that $b > b_H^*$. With probability of at least α_{LH} the economy will be in a boom the next period and, again, it will remain in a boom thereafter with probability α_{HH}. If it does remain in a boom, then for as long as the debt level remains above b_H^*, debt will be reduced by at least η in each period. Thus, after N periods, the debt level must have gone below b_H^* in some period and therefore will have reached x_H^*. ■

10.12 Proof of Proposition 10

The primary surplus in state θ is given by:

$$PS_\theta(b) = (1+\rho)b - x_\theta(b).$$

Note first that the primary surplus in state θ is increasing in b. This is immediate if $b < b_\theta^*$ since in that case $x_\theta(b) = x_\theta^*$. To see the result if $b > b_\theta^*$ note first that when the mwc is not providing pork

$$PS_\theta(b) = R_\theta(r_\theta(b)) - pg_\theta(b).$$

Now recall that $r_\theta(b)$ is increasing in b and $g_\theta(b)$ is decreasing in b.

To understand the long run behavior of the primary surplus when the economy enters a boom, let the level of debt when the economy enters a boom be b. By Proposition 5, we know that this debt level must exceed x_H^*. To show that the primary surplus jumps up when the economy enters the boom, we need to show that

$$(1+\rho)b - x_H(b) > (1+\rho)x_L^{-1}(b) - b.$$

We have that, by definition,

$$(1+\rho)x_L^{-1}(b) - b = (1+\rho)x_L^{-1}(b) - x_L(x_L^{-1}(b))$$

Since debt levels are increasing in a recession, we have that $b > x_L^{-1}(b)$. Thus, using the fact that PS_L is increasing, we have that

$$(1+\rho)x_L^{-1}(b) - x_L(x_L^{-1}(b)) < (1+\rho)b - x_L(b).$$

From the fact that $b > x_H^*$, we know that $x_H(b) < x_L(b)$ and hence

$$(1+\rho)b - x_L(b) < (1+\rho)b - x_H(b).$$

The fact that, after the initial jump, the primary surplus starts gradually declining until either it reaches a minimal level of ρx_H^* or the boom ends follows from Proposition 5 and the fact that $PS_H(b)$ is increasing in b.

To understand the long run behavior of the primary surplus when the economy enters a recession, let the level of debt when the economy enters a recession be b. By Proposition 5, we know that this debt level must be at least as big as x_H^*. To show that the primary surplus jumps down when the economy enters the boom, we need to show that

$$(1+\rho)b - x_L(b) < (1+\rho)x_H^{-1}(b) - b.$$

We have that, by definition,

$$(1+\rho)x_H^{-1}(b) - b = (1+\rho)x_H^{-1}(b) - x_H(x_H^{-1}(b)).$$

Since, in the long run, debt levels are decreasing or constant in a recession, we have that $b \leq x_H^{-1}(b)$. Thus, using the fact that PS_H is increasing, we have that

$$(1+\rho)x_H^{-1}(b) - x_H(x_H^{-1}(b)) \geq (1+\rho)b - x_H(b).$$

From the fact that $b > x_H^*$, we know that $x_H(b) < x_L(b)$ and hence that

$$(1+\rho)b - x_H(b) > (1+\rho)b - x_L(b).$$

The fact that, after the initial jump, the primary surplus starts increasing follows from Proposition 5 and the fact that $PS_L(b)$ is increasing in b. ■

10.13 Completion of proof of Proposition 1

The first task is to solve for r^*, g^*, x_L^* and x_H^* when $q = n$. From (16) and (17), we see that r^* equals 0 and g^* equals the Samuelson level g_S. For x_L^* and x_H^*, note first that the value function v_θ is differentiable everywhere since at $x = b_\theta^*$ the left hand derivative is equal to the right hand derivative. We can therefore use first order conditions to characterize x_L^* and x_H^*. When $q = n$, the first order condition for x_θ^* (26) requires that

$$\frac{1}{n} = -\delta[\alpha_{\theta H} v_H'(x_\theta^*) + \alpha_{\theta L} v_L'(x_\theta^*)].$$

For this equation to be satisfied, we must have that $-\delta v_H'(x_\theta^*) = -\delta v_L'(x_\theta^*) = (1+\rho)/n$ which implies that x_θ^* must be less than or equal to b_L^*. Since $r^* = 0$ and $g^* = g_S$, this implies that $\rho x_\theta^* \leq -pg_S$. It follows that $x_L^* = x_H^* = \underline{x}$. This, in turn, implies that $b_L^* = b_H^* = \underline{x}$.

It remains to understand the dynamics of the planner's solution. Note first that since $b_L^* = b_H^* = \underline{x}$ and $x_L^* = x_H^* = \underline{x}$, if the debt level ever got to $\underline{x}$, it would remain there forever. Lemma 3 therefore needs to be modified to say (i) that for all $b \in (\underline{x}, \overline{x})$, $x_L(b) > b$ and (ii) for all $b \in (\underline{x}, \overline{x}]$, $x_H(b) < b$. Thus, if the debt level exceeds $\underline{x}$, debt increases in recessions and decreases in booms. In the long run, however, the debt level must reach $\underline{x}$ with probability one and at that point the cyclical fluctuation in debt stops. A steady state is reached in which the debt level is $\underline{x}$, the tax rate is 0, and the public good level is g_S. This is precisely the claim made in Proposition 1. ■