Security Pools and Efficiency*

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Abstract Collateralized mortgage obligations, collateralized debt obligations and other bundles of securities that are collateralized by other securities (rather than directly by physical objects) have received a great deal of attention in the the popular press, where it it frequently asserted that such securities serve no social function. This paper builds an extension of general equilibrium theory that incorporates durable goods, collateralized securities and pools of collateralized securities to argue that such pools do in fact serve an important social function. In particular, such pools make it possible for the economy to reach an efficient allocation in robust situations where it would otherwise be impossible.

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PRELIMINARY – DO NOT QUOTE

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1 Introduction

Recent events in financial markets provide a sharp reminder that much of the lending in modern economies is secured, directly or indirectly, by physical capital. Residential and commercial mortgages are secured directly by the mortgaged property itself; corporate bonds are secured directly by the physical assets of the firm; collateralized mortgage obligations and debt obligations and other similar instruments are secured by pools of mortgages and loans and hence secured indirectly by the physical capital that secures these mortgages and loans. The total of indirectly collateralized lending is enormous: in 2007, the (notional) value of collateralized debt obligations was estimated to exceed \$50 trillion – more than three times U.S. GDP. Surprisingly, there has been little discussion of the effects of direct and indirect collateralization in the scholarly literature. The discussion in the popular press has focused on the negative effects and often suggests that collateralized debt obligations serve no social function.

The central purpose of this paper is to argue to the contrary that collateralized debt obligations and other pools of securities serve a very important social function. In particular they enable the realization of Walrasian outcomes in situations where such outcomes would otherwise not be obtainable. **should we say more here?**

need a better segue In an environment in which all lending must be collateralized, the supply of collateral becomes an important financial constraint. If collateral is in short supply the necessity of using collateral to back promises creates incentives to create collateral and to stretch existing collateral. The state creates collateral by issuing bonds that can be used as collateral and by promulgating law and regulation that make it easier to seize goods used as collateral. The market creates collateral through financial engineering, which has rapidly accelerated over the last three-and-a-half decades (beginning with the introduction of mortgage-backed securities in the early 1970's) and that stretches collateral by making it possible for the same collateral to be used several times: allowing agents to collateralize their promises with other agents' promises (pyramiding) and allowing the same collateral to back many different promises (tranching). **say something about pooling** These innovations are at the bottom of the securitization and derivatives boom on Wall Street, and have greatly expanded the scope of financial markets.

To make this point – and others – we formulate an extension of general equilibrium theory that incorporates durable goods, securities that are collateralized by durable goods and securities that are collateralized by other securities that are in turn collateralized by durable goods. To focus the discussion, we restrict attention to a pure exchange framework with two dates but many possible states of nature (representing the uncertainty at time 0 about exogenous shocks at time 1). As is usual in general equilibrium theory, we view individuals as anonymous price-takers.[1] For simplicity, we use a framework with a finite number of agents and divisible loans.[2] Central to the model are that the definition of a security must include not only its promised deliveries but also the collateral that backs those promises and the fact that the actual deliveries will depend on the value of the collateral and not only on the promises.

The requirement that borrowing be collateralized implies an endogenous bound on short sales, so that equilibrium always exists (Theorem 1). Although the existence of more complicated securities expands the set of possible market outcomes, it may still fail to yield Walrasian allocations. In particular, no collateral equilibrium can ever achieve an allocation in which some agent's consumption in some terminal state has less value than his/her initial (unpledgeable) endowment in that state (Theorem 2). As a consequence, even with pooling, pyramiding and tranching, collateral equilibrium is robustly inefficient: given any array of consumer preferences and any social endowment, there is always an open set of distributions

[1] Anonymity and price-taking might appear strange in an environment in which individuals might default. In our context, however, individuals will default when the value of promises exceeds the value of collateral and not otherwise; thus lenders do not care about the identity of borrowers, but only about the collateral they bring.

[2] The assumptions of anonymity and price-taking might be made more convincing by building a model that incorporates a continuum of individuals, and the realism of the model might be enhanced by allowing for indivisible loans, but doing so would complicate the model without qualitatively changing the conclusions.

of that endowment with the property that collateral equilibrium from those endowments fails to be Pareto optimal (Theorem 3) – no matter what securities are available for trade. On the other hand, any Walrasian equilibrium in which every agent's consumption in each terminal state has greater value than his/her initial (unpledgeable) endowment in that state can be obtained as a collateral equilibrium whenever a complete set of tranched Arrow securities is available (Theorem 4). Absent tranching, this conclusion *does not* hold (Example 1); thus, pooling pyramiding and tranching served an important role in furthering social welfare.

For a more basic model in which all securities are collateralized directly by physical goods and a discussion of related literature, see Geanakoplos and Zame (2010).

2 Model

As in the canonical model of securities trading, we consider a world with two dates; agents know the present but face an uncertain future. At date 0 (the present) agents trade a finite set of commodities and securities. Between date 0 and date 1 (the future) the state of nature is revealed. At date 1 securities pay off and commodities are traded again.

2.1 Time & Uncertainty

There are two dates, 0 and 1, and S possible states of nature at date 1. We frequently refer to $0, 1, \ldots, S$ as *spots*.

2.2 Commodities, Markets & Prices

There are $L \geq 1$ commodities available for consumption and trade in spot markets at each date and state of nature; the commodity space is $\mathbb{R}^{L(1+S)} = \mathbb{R}^L \times \mathbb{R}^{LS}$. We interpret $x \in \mathbb{R}^{L(1+S)}$ as a claim to consumption at each date and state of the world. For a bundle

$x \in \mathbb{R}^{L(1+S)}$ and indices s, ℓ, we write x_s for the vector of spot s consumption specified by x, and $x_{s\ell}$ for the quantity of commodity ℓ specified in spot s. We abuse notation and view $\mathbb{R}^L$ as the subspace of $\mathbb{R}^{L(1+S)}$ consisting of those vectors which are 0 in the last LS coordinates; thus we identify a vector $x \in \mathbb{R}^L$ with $(x, 0, \ldots, 0) \in \mathbb{R}^{L(1+S)}$. Similarly we view $\mathbb{R}^{LS}$ as the subspace of $\mathbb{R}^{L(1+S)}$ consisting of those vectors which are 0 in the first L coordinates. We write $\delta_{s\ell} \in \mathbb{R}^{L(1+S)}$ for the commodity bundle consisting of one unit of commodity ℓ in spot s and nothing else. We write $x \geq y$ to mean that $x_{s\ell} \geq y_{s\ell}$ for each s, ℓ; $x > y$ to mean that $x \geq y$ and $x \neq y$; and $x \gg y$ to mean that $x_{s\ell} > y_{s\ell}$ for each s, ℓ.

We depart from the usual intertemporal models by allowing for the possibility that goods are durable. If $x_0 \in \mathbb{R}^L$ is consumed (used) at date 0 we write $F_s(x_0)$ for what remains in state s at date 1. We assume the map $F : S \times \mathbb{R}^L \to \mathbb{R}^L$ is continuous and is linear and positive in consumption. The commodity 0ℓ is *perishable* if $F(\delta_{0\ell}) \equiv 0$ and *durable* otherwise. It may be helpful to think of F as like a production function — except that inputs to production can also be consumed.

For each s, there is a spot market for consumption at spot s. Prices at each spot lie in $\mathbb{R}^L_{++}$, so $\mathbb{R}^{L(1+S)}_{++}$ is the space of spot price vectors. For $p \in \mathbb{R}^{L(1+S)}$, p_s are the prices in spot s and $p_{s\ell}$ is the price of commodity ℓ in spot s.

2.3 Consumers

There are I consumers (or types of consumers). Consumer i is described by a consumption set, which we take to be $\mathbb{R}^{L(1+S)}_{+}$, an endowment $e^i \in \mathbb{R}^{L(1+S)}_{+}$, and a utility function $u^i : \mathbb{R}^{L(1+S)}_{+} \to \mathbb{R}$.

2.4 Collateralized Securities

A *collateralized security* (*security* for short) is a pair $\mathbf{A} = (A, c)$, where $A : S \times \mathbb{R}^L_{++} \times \mathbb{R}^L_{++} \to \mathbb{R}_+$ is the *promise* or *face value*, and $c \in \mathbb{R}^L_+$ is the *collateral requirement*. We allow for the possibility that the amount promised in each state depends on spot prices in that state; hence A is a function (assumed continuous) of the state and of prices in that state. The collateral requirement c is a bundle of date 0 commodities; an agent wishing to sell one share of (A, c) must hold the commodity bundle c. (Recall that selling a security is borrowing.)

In our framework, the collateral requirement is the only means of enforcing promises. Hence, if agents optimize, the *delivery* per share of security (A, c) in state s will not be the face value $A_s(p_0, p_s)$ but rather the minimum of the face value and the value of the collateral in state s:

$$\textsc{Del}((A, c), s, p) = \min\{A_s(p_0, p_s),\ p_s \cdot F_s(c)\}$$

The delivery on a portfolio $\theta = (\theta^1, \ldots, \theta^J) \in \mathbb{R}^J$ is

$$\textsc{Del}(\theta, s, p) = \sum_j \theta^j \textsc{Del}((A^j, c^j); s, p)$$

We take as given a finite (but perhaps very large) set of securities $\mathcal{A} = \{(A^1, c^1), \ldots, (A^J, c^J)\}$. Because deliveries never exceed the value of collateral, we assume without loss of generality that $F_s(c^j) \neq 0$ for some s. (Securities that fail this requirement will deliver nothing; in equilibrium such securities will have 0 price and purchases or sales of such securities will be irrelevant.) We find it convenient to distinguish between security purchases and sales; we typically write $\varphi, \psi \in \mathbb{R}^J_+$ for portfolios of security purchases and sales, respectively. We assume that buying and selling prices for securities are identical; we write $q \in \mathbb{R}^J_+$ for the vector of security prices. An agent who sells the portfolio $\psi \in \mathbb{R}^J_+$ will have to hold (and will enjoy) the collateral bundle $\textsc{Coll}(\psi) = \sum \psi^j c^j$.

This formulation allows for nominal securities, for real securities, for options and for complicated derivatives.

2.5 Securitization

Securitization usually refers to the process of converting non-tradable assets into tradable securities through the repacking of their cash flows (Elul, 2005). We view securitization more generally, as the process of creating securities – we shall refer to them as *security pools* – that are collateralized by other securities. In general, the securities used as collateral might in turn be collateralized by other securities, and so forth through many layers, but for our purposes it shall be enough to allow for only a single layer; we leave the straightforward generalization to the interested reader. This section presents the formal model; discussion and applications to welfare are discussed in Section 5

Fix commodities and a family $\mathcal{A} = \{(A^1, c^1), \ldots, (A^J, c^J)\}$ of collateralized securities. A *security pool* is a tuple $\mathbf{B} = (B^1, \ldots, B^T; \chi)$ where each *tranche* B^t is a promise of delivery as a function of prices, and $\chi = (\chi_0, \chi_1) \in \mathbb{R}^L_+ \times \mathbb{R}^J_+$ (a bundle of commodities and a portfolio of securities) is the collateral requirement. It is convenient to write:

$$\textsc{Del}(\chi; s, p) = p \cdot \chi_0 + \textsc{Del}(\chi_1; s, p)$$

for the delivery of the collateral requirement $\chi = (\chi_0, \chi_1)$. We interpret the promise B^t as senior to B^{t+1}, so actual deliveries may be defined by induction:

$$\textsc{Del}(B^1; s, p) = \min\Big\{B^1(s, p_s),\ \textsc{Del}(\chi; s, p)\Big\}$$
$$\textsc{Del}(B^{t+1}; s, p) = \min\left\{B^{t+1}(s, p_s),\ \textsc{Del}(\chi; s, p) - \sum_{t'=1}^{t} \textsc{Del}(B^{t'}; s, p)\right\}$$

Note that the delivery on each of the promises lies (weakly) between 0 and the delivery on the collateral. There is no loss in assuming that all pools have the same number of tranches (because we can always add tranches that promise 0 delivery).

If $\mathcal{B} = \{\mathbf{B}^1, \ldots, \mathbf{B}^\mathbf{K}\}$ is the set of available security pools, a portfolio of security tranches is a vector $\Theta \in \mathbb{R}^{KT}_+$; the delivery on Θ is

$$\textsc{Del}(\Theta; s, p) = \sum_k \sum_t \Theta^{kt} \textsc{Del}(\mathbf{B}^{kt}; s, p)$$

2.6 The Economy

An *economy with collateralized securities and security pools* is a tuple $\mathcal{E} = \langle (e^i, u^i), \mathcal{A}, \mathcal{B} \rangle$. Write $\overline{e} = \sum e^i$ for the social endowment. The following assumptions are always in force:

- Assumption 1 $\overline{e} + F(\overline{e}) \gg 0$
- Assumption 2 For each consumer i: $e^i > 0$
- Assumption 3 For each consumer i:
  - (a) u^i is continuous and quasi-concave
  - (b) if $x \geq y \geq 0$ then $u^i(x) \geq u^i(y)$
  - (c) if $x \geq y \geq 0$ and $x_{s\ell} > y_{s\ell}$ for some $s \neq 0$ and some ℓ, then $u^i(x) > u^i(y)$
  - (d) if $x \geq y \geq 0$, $x_{0\ell} > y_{0\ell}$, and commodity 0ℓ is perishable, then $u^i(x) > u^i(y)$

The first assumption says that all goods are represented in the aggregate (keeping in mind that some date 1 goods may only come into being when date 0 goods are used). The second assumption says that that individual endowments are non-zero. The third assumption says that utility functions are continuous, quasi-concave, weakly monotone, strictly monotone in date 1 consumption of all goods and in date 0 consumption of perishable goods.[3]

2.7 Budget Sets

We write p for spot prices of commodities, q for prices of securities and Q for prices of tranches of pools (so q^j is the price of security j and Q^{kt} is the price of tranche B^{kt} of pool $\mathbf{B}^k$). Write φ^i, ψ^i, and Φ^i, Ψ^i for consumer i's purchases and sales of securities and tranches

[3] We do not require strict monotonicity in durable date 0 goods because we want to allow for the possibility that claims to date 1 consumption are traded at date 0; of course, such claims would typically provide no utility at date 0.

(so $\varphi^{ij}, \psi^{ij} j$ are consumer i's purchases and sales of security j and Φ^{kti}, Ψ^{kti} are consumer i's spot purchases and sales of the tranche t of pool k. Given spot prices p, security prices q and tranche prices Q, the *budget set* of a consumer whose endowment is e is the set of plans $(x, \varphi, \psi, \Phi, \Psi)$ (for consumption, security purchases, security sales, tranche purchases and tranche sales) that satisfy the budget constraints at date 0 and in each state at date 1 and the collateral constraints at date 0:

- **At date 0**

$$p_0 \cdot x_0 + q \cdot \varphi + Q \cdot \Phi \leq p_0 \cdot e_0 + q \cdot \psi + Q \cdot \Psi$$

$$x_0 \geq \sum_j \psi_j c^j$$
$$\varphi \geq \sum_k \max_t \Psi^{kt} \chi^k$$

That is, expenditures for consumption, security purchases and pool purchases do not exceed income from endowment, security sales and pool sales, date 0 consumption includes collateral for all security sales and date 0 security purchases include collateral for all pool sales. Note that, as intended, holding the collateral χ^k is sufficient to collateralize sales of one unit of *each* tranche of pool $\mathbf{B}^k$.

- **In state s**

$$\begin{aligned} p_s \cdot x_s + \textsc{Del}(\varphi; s, p) + \textsc{Del}(\Phi; s, p) &\leq p_s \cdot e_s + p_s \cdot F_s(x_0) \\ &\quad + \ \textsc{Del}(\psi, s, p) + \textsc{Del}(\Psi; s, p) \end{aligned}$$

That is, expenditures for consumption and deliveries on securities and pools do not exceed income from endowment, from the return on durable goods, and from deliveries on security promises and pool promises.

2.8 Pool Equilibrium

Given an economy, $\mathcal{E} = \langle (e^i, u^i), \mathcal{A}, \mathcal{B} \rangle$, a *pool equilibrium* consists of spot prices $p \in \mathbb{R}_+^{L(1+S)}$, security prices $q \in \mathbb{R}_+^J$, pool prices $Q \in \mathbb{R}_+^{KT}$ and consumer plans $(x^i, \varphi^i, \psi^i, \Phi^i, \Psi^i)$ satisfying the obvious conditions:

- **Commodity Markets Clear**

$$\sum_i x^i = \sum_i e^i + \sum_i F(e_0^i)$$

- **Security Markets Clear**

$$\sum_i \varphi^i = \sum_i \psi^i$$

- **Pool Markets Clear**

$$\sum_i \Phi^i = \sum_i \Psi^i$$

- **Plans are Budget Feasible**

$$(x^i, \varphi^i, \psi^i) \in B(p, q, Q; e^i, \mathcal{A}, \mathcal{B})$$

- **Consumers Optimize**

$$(x, \varphi, \psi, \Phi^i, \Psi^i) \in B(p, q, Q; e^i, \mathcal{A}, \mathcal{B}) \Rightarrow u^i(x) \leq u^i(x^i)$$

It is natural to think of security pools as assembled by intermediaries who purchase all the collateral and then sells some of the tranches, holding the rest themselves.

2.9 Walrasian Equilibrium

We compare collateral equilibrium with the benchmark of Walrasian equilibrium so it is useful to recall the definition of Walrasian equilibrium in the present context; see Dubey, Geanakoplos, and Shubik (2005) for further details. We maintain the same structure of commodities and preferences. In particular, date 0 commodities are durable, and $F_s(x_0)$ is what remains in state s if the bundle x_0 is consumed at date 0. Suppressing commodities and the nature of durability, the data of a *durable goods economy* is thus a set (e^i, u^i) of consumers, specified by endowments and utility functions. We use notation in which a purchase at date 0 conveys the rights to what remains at date 1; hence if commodity prices are $p \in \mathbb{R}^{(1+S)L}_{++}$, the Walrasian budget set for a consumer whose endowment is e is

$$B^W(e, p) = \{x : p \cdot x \leq p \cdot e + p \cdot F(x_0)\}$$

A *Walrasian equilibrium* consists of commodity prices p and consumption choices x^i such that

- **Commodity Markets Clear**

$$\sum_i x^i = \sum_i e^i + \sum_i F(e^i_0)$$

- **Plans are Budget Feasible**

$$x^i \in B^w(e^i, p)$$

- **Consumers Optimize**

$$y^i \in B(e^i, p) \Rightarrow u^i(y^i) \leq u^i(x^i)$$

3 Existence of Pool Equilibrium

Our model of security pools satisfies the basic consistency requirement that equilibrium exists.

Theorem 1 ***(Existence of Pool Equilibrium)*** *Every economy with collateralized securities and security pools, satisfying Assumptions 1-3 (in Section 2) admits an equilibrium.*

Proof The proof follows exactly the proof of Theorem 1 in Geanakoplos and Zame (2010) with the obvious addition of pools, pool prices, and pool purchases and sales. We leave the (messy) details to the reader. ■

4 Pyramiding and Pooling

Our model incorporates three distinct processes: *pyramiding* (the use collateralized securities to collateralize further securities), *pooling* (the combining of bundling of collateral goods and securities to collateralize different loans) and *tranching* (the using collateral goods and securities to collateralize several securities). Section 5 shows how these processes operate when used together (in our environment) but a brief informal discussion may guide the reader.

- To see how pyramiding could be useful, imagine an economy with one consumption good and three states at date 1. Suppose there is a durable good (houses today) yields consumption in quantities (2,1,1) in the three states. Agent 0 has utility for date 0 housing, agent 1 only wants to consume in state 1, and agent 2 (who is very risk averse) wants to smooth consumption perfectly in date 1. Suppose further that in the initial condition of society, only riskless promises (i.e., promises of the form (a, a, a) can be written. If agent 0 owns the house and sells off a promise of (1,1,1) to agent 2, then agent 0 gets stuck consuming 1 in state 1 tomorrow. On the other hand, if agent 1 owns the house and sells of the promise (1,1,1) to agent 2, then the right agent gets consumption of 1 in state 1 tomorrow, but the house is in the wrong hands. With pyramiding, agent 0 could own the house and sell off promise (2,2,2) to agent 1. Agent 1 could use that promise – which delivers (2,1,1) – as collateral for a futher promise of (1,1,1) to agent 2. This achieves the efficient allocation of getting 0 to live in the house, agent 1 to consume 1 in state 1, and agent 2 to consume (1,1,1) in the three states tomorrow. (We might think of agent h=0 as a homeowner, agent 1 as a speculator, and agent 2 as the risk averse lender.) We see that pyramiding, combined with default, allows for a socially superior allocation.

- **why is pooling useful if everything is divisible?** To see how pooling is useful, imagine a variant of the previous example in which there are two houses and two potential homeowners 0 and 0'. Suppose the first house pays (1,1,0) and the second house pays (1,0,1) in the three states. The optimal allocation is achieved when 0 buys the first house and using it as collateral sells the promise (1,1,1), thereby delivering (1,1,0). Agent 0' buys the second house and using it as collateral sells the promise (1,1,1), delivering (1,0,1). Agent 1 buys both promises, pooling them together as collateral to back the promise (1,1,1), which delivers fully and is sold to agent 2, leaving agent 1 with the residual payoff of (1,0,0). Pooling the promises allowed for the diversification that made the pool able to fully cover the (1,1,1) promise. Note that the houses could not directly be pooled together, because they need to be owned by separate homeowners. This example illustrates the power of say subprime mortgage pools to enable homeowners to borrow the money to buy houses to live in, while dividing the mortgage cash flows between speculators and risk averse agents. In states 2 and 3 one homeowner defaults, but at the pool level the promise is kept.[4]

[4]An example in keeping with how events unfolded over the past two years, as opposed to how they were meant to unfold in theory, would involve a fourth state in which many both house payoffs are 0, forcing two defaults at the homeowner level as well as default at the pool level.

- Tranching allows the same collateral is used to back more than one loan or tranche. With more than one loan depending on the same collateral, a seniority is required to define the payoffs. Consider the first example in which the homeowner buys the house and using it as collateral issues a senior promise (first mortgage) promising (1,1,1) and a junior tranche (second mortgage) also promising (1,1,1). The senior tranche will fully deliver (1,1,1) and be bought by agent 2, and the junior tranche will deliver (1,0,0) and be bought by agent 1.

5 Securitization and Efficiency

We argue here that, in a world in which all securities must be collateralized, securitization promotes efficiency but that there are robust situations in which efficiency cannot be obtained. To make these points we begin with a simple observation.

Theorem 2 (Net Savers) *If $\langle p, q, Q, (x^i, \varphi^i, \psi^i, \Phi^i, \Psi^i\rangle$ is a pool equilibrium for the economy $\langle (e^i, u^i), \mathcal{A}, \mathcal{B}\rangle$ then each consumer's future expenditures must exceed his/her unpledgeable income in every future state; that is,*

$$p_s \cdot x^i_s \geq p_s \cdot e^i_s$$

for each consumer i and state s.

Proof If $p_s \cdot x^i_s < p_s \cdot e^i_s$ for some consumer i and state s, then in state s, consumer i could default on all the promises of the securities s/he sold at date 0, surrender the collateral backing these promises, and still afford *more* than x^i_s. This would contradict the requirement that i's equilibrium plan be optimal in i's budget set. Hence $p_s \cdot x^i_s \geq p_s \cdot e^i_s$, as asserted. ■

This simple theorem has a striking negative consequence for efficiency: provided we rule out avoid corner solutions, inefficiency is a robust phenomenon – independently of consumer preferences and the availability of securities and security pools.

Theorem 3 (Robust Inefficiency) *Fix a positive social endowment $e \geq 0$ and smooth utility functions (u^i) that are strictly monotone and satisfy the boundary condition.*[5] *There is an non-empty open subset Ω of the set of endowment profiles $\{(e^i) : \sum e^i = e\}$ with the property that* no *collateral equilibrium from any endowment profile in Ω can be Pareto optimal, no matter what securities and security pools are available for trade.*

[5] That is, indifference curves through any point in the strictly positive orthant lie entirely in the strictly positive orthant; Debreu (1972).

Proof To be added. ■

On the other hand, any allocation that can be supported as a Walrasian equilibrium and that Theorem 2 does not rule out as occurring in a collateral equilibrium can in fact be obtained whenever "enough" securities and security pools are available. Moreover we need only a particularly simple set of collateralized securities and a single properly constructed pool.

For each good 0ℓ define the security

$$A^\ell = (p_1 \cdot F(\bar{e}_{0\ell}), \bar{e}_{0\ell})$$

A^ℓ promises to pay the date 1 value of what a society's endowment of 0ℓ becomes at date 1, collateralized by society's endowment of 0ℓ itself. Note that there will never be any default on A^ℓ: deliveries will always equal promises. We refer to the securities A^ℓ as the *primary loans.*

Now define a pool $\mathbf{B} = (B^1, \ldots, B^S; \chi)$ follows:

- $\chi_0 = 0$, $\chi_1 = \{A^1, \ldots, A^\ell\}$
- $$B^t(p_t) = \begin{cases} \sum_{\ell=1}^L A_s^\ell(p_s) & \text{if} \quad t = s \\ 0 & \text{if} \quad t \neq s \end{cases}$$

That is, the t-tranche of $\mathbf{B}$ promises to pay the total of all promises on primary loans in state $s = t$ and nothing in any other state. For obvious reasons we refer to $\mathbf{B}$ as the *pool of primary loans with Arrow tranches.*

Theorem 4 (Supporting Walrasian Equilibrium) *If $\langle \tilde{p}, (x^i) \rangle$ is a Walrasian equilibrium for the economy $\langle (e^i, u^i) \rangle$ at which each consumer is a net saver in the sense that*

$$\tilde{p}_s \cdot (x_s^i \geq \tilde{p}_s \cdot e_s^i)$$

for each consumer i and state s and $\mathcal{A}$ is any family of collateralized securities that contains all the primary loans $\mathcal{B}$ is any family of security pools that contains the pool of primary loans with Arrow tranches, then there is an equilibrium $\langle p, q, Q, (x^i, \varphi^i, \psi^i, \Phi^i, \Psi^i)\rangle$ for the economy $\langle (e^i, u^i), \mathcal{A}, , \mathcal{B}\rangle$ with the same consumptions (and the same commodity prices) as the given Walrasian equilibrium.

Proof Suppose each consumer is a net saver. For $\ell = 1, \ldots, L$ let $\mathbf{B}^\ell = (B^{\ell 1}, \ldots, B^{\ell S}; \delta_{0\ell})$ be the security pool which is collateralized by one unit of the commodity 0ℓ and for which the s tranche $B^{\ell s}$ promises to deliver in state s the value of what one unit of commodity 0ℓ becomes in state s and promises to deliver nothing in states $\sigma \neq s$. That is:

$$B^{\ell s}_\sigma = \begin{cases} p_s \cdot F_s(\delta_{0\ell}) & \text{if} \quad \sigma = s \\ 0 & \text{if} \quad \sigma \neq s \end{cases}$$

Define prices for commodities and tranches as follows:

$$\begin{aligned} p_{0\ell} &= \tilde{p}_{0\ell} + \sum_s \tilde{p}_s \cdot F_s(\delta_{0\ell}) \\ p_{s\ell} &= \tilde{p}_{s\ell} \\ Q^{\ell s} &= p_s \cdot F_s(\delta_{0\ell}) \end{aligned} \tag{1}$$

For each consumer i and each state s define

$$r^i_s = p_s \cdot [x^i_s - F_i(x^i_0)] - p_s \cdot [e^i_s - F_s(e^i_0)]$$

Note that this quantity could be positive, negative or zero. For each consumer i define the portfolios Φ^i, Ψ^i of purchases and sales of tranches as follows:

$$\begin{aligned} \varphi^i &= x^i_0 \\ \psi^i &= x^i_0 \\ \Phi^{i\ell s} &= \frac{x^1_0}{p_s \cdot F_s(x^1_0} (r^1_s)^+ \\ \Psi^{i\ell s} &= \frac{x^1_0}{p_s \cdot F_s(x^1_0} (-r^1_s)^+ \end{aligned} \tag{2}$$

We claim $\langle p, , Q, (x^i, \Phi^i, \Psi^i)\rangle$ is an equilibrium for the economy $\mathcal{E} = \langle (e^i, u^i), (\mathbf{F}^\ell)\rangle$.

To see this note first that deliveries on tranches coincide with promises: this follows immediately from the definitions. Moreover, for each consumer i the plan (x^i, Φ^i, Ψ^i) is in consumer i's collateral budget set $B(p, Q; e^i)$: this follows immediately by substituting the definitions of prices (1) and of portfolios (2) into the Walrasian budget constraints.[6] We assert that, for each i, all consumption plans that that can be financed by purchases and sales of security pools are in the Walrasian budget set $B^W(\tilde{p}; e^i)$. To see this, suppose $(\hat{x}^i, \hat{\varphi}^i, \hat{\Phi}^i, \hat{\Psi}^i)$ is in consumer i's budget set $B(p, Q; e^i)$. The date 0 and state s budget constraints are

$$
\begin{aligned}
p_0 \cdot \hat{x}_0 + Q \cdot \hat{\Phi} &\leq p_0 \cdot e_0 + Q \cdot \hat{\Psi} \\
p_s \cdot \hat{x}_s + \textsc{Del}(\hat{\Phi}; s, p) &\leq p_s \cdot e_s + p_s \cdot F_s(\hat{x}_0) \\
&\quad + \ \textsc{Del}(\hat{\Psi}; s, p)
\end{aligned}
$$

Substituting the definitions of spot prices and security deliveries, summing and doing some algebra yields

$$
\tilde{p}_0 \cdot \hat{x}_0 + \sum \tilde{p}_s \cdot \hat{x}_s \leq \tilde{p}_0 \cdot e_0 + \sum \tilde{p}_s \cdot e_s
$$

which is the Walrasian budget constraint. ■

A simple example illustrates Theorem 4 and the reason why pools are required.

Example 1 (Pools and Walrasian Equilibrium) There are two states of nature, two goods (Food and Housing), and four consumers. Each consumer assigns equal probability to the two states in date 1. Consumer 1 owns the housing and is risk neutral; Consumer 2 likes housing much more than other consumers; Consumers 3, 4 care only about food and have

[6] We do *not* assert that *every* consumption plan in the Walrasian budget sets can be financed by appropriate portfolios of security purchases and sales – and in general, this is not so – but only that these *particular* consumption plans can be so financed.

an insurance motive. We take the supply of housing $h \in [0, 4]$ as a parameter. Endowments and utilities are:

$$
\begin{aligned}
e^1 &= (8, h; 32, 0; 32, 0) \\
u^1 &= x_{0F} + x_{0H} + \frac{1}{2}\left[x_{1F} + x_{1H}\right] + \frac{1}{2}\left[x_{2F} + x_{2H}\right] \\
e^2 &= (9, 0; 72, 0; 72, 0) \\
u^2 &= \log(x_{0F}) + 4x_{0H} + \frac{1}{2}\left[x_{1F} + 4x_{1H}\right] + \frac{1}{2}\left[x_{2F} + 4x_{2H}\right] \\
e^3 &= (12, 0; 8, 0; 0, 0) \\
u^3 &= \log(x_{0F}) + \frac{1}{2}\log(x_{1F}) + \frac{1}{2}\log(x_{2F}) \\
e^4 &= (12, 0; 0, 0; 8, 0) \\
u^4 &= \log(x_{0F}) + \frac{1}{2}\log(x_{1F}) + \frac{1}{2}\log(x_{2F})
\end{aligned}
$$

A simple computation shows that Walrasian prices and utilities are unique but equilibrium allocations are indeterminate:

$$
\begin{aligned}
\tilde{p} &= (1, 8; 1/2, 2; 1/2, 2)) \\
x^1 &= (24, 0; 24 + 8h + \zeta, 0; 24 + 8h - \zeta, 0) \\
x^2 &= (1, h; 72 - 8h - \zeta, h; 72 - 8h + \zeta, h) \\
x^3 &= (8, 0; 8, 0; 8, 0) \\
x^4 &= (8, 0; 8, 0; 8, 0)
\end{aligned}
$$

for $\zeta \in [-\min(24 + 8h, 72 - 8h), \min(24 + 8h, 72 - 8h)]$.

For which values of h, ζ can this equilibrium be supported as a pool equilibrium for an appropriate choice of securities and security pools? In view of Theorems 2 and 4, it is necessary and sufficient that each consumer be a net saver. For Consumers 3, 4 this requirement is satisfied independently of h, ζ; for Consumers 1, 2 this requirement entails the inequalities:

$$
\begin{aligned}
24 + 8h + \zeta &\geq 32 \\
24 + 8h - \zeta &\geq 32 \\
(1/2)(72 - 8h - \zeta) + 2h &\geq 32 \\
(1/2)(72 - 8h + \zeta) + 2h &\geq 32
\end{aligned}
$$

which simplify to

$$
\begin{aligned}
8h + \zeta &\geq 8 \\
8h - \zeta &\geq 8 \\
8 &\geq 4h - \zeta \\
8 &\geq 4h + \zeta
\end{aligned}
$$

This region in h, ζ space is shown as the shaded rhombus in Figure 1. Notice in particular that if $h \notin [1, 2]$ then *no* Walrasian equilibrium can be supported as a pool equilibrium no matter what securities and security pools are available.

Figure 1 about here

To understand why these inequalities characterize supportability as a pool equilibrium, let us focus on the case $\zeta = 0$. Note first that for $h \in [1, 2]$ and $\zeta = 0$ the pool equilibrium is easy to describe: Consumer 2 borrows to buys all the housing, using the housing to collateralize the loan; Consumer 1 uses the housing loans to collateralize a security pool with two tranches, each promising to deliver the value of 8 units of food in each state; Consumers 3 and 4 each buy one of these tranches.

Now consider what happens if $h \notin [1, 2]$. If $h < 1$ then Consumer 2 is rich enough at date 0 to pay for all the housing from her endowment and still have more than 1 unit of food, so she will wish to save. For Consumer 2 to save some other Consumer must lend. Because lending must be collateralized by housing, some other Consumer must hold housing – which no other Consumer wishes to do. If $h > 2$ Consumer 2 will need to borrow $8h - 8$ units of account to buy housing in date 0. Because borrowing must be collateralized by housing and the date 1 value of housing is 4, no one will lend more than 4 units of account per unit of housing collateral, so Consumer 2 will not be able to borrow more than $4h$ units of account; this is not enough. In short, if $h < 1$ then Consumer 2 is too rich and if $h > 2$ then Consumer 2 is too poor.

Finally, to see why collateralized securities alone are not sufficient – and pools are necessary – note that Consumers 3, 4 must buy securities that pay in different states; hence some other agent(s) must sell these securities. Because selling securities must be collateralized, *each* of these loans must be collateralized by at least two *distinct* houses. Security pools " solve" this problem by making it possible for the same houses to collateralize both sales. $\diamondsuit$

6 Conclusion

When securities must be collateralized, the shortage of collateral leads to financial innovations that stretch the available collateral. These financial innovations (pooling, pyramiding and tranching) promote social welfare. However, even after all possible financial innovations, the requirement that lending be collateralized means that robust inefficiency is an inescapable possibility.

The model offered here abstracts away from transaction costs, informational asymmetries, and many other frictions that play an important role in real markets. It also restricts attention to a two-date world, and so does not address issues such as default at intermediate dates. All these are important questions for later work.

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