ON THE EFFICIENCY OF MONETARY EQUILIBRIUM WHEN AGENTS ARE WARY*

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October 4, 2014

*We gratefully acknowledge the financial support from CAPES, CNPq and FAPERJ (Brazil), FCT/FEDER and NOVA FORUM (Portugal). Gama-Torres benefited from a doctoral grant by FAPERJ (Brazil). Pascoa and Gama-Torres acknowledge support from project EGE-ECO/114496/2009 (FCT, Portugal). We are grateful to comments by seminar audiences at UCLA, the University of Chicago Money Workshop, the Harvard-MIT Theory seminar, 2nd BWGT at USP and SAET 2011.

Abstract

Wary agents tend to neglect gains at distant dates but not the loses that occur at those far away dates. Contrary to what happens under pure discounted utility, efficient allocations among wary agents are implemented with a non-vanishing money supply. The hedging rule of money does not disappear over time and the transversality condition allows for consumers to have limiting long positions. The implementation scheme starts by allocating money and then, at subsequent dates, taxes money balances that deviate from the efficient path. We address also why fiat money does not lose its value when Lucas trees are available and why we might not want to replace money by a tree.

1 Introduction

This paper reexamines some core questions in monetary economics in the light of a reformulation of the way infinite lived agents may discount the future. We depart from the classical impatience assumption and assume instead that consumers are *wary*, that is, they tend to ignore distant gains and are not willing to overlook loses that occur at far away dates. To do this, we add to the usual series of discounted utilities a new term which is the infimum of lifetime utilities. Wariness is inspired by recent developments in decision theory, pioneered by the work of Gilboa and Schmeidler (1989) and Schmeidler (1989). Analogously to the literature on ambiguity aversion, a wary consumer can be thought of as being unsure how to discount the future, so that he ends up picking for each consumption plan the most penalizing discount process and, therefore, ends up maximizing the minimal utility, over some set of discount factors. This characterization of wariness brings it close to the macro literature on robust decision-making (see Hansen and Sargent (2001) and Hansen and Sargent (2008)).

When wary consumers face persistent fluctuations in endowments, saving plans can have a limiting hedging effect, of raising a distant infimum in consumption. Efficient allocations can be implemented sequentially by trading assets that provide this hedging role. One such asset is fiat money. Our *first theorem* shows that, contrary to what happens under pure discounting, monetary equilibria become efficient when consumers are wary. Actually, under these new preferences, the optimal quantity of money should not fall to zero and does not need to contract at all, as established in our *second theorem*. While we still follow Friedman's rule as far as deflation being efficient, we depart from the claim that money supply should contract at a steady rate.

The sequential implementation of efficient allocations requires combining money with a fiscal policy that discourages inefficient savings. Equilibrium money balances are not taxed but other plans might be taxed. Impersonal non-lump-sum taxes raise the opportunity cost of carrying on cash for plans whose opportunity cost was lower than the hedging benefit at infinity. These taxes make the individual optimization problem become solvable but, at the same time, eliminate plans that would lead to excessive savings, from a collective point of view.

We are not claiming that money plays an irreplaceable hedging role[1]. A Lucas tree could also play the same hedging role at infinity but the sequential market completeness might not be attained so easily. In fact, short positions in money can be easily avoided by raising the initial holdings high enough, whereas in the case of a Lucas tree such increase in initial holdings would be incompatible with the given Arrow-Debreu endowments (which must be equal to the sequential endowments plus the returns from the initial holdings of the tree). It is true that short positions in long lived assets can be avoided in an alternative way, by adding other financial instruments that can take negative positions. Our *third theorem* states that we could have implemented using Lucas trees but with the help of zero-net-supply promises. Moreover, such promises should not be secured by the Lucas trees, otherwise the markets might become incomplete due to the friction created by the collateral constraint[2]. The drawback of relying on unsecured credit is that full commitment of debtors would have to be assumed, which might clash with incentive compatibility.

We address also a related, less demanding, classical monetary theme. Our *fourth theorem* establishes that, in a stochastic environment, coexistence of money and

[1]We were asked this question by Nancy Stokey at a presentation at the University of Chicago in 2012.

[2]See, for example, Gottardi and Kubler (2012) on this issue and on weaker notions of efficiency, that depart from the full efficiency we are interested in.

other long-lived assets, paying dividends, does not make money lose its positive price or its efficient role. Money actually widens the hedging that the other long-lived assets can do with non-negative positions. In fact, non-negativity of positive-net-supply long-lived assets is ensured at no cost since the initial holdings of money can always be increased so that all long-lived assets can complete the markets without any short sales.

Our results illustrate how a new approach to the preferences of infinite-lived consumers yields quite different answers to long-standing monetary themes. The idea that money plays a crucial reserve role has captured a lot of attention in the literature. Friedman (1953, 1969) put forward the idea that consumers should not economize unnecessarily on money balances as these holdings are "a reserve against future emergencies". The wasteful economizing of cash should be avoided by deflation or by providing money with an explicit real rate of interest. This proposition has been often associated with the much stronger recommendation of a steady contraction of the money supply. However, the latter seemed to reduce the full impact of the former. An asymptotically null money supply would imply that money could not have a persistent efficient role.

In this paper we reconcile Friedman's appraisal of the hedging role of money with an optimal non-zero limit for the money supply. We do this by resuming Bewley's (1980) idea that the "devise to give money a value is infinite horizon (together with the need for insurance)", but we take a step forward. Our devise is not simply the use of infinite lived consumers, but consists in taking into account specifically the hedging role of money at infinity, that is, at arbitrarily distant dates. For wary consumers, when the worst outcome is never attained in finite time (there is always a worse outcome sometime ahead), there is a marginal utility at infinity, of raising the infimum of consumption. Under such a precautionary role of money at infinity, it is not surprising why our efficient monetary equilibrium requires a positive limit for the money supply. Our result is reminiscent of the persistent role of money in Samuelson (1958) overlapping generations model, where the young hoard money to sell when old, allowing for an inter-temporal transfer of wealth that could not be done by trading a zero net supply promise (as the old would no longer be alive afterwards to pay back the debt). Such persistent role of money seemed until now impossible to observe in models with immortal agents. For impatient agents, Bewley (1980, 1983) showed that a non-vanishing money supply, together with interior consumption, had to be inefficient. Levine (1986, 1988, 1989) confirmed this under Inada's condition and observed that efficiency might prevail under non-interior consumption (see also Woodford (1990), Kehoe, Levine and Woodford (1992), Pascoa, Petrassi and Torres-Martinez (2010) and Wallace (2014)).

Deflation, while being necessary for Pareto optimality, also creates a difficulty for the implementation of efficient allocations: wary agents may get unbounded utility gains by hoarding too much and then taking advantage of deflation to raise a distant infimum of consumption. This problem is a new instance of a difficulty already noticed by Friedman and Bewley, known as the *insatiable demand for precautionary liquidity.* In our context the benefits-costs gap is not a short run gap but rather a (arbitrarily-)long run gap.

When the limit of the cost of carrying on cash (measured by the marginal utilities of forsaken consumption) is lower than the hedging gain from raising a distant infimum of consumption, there is a long-run arbitrage opportunity, just like for a non-wary, impatient, consumer a Ponzi scheme would constitute a long-run arbitrage. Under impatience, a positive limit in a consumer's deflated real balances would be a waste, whereas a negative limit would constitute a Ponzi scheme but was always ruled out as money cannot be short sold. Under wariness, a positive limit for the deflated real value of hoarded funds is not a waste and an improvement strategy, akin to a new type of infinite horizon arbitrage opportunity, becomes available when that limit is lower than the hedging benefit at infinity.

The paper is related to our earlier work (Araujo, Novinski and Pascoa (2011)) on sequential implementation of Arrow-Debreu (AD) allocations using long-lived assets paying dividends. However, differently from what we did before, instead of imposing portfolio constraints that prevent excessive savings, we have just a plain no-short-sales constraint, but look for taxes that discourage excessive savings. We believe this approach is quite novel and illustrates well what can be done differently when the implementing asset is money.

Finally, it should be pointed out that time consistency is compatible with wariness. When the series of discounted utilities describes a time-consistent behavior (say, under exponential discounting), then adding a term dealing with the infimum of the utilities could introduce an inconsistency but it does not in equilibrium as long as the infimum is not attained in finite time, which is precisely the case we are interested in. More precisely, we focus on allocations such that (a) *at any moment in time there is always some later date where consumption will be lower* (which is much weaker than requiring consumption plans or endowments to be decreasing sequences) and (b) AD *net trades are not converging sequences*[3] Roughly speaking, endowments oscillate around some trend that always has a worse outcome later on. As illustrated in our examples, fiat money manages to hedge such persistent fluctuations by implementing efficient consumption allocations that are inside the endowment span.

The rest of the paper is organized as follows. Section 2 describes the model and Section 3 introduces the leading example. Section 4 presents the result on efficient monetary equilibrium. Section 5 proves this result and Section 6 addresses implementation when Lucas trees are also available.

2 Sequential Economy with Fiat Money and Wariness: the Model and Preliminary Results

In this section we describe a deterministic economy with infinitely many dates and a single asset, fiat money, which is used to transfer wealth across dates. Government provides money endowments to the consumers at the initial date and then their money holdings may be taxed at subsequent dates.

2.1 Fiat Money and the Budget Constraints

There is a finite set of infinite lived agents $\mathcal{I} = \{1, ..., I\}$. Their set of consumption plans is[4] ℓ^∞_+. We denote by $\omega^i = (\omega^i_t)_{t\in\mathbb{N}} \in \ell^\infty_+$ the commodity endowments of agent i and suppose that his preferences are representable by a utility function $U^i : \ell^\infty_+ \to I\!R$ which will be specified in subsection (2.2).

[3] The earlier work (Araujo, Novinski and Pascoa (2011)) had already shown that in the case of converging net trades, fiat money could not have a positive price.

[4] See some properties of the space ℓ^∞ in the Appendix A.

Given some initial holdings of money, y_0^i, the purchase of a consumption plan x can be done by allocating consumer's wealth across time through the choice of a sequence $(y_t)_t$ of *non-negative holdings of fiat money* in order to satisfy the following sequential budget constraints, expressed in units of the consumption good (that is, the single good is the *numéraire* at each date):

$$x_t - \omega_t^i \leq q_t \left[y_{t-1} - y_t - \tau_t(y) \right] \quad \forall\, t \in \mathbb{N}, \tag{1}$$

where $q = (q_t)_{t\in\mathbb{N}}$ is the sequence of money prices and $(\tau_t(y))_{t\in\mathbb{N}}$ is a taxation profile that depends *only* on the sequential money holdings $y = (y_t)_{t\in\mathbb{N}}$. More precisely, the fiscal policy τ is a function that maps, in an impersonal way, each y into a sequence of time-indexed taxes $\tau(y) = (\tau_t(y))_{t\in\mathbb{N}}$. When τ is constant over all possible choices y, τ is said to be a *lump-sum taxation profile.* Observe also that $\tau_t(y)$ just has an impact at date t when $q_t > 0$.

We suppose that the tax $\tau_t(y)$, even when non-lump-sum, never depends on values that the plan of money holdings takes at any finite set of dates. That is, different money balances trajectories may be taxed differently only if they differ on some infinite subset of dates. It is only the asymptotic implications of different savings strategies that makes them have different fiscal treatment.

Let us denote by $B(q, y_0^i, \omega^i, \tau)$ the set couples $(x, y) \in \ell_+^\infty \times I\!R_+^\infty$ of consumption and money holdings plans satisfying the sequential budget constraints (1). The goal of agent i is to maximize U^i under $B(q, y_0^i, \omega^i, \tau)$.

For a given fiscal policy τ, the *money supply* is endogenous in the sense that the supply M_t at date $t \in \mathbb{N}$ is equal to M_{t-1} net of the date t taxes which depend on what consumers' money balances $(y^1, ..., y^I)$ are. The initial money supply M_0 is given and equal to $\sum_{i=1}^{I} y_0^i$. Then, at each date $t \geq 1$

$$M_t(y^1, ..., y^I) = M_{t-1}(y^1, ..., y^I) - \sum_{i=1}^{I} \tau_t(y^i),$$

This condition can be interpreted as the government sequential budget constraint. Equivalently

$$M_t(y^1, ..., y^I) = \sum_{i=1}^{I} \left(y_0^i - \sum_{s=1}^{t} \tau_s(y^i) \right). \tag{2}$$

Definition: A vector $(q,(x^i,y^i)_{i\in\mathcal{I}}) \in I\!R^\infty_+ \times (\ell^\infty_+ \times I\!R^\infty_+)^I$ is said to be an *equilibrium* for the economy with initial fiat money holdings $(y^1_0, ..., y^I_0)$ and a fiscal policy τ when:

(a) $(x^i, y^i) \in \operatorname{argmax}\{U^i(x) : (x,y) \in B(q, y^i_0, \omega^i, \tau)\}$;

(b) $\sum_{i=1}^I x^i = \sum_{i=1}^I \omega^i$;

(c) $M_t(y^1, ..., y^I) = \sum_{i=1}^I y^i_t \;\; \forall\, t \in \mathbb{N}$.

Definition: An equilibrium $(q,(x^i,y^i))$ is a *monetary equilibrium* if $q \neq 0$.

If $q_{t_0} > 0$ for some date t_0, it will be true by non-arbitrage that $q_t > 0 \;\; \forall\, t$,. It is easy to see that for U^i monotonous[5] $\forall\, i$, summing over i the equalities in 1, we get that (b) implies (c) (this is a version of Walras law).

We introduce next the class of preferences that will be assumed in our main results and examples. It is rich enough to accommodate standard impatience preferences as well as patience driven by wariness. Efficient monetary policies turn out to be quite different depending on whether impatience holds or not, as money supply must be entirely withdrawn under impatience but can be persistently positive otherwise.

2.2 Consumer Preferences: Wariness and Ambiguity on Discount Factors

We assume that each agent $i \in \mathcal{I}$ has a utility function U^i of the form

$$U^i(x) = \sum_{t=1}^{\infty} \zeta^i_t u^i(x_t) + \beta^i \inf_{t\geq 1}, u^i(x_t) \tag{3}$$

with $\zeta^i \in \ell^1_{++}$, $\beta^i \geq 0$ and $u^i : \mathbb{R}_+ \to \mathbb{R}$ increasing, concave and continuously differentiable.

When $\beta^i = 0$, U^i is a standard time-separable utility and the agent i is impatient. More precisely, take any plan x. Let us denote by $1\!\!1^n$ the sequence which is null up to component n and equal to 1 otherwise. The agent is upper semi-impatient at x if for each $\tilde{x}$ such that $x \succ \tilde{x}$ we have $x \succ \tilde{x} + k1\!\!1^n$ for $k > 0$ and n large enough; the agent is lower semi-impatient at x if for each $\overline{x}$ such that $\overline{x} \succ x$ we

[5] That is, if $h > 0$ and e_t is the sequence whose s-th coordinate is equal to 1 if $s = t$ and equal to zero otherwise, then $U^i(x + he_t) > U^i(x)$ for all $x \in \ell^\infty_+$.

have $\tilde{x} - k\mathbb{1}^n \succ x$ with $k > 0$ for which $\tilde{x} - k\mathbb{1}^n \in \ell^\infty_+$ and n large enough. It is clear that both upper and lower semi-impatience hold at any plan when $\beta^i = 0$. Since preferences are described by (3) in the deterministic case, by impatience we mean henceforth that $\beta^i = 0$.

However, when $\beta^i > 0$, the agent is upper but not lower semi-impatient, that is, he tends to overlook gains but not losses at far away dates.[6] In fact, take $x = \mathbb{1}$, the sequence whose terms are all equal to one and $\overline{x} = (1+\varepsilon)\mathbb{1}$, with $\varepsilon > 0$. So $\overline{x} \succ x$ but $x \succ \overline{x} - (1/2)\,\mathbb{1}^n$ for all n large enough and ε small enough, as $\inf\left(\overline{x} - (1/2)\,\mathbb{1}^n\right) = (1/2) + \varepsilon$ whereas $\inf x = 1$.

If $\beta^i > 0$, we say that the agent i is *wary*[7]. This utility function has a nice interpretation in terms of *ambiguity aversion* in the way of discounting the future. Not being sure how to do this, consumers use the worst discounting factor within all that have a certain lower bound at each date (see Dow and Werlang (1992)). In fact, the utility given by (3) can be written as

$$\inf_{(\eta_t)\,\in\mathcal{D}} \sum_{t=1}^{\infty} \eta_t u\left(x_t\right), \tag{4}$$

where $\mathcal{D}$ is the set of all real sequences (η_t) such that $\sum_t \eta_t = 1 + \beta$ and $\eta_t \geq \zeta_t$ $\forall t$. This utility is a particular case of the Maxmin Expected Utility model by Schmeidler (1989). The use of this maxmin approach in deterministic dynamic settings had already been suggested also by as a way to represent preferences that are averse to fluctuations in consumption by Gilboa (1989).

Let us see how do supporting prices (supergradients[8]) look like for such preferences. This result builds on Bewley (1972), which already allowed for AD prices outside of ℓ^1, but goes beyond by finding a condition ensuring that AD prices cannot be in ℓ^1 and by characterizing these prices. The crucial condition is that the infimum of the consumption plan is the limit of some subsequence of consumption and that it is actually never attained in finite time. In this case, the supporting

[6] Upper but not lower impatience had been studied already by Brown and Lewis (1981) and Araujo (1985).

[7] More generally, a consumer is wary when the preferences are upper but not lower semi-continuous for the Mackey topology $\tau(\ell^\infty, \ell^1)$, as it is the case if $\beta^i > 0$ (see Araujo, Novinski and Pascoa (2011)).

[8] Supergradients are the generalization of $\nabla U(x)$ for concave functions, more precisely, $f \in (\ell^\infty)^*$ is a supergradient of U at x if $U(y) \leq U(x) + \langle f, y - x\rangle$ $\forall y$.

price must have a pure charge component. Let $\underline{x} = \inf_t x_t$.

Lemma 1: *Let $x \in \ell^\infty_+$ so that $\underline{x} > 0$ is a cluster point never attained of x, then any Arrow-Debreu supporting price π for x takes the following value at any $c \in \ell^\infty$*

$$\pi c = \sum_{t \geq 1} \zeta_t \, (u^i)'(x_t) \, c_t \; + \; \beta^i \, (u^i)'(\underline{x}) \, \mathrm{LIM}(c)$$

where LIM *is a bounded linear functional such that* $\mathrm{LIM}(c) \in [\liminf c, \limsup c]$ *and satisfies* $\mathrm{LIM}(x) = \underline{x}$.

(for a proof see Araujo, Novinski and Pascoa (2011), where the general characterization, when $\underline{x}$ may be attained, is also given; notice the multiplicity of supporting prices due to the freedom in choosing the *generalized limit* LIM; such multiplicity gives rise to a real indeterminacy of AD equilibria)

The functional mapping each $c \in \ell^\infty$ into $\beta^i \, (u^i)'(\underline{x}) \, \mathrm{LIM}(c)$ is the *pure charge* component of the supporting price.

Remark 1: Although preferences described by (3) may fail to be *time-consistent*, it is clear that when the infimum of consumption is not attained time-consistency holds. That is, precisely in the case that matters to us, where we manage to implement a monetary equilibrium, preferences are also time-consistent.

2.3 On Friedman's Rule

As a preliminary result, we show that an efficient monetary equilibrium complies with Friedman's rule requiring a zero nominal interest rate. To see this we derive the Euler conditions.

2.3.1 Euler Conditions

We focus on the case where the optimal consumption plan has an infimum which is never attained in finite time. In this case the utility function (3) has partial derivatives $\partial_t U^i(x)$, along the canonical directions e_t, given by $\zeta^i_t (u^i)'(x_t)$[9].

[9]On the contrary, if the infimum were instead attained at $\hat{t}$, then, for the direction $e_{\hat{t}}$, the left hand side derivative might exceed the right hand side derivative (as a reduction in $x_{\hat{t}}$ would lower $\inf_{t \geq 1}, u^i(x_t)$ whereas an increase in $x_{\hat{t}}$ might not affect it).

Lemma 2: *Let $x \gg 0$ be an optimal solution to the sequential problem with preferences described by (3). Assume* $\inf_t x$ *is not attained in finite time. Then*

$$q_t \zeta_t^i (u^i)'(x_t) \geq q_{t+1} \zeta_{t+1}^i (u^i)'(x_{t+1}) \tag{5}$$

The nature of the Euler conditions is independent of whether β^i is zero or positive. Notice also that for a consumer i who holds money at date t the Euler equation holds: $q_t \zeta_t^i (u^i)'(x_t) = q_{t+1} \zeta_{t+1}^i (u^i)'(x_{t+1})$.

Remark 2: Friedman's rule prescribing a zero nominal interest rate (sometimes known as the weak rule) follows from this lemma. To see this, notice that agents who hold money at date t would be willing to buy or sell a one-period bond at a null nominal interest rate, if such bond existed. But agents not holding money would be happy with a positive nominal interest rate. In fact, the nominal interest rate i would be such that $q_{t+1}(1+i)\partial_{t+1} U^i(x) = q_t \partial_t U^i(x)$ and the result follows, depending on whether the Euler equation or just (5) holds.

Now, suppose that for all i, U^i satisfies Inada condition (that is, for a sequence (x^m) in ℓ_+^∞ such that $x_t^m \to 0$ for some t and x_s^m is bounded away than zero for $s \neq t$, we have $\lim_m \partial_t^+ U^i(x^m) = \infty$). Then, *the nominal interest rate is zero* in any efficient monetary equilibrium $(q, (x^i, y^i))$ with $\inf x^i$ not attained, for any i[10].

To put it in an equivalent way, the inflation rate $(\frac{1/q_{t+1}}{1/q_t} - 1)$ should be equal to the consumers' rate of time preference $(\partial_{t+1} U^i(x)/\partial_t U^i(x))$ minus one. Hence, *efficiency requires deflation*, at least at infinitely many dates (as $(\partial_t U^i(x))_t \in \ell^1$). Actually, deflation occurs always beyond some date when consumption converges to some positive level.

We will examine next a strong rule, that money supply should tend to zero at an efficient monetary equilibrium when $\beta = 0$.

2.3.2 On the Money Supply when Agents are Impatient

We start by recalling what can be said about the efficient money supply when agents are impatient.

[10] In fact, as $\inf x^i$ not attained, $\partial_t U^i(x)$ exist, and, by Inada's condition, consumers' marginal rates of intertemporal substitution should be equal. Hence, as someone must be purchasing money, no one can have a shadow price for the no-short-sales constraint, that is, equalities hold in (5).

Proposition 1: *If for each agent $i \in \mathcal{I}$ we have $\beta^i = 0$ and Inada condition is satisfied, then a lump-sum fiscal policy $(\tau^i)_i$ induces an efficient monetary equilibrium $(q, (x^i, y^i))$, only if $M_t \to 0$.*

In fact, lump-sum taxes do not affect the necessary conditions for individual optimality and the result follows as in Proposition 5 in Pascoa, Petrassi and Torres-Martinez (2010).

Remark 3: Proposition 1 is a strong variation upon a claim made by Friedman (1969), although his claim actually just required a zero nominal interest rate and that, for that purpose, money supply should contract at a rate equal to the equilibrium real interest rate. Bewley's work (1980,1983) on impatient preferences not satisfying Inada had already shown that when consumption is always positive, a constant money supply is inefficient, whereas a money supply decreasing to zero at a constant rate can be made efficient when combined with lump-sum taxes. Levine (1986) gave interesting examples of efficient non-vanishing money supply for impatient agents with linear utilities, where corner solutions were crucial for building up large money balances[11]. For the preferences described by (3) with $\beta^i > 0$, agents have an incentive to keep large money balances for a long-run hedging which affect transversality conditions, and in this case, Inada conditions will not prevent the implementation of efficient monetary equilibrium with constant money supply. The following leading example illustrates it.

3 The leading example

We consider two-agent economies where endowments suffer shocks that alternate in sign along time. When one consumer gets a positive shock, the other suffers a negative one. Money can be used to hedge against these shocks. Consumers would like to hold money forever (or at least, along some subsequence) in order to find a consumption path in between the upper and the lower endowment subpath. That is, consumers would like to raise the infimum of consumption, but there is a trade-off due to the cost of carrying on cash (the forsaken consumption along the sequence).

[11] See also Levine's (1989) later results under differentiable preferences not satisfying Inada.

Take the utility function (3) with $u^i(.) = \sqrt{.}$ and $\beta = 6$. Take, for both agents, $\zeta_t = (1/2)^{t-1}\sqrt{1+1/t}$. Endowments are $\omega^i_t = 16\frac{t+1}{t} + G^i_t$, where G^1_t is given by $G^1_t = 13$ if t is even and $G^1_t = -11$ if t is odd, and $G^2_t = -G^1_t$. Recall that the indeterminacy in the generalized limit considered in the AD price leads to a real indeterminacy in AD equilibrium allocations. Take the equilibrium allocation that results from using a Banach limit[12] B. Consider the allocation $x^i_t = 16\frac{t+1}{t}$ and its supporting price, which (consistently with Lemma (1)) is of the form $\pi^i c = \sum_{t=1}^{\infty}(\frac{1}{2})^{t+2}c_t + \frac{3}{4}B(c)$. Let us normalize the price functional so that the coefficient of the Banach limit is one: $\pi = \frac{4}{3}\pi^i$. We denote by p the summable component of π, the deflator $p_t = \frac{4}{3}2^{-t-2}$.

Taking the AD Lagrange multipliers to be 3/4, the pair $((x^i)_i, \pi)$ constitutes an AD equilibrium, as AD budget equations hold since $\pi(G^1) = 0$ follows from $B(G^1) = 1$ and $p(G^1) = -1$

For $y^i_0 = 9$, make $q_t = \frac{3}{4}2^{t+2}$, the inverse of the deflator p_t. Let z_t be the funds put aside by a consumer at date t, which will be decomposed as a sum of his money balances and the cumulated taxes on his money balances: $z_t = y_t + \sum_{s\leq t}\tau^i_s(y)$.

Now, the implementation is achieved (as explained in detail in Section 5) with $(z^i)_i$ if we (i) make $\lim_t z^i_t = \limsup(x^i - \omega^i)$, that is, the limiting cost of carrying on cash equals the marginal gain of hedging at infinity, given by the highest possible value that any pure charge of a supporting price (not necessarily the above π^i), applied to the net trade, can take (see Lemma (1)) and (ii) require all other plans $\hat{z}$ to satisfy $\lim_t \hat{z}^i_t \geq \limsup(x(\hat{z}^i) - \omega^i)$ (a limiting cost of funds not below the marginal gain at infinity).

The latter can be achieved by designing taxes so that the inequality holds. More precisely, a money holdings plan y must end up paying cumulated taxes $\sum_{t=1}^{\infty}\tau^i_t(y) = \limsup(x(\hat{y} - \omega^i) - \lim_t y_y$, which implies (ii).

The former, together with the AD budget equation, determine what z^i_0 should be and imply $z^i_t = 12 + \sum_{s=1}^{t} p_s G^i_s$. Then, $\lim z^1_t = 11$ whereas $\lim z^2_t = 13$.

Now, take $\theta = 0$ and $y^i = z^i$ so that equilibrium cash balances are not taxed and money supply remains constant. But we could have taken instead $0 < \sum_{s\leq t}\theta_s < z^i_t$

[12] We say that a generalized limit B is a Banach limit if $B(c) = \lim_n \frac{1}{n}\sum_{t=1}^{n} c_t$ whenever this limit exists.

for $i = 1, 2$ and obtain $y_t^i = z_t^i - \sum_{s \le t} \theta_s < z_t^i$. For instance, let $\sum_{s \le t} \theta_s = 12 + \sum_{s=1}^t p_s \min\{G_s^1, G_s^2\}$, then $\sum_{t=1}^\infty \theta_t = 6.814$, $\lim y_t^1 = 4.186$, $\lim y_t^2 = 6.186$ and the limiting money supply is 10.372. In any case, real money balances $q_t y_t^i$ explode and deflated money balances $p_t q_t y_t^i = y_t^i$ tend to a positive constant.

As $\lim z_t^1$ is different from $\lim z_t^2$ we could not make $\sum_{t=1}^\infty \theta_t = \lim z_t^i$ for all i, so that money supply would tend to zero. Impersonal taxes are incompatible with a limiting zero money supply, except in the symmetric case where $\limsup(x^i - \omega^i)$ is the same for all agents, as will be claimed below.

Remark 4: This example can be modified to include the case in which the economy does not necessarily decrease at each date. In fact, the only condition that must be satisfied is that for any date t there exists $T > t$ such that the aggregate endowment at T is lower than in t. To see how the example could be modified, suppose that at even dates consumers' endowments follow increasing sequences and that at odd dates endowments are described as in the example. More precisely, for $t \ge 1$ we have $\omega_{2t-1}^i = 16\frac{t+1}{t} + G_{2t-1}^i$, where G_{2t-1}^1 is given by $G_{2t-1}^1 = 13$ if t is even and $G_{2t-1}^1 = -11$ if t is odd, and $G_{2t-1}^2 = -G_{2t-1}^1$. That is, along the odd dates subsequence endowments are oscillating around a decreasing trend. But the even dates subsequence can be chosen to be increasing or constant, say that for $t \ge 1$, we have $\omega_{2t}^i = 32 + G_1^i + a(t)$, where $a(t)$ is an increasing bounded sequence of positive numbers.

This variant of the example suggests that what is driving the bubble in money is a pattern of endowments showing some subsequence where aggregate endowments fall (this can be achieved through increasing aggregate oscillations, between odd and even dates) together with idiosyncratic shocks around that decreasing subsequence (the individual shocks G^i). The former ensures that in AD allocations the infimum of consumption is never attained, this guarantees that AD prices (and other supporting prices) have a pure charge. The latter ensures that AD net trades do not converge, otherwise money could not have a positive price (as already pointed out in (Araujo, Novinski and Pascoa (2011)), see also the discussion below, in subsection 5.1).

As a second remark, notice that the discount factors in the leading example are a product of exponential and hyperbolic discounting. Preferences fail to be time-consistent, not as a consequence of β^i being positive, but as a result of the somehow hyperbolic discounting that was already assumed for convenience reasons. In fact, the choice of the discount factor allowed us to pick equilibrium consumption plans that are equal to the endowment trend, common to both agents. The example could be redone with longer computations (along the lines of Example 1 in (Araujo, Novinski and Pascoa (2011))) under exponential discounting and consumption plans that differ from the endowment trend (but are in between the upper and the lower endowment subsequences).

4 Main Results

In this section we state our result on the sequential implementation of efficient allocations, using fiat money and taxes, and we also address why an efficient monetary equilibrium requires a positive limiting money supply, except for some degenerate equilibria (where consumers' net trades have the same highest cluster point).

We allow for taxes on money holdings but we will see that equilibrium portfolios might not need to be taxed and, in the non-degenerate case, the tax should not completely erode the cash balances. In the case of impatient agents, knowing that taxes will have to be paid later makes consumers hoard but there is no reason to carry on cash to infinity. However, when agents are wary, the incentive to hoard may be too strong. As deflation is necessary for an efficient outcome (as we saw in the previous section), the return from savings may become unbounded and a finite optimum may not exist. The no-short-sales constraint on money does not suffice to guarantee that a finite optimum exists. It is no longer the case that optimality can be achieved among portfolio plans with limiting non-negative deflated positions, as was the case under impatience.

As we will see below, when $\beta^i > 0$ and the infimum of the consumption plan x^i is never attained in finite time, consumers have a marginal benefit at infinity by raising $\inf x^i$. An improvement strategy, akin to a long-run arbitrage, becomes available to wary agents when marginal utility benefits at infinity outweigh the limiting cost of carrying on cash. We consider taxes that, while being impersonal, may be non-lump-sum, at least beyond some distant date, and eliminate such improvement opportunities.

4.1 On Long-Run Improvement Opportunities

To be more precise about what we mean by long-run improvement opportunities, given any consumer's plan y of money holdings, let $z_t = y_t + \sum_{s\le t}\tau_s(y)$ be the sum of funds put aside up to date t. Consider an AD equilibrium (x,π) and the portfolio plan z^i that makes (x^i,z^i) satisfy the sequential budget constraints (1) for consumer i given y_0^i and q. Take any other plan (X,Z) that verifies the sequential budget constraints (1) with equality at (y_0^i,q) and satisfies $X\in\ell_+^\infty$. What happens if we move from z^i in the direction of Z starting at some date n? Notice that, as far as the non-negativity constraint is concerned, this direction is admissible for positive changes (it is said to be $d(n)$ right-admissible): at any $t\ge n$ we raise z_t^i to $z_t^i+hZ_t$.

What is the resulting change in consumption, so that the sequential budget constraint remains satisfied? The direction of changes in consumption is $d(n)$ defined by $d(n)_t=0$ if $t<n$, $d(n)_n=-q_nZ_n$ and $d(n)_t=q_t(Z_{t-1}-Z_t)=X_t-\omega_t^i$ if $t>n$. By moving on the right along this direction, we hoard more at date n and at subsequent dates for which $\omega_t^i>X_t$, in order to increase consumption at subsequent dates where $\omega_t^i<X_t$. We will evaluate the marginal effect in order to see if there are improvement opportunities that should be ruled out.

More formally, denoting by $\delta^+U(x^i,d(n))$ the right hand side derivative[13] of U^i along the direction $d(n)$ evaluated at x^i, we want to rule out $\delta^+U(x^i,d(n))>0$.

Let us see how does the right hand side directional derivative $\delta^+U(x^i,d(n))$ look like. We can write it as the minimal value taken at $d(n)$ by the supporting prices (supergradients) of U^i at x^i, that is, $\delta^+U(x^i,d(n))=\min\{Td(n):T\in\partial U^i(x^i)\}$.

Then, by Lemma 1, for some generalized limit LIM, we have $\delta^+U(x^i,d(n)) = -\zeta_n(u^i)'(x_n)q_nZ_n+\sum_{t>n}\zeta_t(u^i)'(x_t)q_t(Z_{t-1}-Z_t)+\beta^i(u^i)'(\underline{x})\mathrm{LIM}(X-\omega^i)$. Now, efficiency requires Euler conditions to hold as equalities for every agent and every date (otherwise agents holding money might have marginal rates of substitution different from those of agents that might not hold money), so $\zeta_t(u^i)'(x_t^i)q_t$

[13] Given $x\in\ell_+^\infty$ and $v\in\ell^\infty$, $\lim_{h\downarrow 0}\frac{U(x+hv)-U(x)}{h}$ is called the *right-directional derivative* of U at x along (the direction) v and it is denoted by $\delta^+U(x;v)$. The *left-directional derivative* $\delta^-U(x;v)$ is defined analogously.

is constant, and it follows that $\sum_{t>n} \zeta_t (u^i)'(x_t) q_t (Z_{t-1} - Z_t) = \zeta_n (u^i)'(x_n) q_n Z_n - \lim \zeta_t (u^i)'(x_t) q_t Z_t$.

Hence, $\delta^+ U(x^i, d(n)) = -\lim_t \zeta_t (u^i)'(x_t) q_t Z_t + \beta^i (u^i)'(\underline{x}) \mathrm{LIM}(X - \omega^i)$. So, independently of what the generalized limit LIM might be, $\delta^+ U(x^i, d(n)) \leq 0$ if $\beta^i (u^i)'(\underline{x}^i) \limsup (X - \omega^i) \leq \lim \zeta_t^i (u^i)'(x_t^i) q_t Z_t$.

We would like to design a fiscal policy that guarantees this condition and, therefore, eliminates the above long-run improvement opportunities.

Remark 5: It is important to notice that the condition that we just derived is not too strong, in the sense that it does not collide with the *transversality condition*, which is a necessary condition for individual optimality. In fact, by Proposition 6 in Araujo, Novinski and Pascoa (2011), the latter says that there are two supergradients of U^i at x^i whose pure charges η and ν satisfy $\eta(x^i - \omega^i) \leq \lim \zeta_t^i (u^i)'(x_t^i) q_t z_t^i \leq \nu(x^i - \omega^i)$. Notice that the transversality condition allows for limiting long positions. Now, $\eta(.) = \beta^i (u^i)'(\underline{x}^i) \mathrm{LIM}^1(.)$ and $\nu(.) = \beta^i (u^i)'(\underline{x}^i) \mathrm{LIM}^2(.)$, where LIM^1 and LIM^2 are two generalized limits. Hence, the proposed sufficiency condition is compatible with the transversality condition if at the optimal plan (z^i, x^i) we have $\beta^i (u^i)'(\underline{x}^i) \limsup (x^i - \omega^i) = \lim \zeta_t^i (u^i)'(x_t^i) q_t z_t^i$. This will be the case in the monetary equilibrium that we will construct (see Section 5) and, therefore, the transversality condition actually requires positive limiting positions for consumers that have a positive cluster point for the AD net trade.

4.2 Taxes that eliminate the marginal benefit - marginal cost gap

Given a plan of money balances y and a sequence of taxes τ, let $x(y, \tau)_t = \omega_t^i + q_t (y_{t-1} - y_t - \tau_t)$ be the associated consumption plan satisfying the sequential budget equations. The tax $\tau_t(y)$ levied at date t upon a plan y of money holdings consists of a fixed, summable, component θ_t, which is the tax that may be imposed on the efficient plans y^i of all agents, and another component that eliminates the above long-run improvement opportunities. Now, the marginal benefit at infinity when taxes are $\tau(y)$ is $\beta^i (u^i)'(\underline{x}(i)) \limsup (x(y, \tau(y)) - \omega^i)$, which is less than or equal to the analogous marginal benefit when taxes are just θ. Then, we rule out such long-run improvement strategies if $\beta^i (u^i)'(\underline{x}(i)) \limsup (x(y, \theta) - \omega^i) \leq$

$\lim_t \zeta_t^i (u^i)'(x_t^i) q_t (y_t + \sum_{s \le t} \tau(y)_s)$.

To write this in an impersonal way, consider the Lagrange multiplier ρ^i of the AD budget constraint of consumer i, at AD prices π. Now, as seen in Lemma (1), $\rho^i \pi = (\zeta_t^i (u^i)'(x_t)_t)_t + \beta^i (u^i)'(\underline{x}(i))\mathrm{LIM}(.)$, when the infimum of consumption is not attained in finite time. Hence, the AD price can be written as $\pi = p + \alpha \mathrm{LIM}(.)$ such that $\rho^i p = (\zeta_t^i (u^i)'(x_t)_t)_t$ and $\rho^i \alpha = \beta^i (u^i)'(\underline{x}(i))$, for all i.

Now, we saw that $\zeta_t (u^i)'(x_t^i) q_t$ is constant and, therefore, $p_t q_t$ has to be constant and can be made equal to α. Moreover, the AD price can be normalized so that $\alpha = 1$ (with the Lagrange multipliers ρ^i chosen accordingly). The benefits-cost gap is then, up to a scalar multiple (the inverse of ρ^i), given by $\limsup q_t (y_{t-1} - y_t - \theta_t) - \lim y_t - \sum_{t=1}^{\infty} \tau(y)_s$, which suggests the following tax scheme.

The lump-sum taxes sequence θ_t is chosen as the tax imposed on the efficient plans y^i of all agents and such that $\sum_{t=1}^{\infty} \theta_t < \infty$.

The tax $\tau_t(y)$ levied at date t upon a plan y of money holdings for which $x(y, \theta, i) \in \ell_+^{\infty}$ consists of the fixed component θ and another component that increases with the "arbitrage" that would be done if the tax were just that fixed part. More precisely,

$$\tau_t(y) \ge \theta_t + \frac{p_t}{\|p\|_1} \left[\limsup (q_t (y_{t-1} - y_t - \theta_t) - \lim y_t - \sum_{t=1}^{\infty} \theta_t \right]^+ . \qquad (6)$$

As we claim below, equilibrium money balances do not need to be taxed (we can set $\theta = 0$), unless we use the lump-sum tax θ to withdraw additional initial holdings A that just had the purpose of making $y \ge 0$ (see subsection 5.2). For such tax policy we have (6) holding with equality[14]. Alternatively, to avoid ever taxing equilibrium money balances, the tax assessment can ignore the cost of carrying on such additional initial holdings A common to all consumers. In this second case, (6) might hold with a strict inequality (see the tax formula (16) reported in subsection 5.2).

It is important to notice also that these non-lump-sum taxes are invariant to changes in money balances at a finite set of dates and, therefore, Euler conditions (5) hold. Moreover, we can make taxes lump-sum up to some date T by replacing the coefficient $p_t/\|p\|_1$ by zero for $t \le T$ and by $p_t / \sum_{s>T} p_s$ otherwise.

[14]Observe that $[\limsup (q_t (y_{t-1} - y_t - \theta_t) - \lim y_t - \sum_{t=s}^{\infty} \theta_s]^+ < \infty$ as $x(y, \theta, i) \in \ell_+^{\infty}$. Hence, τ is well defined.

4.3 Optimal Monetary Policy

Our next result show that the above taxes implement efficient allocations. We focus on efficient allocations that are uniformly bounded away from zero, never attain the infimum in finite time and, at least for some consumer, have a non converging net trade.

Let us formalize our first assumption.

Assumption H: The consumption plan $(x^i)_i$ of agent i is such that $x^i \ggg 0$, $\underline{x}^i$ is never attained and there is a subsequence S of dates such that $x_t - \omega^i_t > 0$ on S, $\lim_S x = \underline{x}^i$ and $\limsup_S(x^i - \omega^i) = \limsup(x^i - \omega^i)$.

This assumption says that the infimum of consumption, never attained in finite time, can be approached along a subsequence where net trades are positive and that the highest asymptotic dishoard occurs precisely along such subsequence. That is, the consumer dishoards more when he is raising the infimum of consumption in the face of very low endowments.

Theorem 1: ***(Efficient Monetary Equilibrium)*** *For preferences given by (3), let $(x^i)_i$ be an efficient allocation such that (i) for each i, x^i satisfies assumption H and (ii) for some agent i, $x^i - W^i$ does not converge. Then, there exist initial holdings y^i_0 that implement $(x^i)_i$ as a monetary equilibrium with taxes.*

(this theorem is proven in Section 5)

Our second theorem says that, in the efficient monetary equilibrium, the money supply cannot go to zero, apart from an exceptional configuration of the AD net trades.

Theorem 2: ***(Non-vanishing Money Supply)*** *Under the assumptions of Theorem 1, impersonal taxes are incompatible with a limiting zero money supply, except in the symmetric case where* $\limsup(x^i - \omega^i)$ *is the same for all agents.*

(this theorem is also proven in Section 5)

Remark 6: When some agents are impatient and the others are wary, the implementation of efficient allocations can be done under the same fiscal policy for all agents or by imposing lump-sum taxes on impatient agents and that policy on the others. The Theorem's assumption that $\limsup(x^i - \omega^i) > 0$ should be imposed only on wary agents. For an implementation with non-vanishing money supply, allowed by AD prices outside of ℓ^1, the consumption plan of impatient agents should not be bounded away from zero (see Araujo, Novinski and Pascoa (2011) for the case of assets paying dividends).

5 The Implementation Argument: Proof of Theorems 1 and 2; Detailed Example

We prove now Theorem 1 and provide also details on the computation of the leading example. We construct an auxiliary economy, with sequential budget constraints as the original economy, but where intertemporal transfers of wealth are achieved by trading a no-dividends asset in constant positive net supply, not subject to taxes but subject to portfolio constraints. We show that AD equilibria can be implemented as sequential equilibria for the auxiliary economy and, then, that the latter are in one-to-one correspondence with sequential equilibria with money and taxes. The tax policy ensures that the portfolio constraints of the auxiliary economy are satisfied.

5.1 An Auxiliary Economy

The sum of the money position and the accumulated taxes, at date t, are the total funds that were put aside (deviated from current consumption) at this date. We can think of this sum as if it were the long position (z_t) on a no-dividends asset in constant positive net supply, subject to transversality constraints. We refer to this asset as the asset z. Positions are related by $z_t = y_t + \sum_{s \leq t} \tau_s^i(y)$. This implies that $z_{t-1}^i - z_t^i = y_{t-1}^i - y_t^i - \tau_t(y)$, which suggests defining an auxiliary sequential economy, whose asset is the portfolio z, with budget constraints as follows

$$x_t - \omega_t^i \leq q_t(z_{t-1} - z_t) \quad \forall\, t \in \mathbb{N}, \tag{7}$$

Let us see how an efficient allocation can be implemented using the portfolio z. By the results in Araujo, Novinski and Pascoa (2011), for a dividends-less asset to implement an AD allocation $(x^i)_i \ggg 0$, the net trades $x^i - \omega^i$ can not converge for all agents (see Proposition 6) and the implementation can not be done by forcing the sequential choice set $B_P(q,\omega^i,z_0^i)$ to be contained in the AD budget set (as had been done in Theorem 2 of Araujo, Novinski and Pascoa (2011))[15].

So the implementation using the portfolio z has to follow a new strategy. A very useful sufficient condition for individual optimality is given as follows[16]. Denote by pch the space of pure charges, that is, the non-summable components of elements in the dual of ℓ^∞ (see Subsection A.1 in the Appendix). We give here the statement with more generality, allowing also for assets paying dividends R, as we will later use this sufficiency condition in context of a Lucas tree. Let $x(z)$ be the consumption plan that a portfolio plan z induces so that (x,z) satisfies $x_t - \omega_t^i \leq q_t(z_{t-1} - z_t) + R_t z_{t-1}$ where $R_1 = 0$.

Lemma 3: *Let z^* be a feasible portfolio and let $x^* = x(z^*)$. (i) Suppose there exists $T \in \partial U(x^*)$ with $T = \mu + \nu$, $\mu \in \ell^1_+$ and $\nu \in \mathrm{pch}_+$ such that, for every t,*

$$\mu_t q_t = \mu_{t+1}\left(R_{t+1} + q_{t+1}\right) \tag{8}$$

and

$$\lim \mu_t q_t z_t^* = \nu(x^* - \omega). \tag{9}$$

(ii) Suppose also that every feasible portfolio z satisfies the condition

$$\lim_t \mu_t q_t z_t \geq \nu(x(z) - \omega), \tag{10}$$

Then z^* is an optimal solution for the problem with constraints (7).

Proof: Given a feasible portfolio z, $U(x(z)) - U(x^*) \leq T(x(z) - x^*) = T(x(z) - \omega) + T(\omega - x^*)$. Moreover, $\mu(x(z) - \omega) = \sum_{t=1}^{\infty}[\mu_t(R_t + q_t)z_{t-1} - \mu_t q_t z_t)]$. By (8), $\mu(x(z) - \omega) = \mu_1 q_1 z_0 - \mu_1 q_1 z_1 + \sum_{t=2}^{\infty}[\mu_{t-1} q_{t-1} z_{t-1} - \mu_t q_t z_t)] = \mu_1 q_1 z_0 - \lim_t \mu_t q_t z_t$. Similarly, $\mu(x^* - \omega) = \mu_1 q_1 z_0 - \lim_t \mu_t q_t z_t^*$. Now by (9), $U(x(z)) - U(x^*) \leq \nu(x(z) - \omega) - \lim_t \mu_t q_t z_t$. Now, by (10), $U(x(z)) - U(x^*) \leq 0$. ∎

The constraint (10) in Lemma 3 can use a supergradient which is not, up to a scalar multiple, equal to the AD price. Let us use a supergradient whose pure charge $\tilde{\nu}^i$ takes the highest value on the direction of the net trade[17]. As shown in

[15] The AD budget set does not have to contain $B_P(q,\omega,z_0)$, even when the latter is convex. This is a consequence of the multiplicity of the generalized limits components of supergradients.

[16] For the moment we ignore the no-short-sales constraint and then we will ensure that the equilibrium that we obtain satisfies this sign constraint.

[17] That is, $\tilde{\nu}^i$ is such that $\delta^- U^i(x^i; x^i - \omega^i) = (\mu^i + \tilde{\nu}^i)(x^i - \omega^i)$, where μ^i is the ℓ^1 component given by $(\partial_t U^i(x))_t$ under the assumptions of the Theorem 1.

Appendix (B), the following property holds for preferences given by (3) under the assumptions of the Theorem 1:

$$\tilde{\nu}^i(x^i - \omega^i) = (u^i)'(\underline{x}^i) \limsup(x^i - \omega^i) \tag{11}$$

This suggests the following portfolio constraint

$$\lim \mu_t^i q_t z_t \geq \alpha^i \limsup(x(z) - \omega^i) \tag{12}$$

where, in the no-dividends case, $x_t(z) = \omega_t^i + q_t(z_{t-1} - z_t)$ and α^i is the norm $\|\tilde{\nu}^i\|$ for some pure charge $\tilde{\nu}^i$ satisfying (11). Let $B^A(q, y_0^i, \omega^i)$ be the set of plans (x, z) satisfying (7), (12) and $z \geq 0$.

DEFINITION: A vector $(q, (x^i, z^i)_{i \in \mathcal{I}}) \in I\!R_+^\infty \times (\ell_+^\infty \times I\!R_+^\infty)^I$ is an *equilibrium for the auxiliary economy* with initial holdings $(z_0^1, ..., z_0^I) = (y_0^1, ..., y_0^I)$ if:

(*a*) $(x^i, z^i) \in \operatorname{argmax}\{U^i(x) : (x, z) \in B^A(q, y_0^i, \omega^i)\}$;

(*b*) $\sum_{i=1}^I x^i = \sum_{i=1}^I \omega^i$;

(*c*) $\sum_{i=1}^I z_t^i = \sum_{i=1}^I y_0^i \ \ \forall\, t \in \mathbb{N}$.

The leading example illustrates the use of (12).

EXAMPLE: DETAILS

Let us go back to the leading example. As we saw, AD equilibrium plans are $x_t^i = 16\frac{t+1}{t}$ and prices π have countably additive component p and pure charge component ν given by $p(y) = \frac{4}{3}\sum_{t=1}^\infty (\frac{1}{2})^{t+2} y_t$ and $\nu(y) = B(y)$. Now $\tilde{\nu}^i(x^i - \omega^i) = \alpha^i \limsup(x^i - \omega^i)$ where $\alpha^i = 3/4$. Constraint(12) becomes $\lim(1/2)^{t+2} q_t z_t \geq \frac{3}{4} \limsup q_t(z_{t-1} - z_t)$.

By Proposition 3 we should find q such that at $(z^i)_i$ implementing $(x^i)_i$ we have $\lim \mu_t^i q_t z_t^i = \tilde{\nu}^i(x^i - \omega^i)$. Now, $x^i(z)$ belongs to the AD budget set if and only if

$$\nu(x^i(z) - \omega^i) - \lim_t p_t q_t z_t^i \leq -z_0^i \lim p_t q_t \tag{13}$$

Moreover, (13) holds with equality if the AD budget equality holds. On the other hand, the first order condition of the AD problem requires[18]

$$\exists \rho^i > 0 : \rho^i \pi \in \partial U^i(x^i), \tag{14}$$

[18]See Zeidler (1984), p.391, Theorem 47.C

Then, $\mu^i + \nu^i = \rho^i(p + \nu)$. So, both requirements are met if for any i

$$\tilde{\nu}^i(x^i - \omega^i) - \nu^i(x^i - \omega^i) = z_0^i \lim \mu_t^i q_t \tag{15}$$

Recall that $p_t q_t = 1$ for any $t \geq 1$. As $\rho^i = 3/4$ for $i = 1, 2$, equations (15) can be rewritten as $\limsup(-G^1) - B(-G^1) = z_0^1$ and $\limsup(G^1) - B(G^1) = z_0^2$, where $B(G^1) = 1$. Since $\limsup(-G^1) = 11$ and $\limsup(G^1) = 13$ we must have $z_0^1 = z_0^2 = 12$. Then, $z_t^i = 12 + \sum_{s=1}^t p_t G_s^i$ and short sales are never done in equilibrium.

In general what can be said? If the pure charge component of the AD supporting price of every agent would already satisfy (11), there would be no room to find a bubble. This can not happen when some agent has a non-converging net trade and the pure charges of all her supergradients have the same norm (the latter holds for the class of preferences given by (3), under the assumptions of Theorem 1). We have actually the following intermediate result (not ensuring yet the non-negativity of portfolios) shown in appendix B,

LEMMA 4: *For preferences given by (3), let $((x^i)_i, \pi)$ be an AD equilibrium such that (i) for each agent i assumption H is satisfied at x^i and (ii) for some agent i, $x^i - \omega^i$ does not converge then, there exist initial holdings z_0^i that implement $(x^i)_i$ as an equilibrium for the auxiliary economy, possibly with short-sales.*

Contrary to what happens with assets paying dividends, the bubble in money is not the AD pure charge evaluated at the dividends stream. What happens instead, so that the AD budget equation holds, is that the bubble in the value of the initial holdings of any agent is the difference between the highest value that a supporting price pure charge (given by Lemma 1) can take at his AD net trades and the value taken by the AD pure charge. When net trades converge, these two values coincide and there is no bubble in fiat money (see Sections 4 and 5 below).

5.2 Mapping back into the original sequential economy

Suppose sequential implementation without taxes was achieved with short sales under the constraint (12). As usual we normalize the AD prices by setting $\alpha = 1$ and can always take also $p_t q_t = 1$. Take the constraint (12) and divide both sides by the Lagrange multiplier ρ^i of the AD budget constraint. We get the requirement $\lim z_t \geq \limsup(x(z) - \omega^i)$ with the equality holding for the equilibrium plan (z^i).

Even if z takes negative values at some dates, we can find money holdings $y_0^i = z_0^i + A$ such that the equilibrium positions z_t^i can be replaced by *non-negative* money balances. To simplify assume that $\theta = 0$. The non-negative plan y^i given by $y_t^i = z_t^i + A$ for $t \geq 0$ with A large enough will be an equilibrium plan if taxes are defined in the following way, still within the class satisfying (6) but possibly with a strict inequality. For any portfolio plan y let

$$\tau_t(y) = (p_t/\|p\|_1) \max\{0, \limsup(q_t(y_{t-1} - y_t)) - \lim y_t + A\} \tag{16}$$

In fact, (9) together with $\limsup(x^i - \omega^i) \geq 0$ (by assumption (i) in the Theorem), imply that z_t^i could be negative just only in a finite number of dates, and as a consequence there exists $A > 0$ such that $z_t^i + A \geq 0$ for all t and all i.

Now, $\sum_{t=1}^{\infty} \tau_t(y) \geq \limsup(q_t(y_{t-1} - y_t)) - \lim y_t + A$. Putting y in one-to-one correspondence with $z = y - A + \tau(y)$, we see that $\sum_{t=1}^{\infty} \tau_t(y) \geq \limsup(x_t(y) - \omega_t^i + q_t\tau_t(y)) - (\lim z + A - \sum_{t=1}^{\infty} \tau_t(y)) + A \geq \limsup(x(y) - \omega^i) - \lim z + \sum_{t=1}^{\infty} \tau_t(y)$. Hence, $\lim z \geq \limsup(x(z) - \omega^i)$. That is, the definition of taxes ensures that any plan y has an image z satisfying constraint (10). As we already knew that (9) holds, it follows that y^i is optimal, for the initial holding $y_0^i = z_0^i + A$, and no taxes are levied in equilibrium. We saw that $(y^i)_i$ manages to implement, under a no-short-sales constraint, the same efficient allocation that $(z^i)_i$ did. This completes the proof of Theorem 1.

Alternatively, we can avoid inserting A in the formula for the non-lump-sum tax but need to consider lump-sum taxes θ such that $\sum_{t=1}^{\infty} \theta_t = A$.

Notice that the constant A is not uniquely defined, it just has a known lower bound. Even if initial holdings $(z_0^i)_i$ are compatible with an implementation using non-negative money balances, we can always increase those initial holdings and then, either take away the excess though lump-sum taxes or keep that extra money as long as out-of-equilibrium plans are taxed taking that extra money into account, as we just described.

The tax formula (16) has the following nice interpretation. Suppose that in order to implement without short selling an AD allocation we need to give to all agents a common extra initial holding of at least A units of money. Then, a money balances plan y will be taxed whenever the marginal benefit at infinity ($\limsup(x(y) - \omega^i)$, of raising a distant infimum of consumption) exceeds the limiting cost ($\lim y_t - A$) of carrying on cash above that common level A. That is, the cost of carrying on that common minimal initial money holding should be ignored in the tax assessment.

Finally, let us prove Theorem 2. Even if equilibrium money balances were taxed with a lump-sum tax θ, it follows from $\lim y^i - A + \sum_{t=1}^{\infty} \theta_t = \limsup(x^i - \omega^i)$ that the impersonal nature of the taxes is compatible with a zero limiting money supply only in the symmetric case where $\limsup(x^i - \omega^i)$ is the same for all agents.

6 On the Implementation in Other Sequential Economies

6.1 Implementation of Efficient Allocations using a Lucas Tree

Could fiat money be replaced by another long lived asset in positive net supply, say a Lucas tree? We consider now an asset with returns in the consumption good at each t, given by $(R_t)_{t\in\mathbb{N}} \in \ell_+^{\infty} \setminus \{0\}$. This asset cannot be shorted. The government would now tax in a different form based on the portfolio of the Lucas tree, these taxes are paid in the *numéraire* since is natural to tax on it in absence of money.

The sequential budget constraint of each agent i is given by:

$$x_t - \omega_t^i \leq q_t (y_{t-1} - y_t) + R_t y_{t-1} - \tau_t(y) \; \forall t \in \mathbb{N}, \tag{17}$$

where $q = (q_t)_{t\in\mathbb{N}}$ is the sequence of Lucas tree prices and τ is the taxation that depends on the Lucas tree positions plan y that the agent may choose. For this economy, the equilibrium is defined analogously to the one considered in the previous sections, adapting the government constraint and the market clearing equations, that must now include the real returns of the Lucas tree. Notice that, as in Araujo, Novinski and Pascoa (2011), AD endowments W^i are now related to sequential endowments ω^i as follows: $W^i = \omega^i + Ry_0^i$.

Let us define an equilibrium for the economy with a Lucas tree and taxes.

Definition: A vector $\left(q, (x^i, y^i)_{i\in\mathcal{I}}\right)$ is an equilibrium for the economy with initial Lucas tree holding $(y_0^i)_{i\in\mathcal{I}}$ and fiscal policy τ when $(x^i, y^i) \in \operatorname{argmax}\{U^i(x) \colon (x, y) \in B\left(q, y_0^i, \omega^i, \tau\right)\}$ and, for every date t, we have

1. $\sum_{i\in\mathcal{I}} x_t^i = \sum_{i\in\mathcal{I}} \omega_t^i + R_t \sum_{i\in\mathcal{I}} y_0^i$,

2. $\sum_{i\in\mathcal{I}} y_t^i = \sum_{i\in\mathcal{I}} y_0^i$,

Note that taxes must be zero in equilibrium, due to item 2.. Since taxes are non-negative, the fiscal policy is in fact a punishment to a deviation from the equilibrium path that takes advantage of the long-run improvement opportunities identified above.

We will see next whether taxes can rule out saving strategies that constitute long-run improvements. We observe first that AD allocations can be implemented if the Lucas tree could be shorted.

Proposition 2: *Let be $(x^i)_i$ be an efficient allocation such that for each i, H holds at x^i. (A) Provided that $\liminf_t (R_t) > 0$, there exist initial holdings $(y_0^i)_i$ of the Lucas tree and fiscal policy τ that implement $(x^i)_i$ as an equilibrium with taxes if the Lucas tree could be shorted. (B) If $(R_t) \geq 0$ and for some agent i, $x_t^i - W_t^i$ does not converge, the same result holds.*

Proposition 2 is proven in Appendix C.

Remark 7: In general and in the absence of other financial instruments, short sales might not be avoided. If we were to create more Lucas trees (increase y_0^i) to overcome such negative positions (as we did in the case of money), then the commodity endowment of each agent in the sequential economy would be reduced, since $\omega_t^i = W_t^i - R_t z_0^i$, and it may happen that the quantity of Lucas tree required to avoid short sales would make ω_t^i become negative.

Remark 8: To avoid short sales we can either (1) impose an additional condition on the AD net trades, such as $\sum_i p_i |x^i - W^i| \leq \beta (u^i)' (\underline{x}) \limsup (x^i - W^i) - \nu^i(x^i - W^i)$, which says that the net trade oscillations are small relative to what is the positive price of money initial holdings (given by the difference between the value that the two pure charges taken on the net trades) or (2) add a *one-period* asset in zero net supply (an I.O.U. promise) that can be shorted[19] at each date t and in this case taxes would depend on the portfolio formed by the Lucas tree and the *one-period* promise. Notice that we do not allow for the I.O.U. promises

[19] With a portfolio constraint to avoid Ponzi schemes.

to be secured by the Lucas tree. In fact, for such collateralized credit, it is not possible to ensure that the markets are sequentially complete, since the collateral constraint could be binding in presence of a low amount of Lucas tree (and we already know that we do not have the freedom to raise its initial holdings).

6.2 Implementation in Stochastic Economies

Let us examine what happens in a stochastic economy. Can fiat money, when properly coupled with other spanning instruments, still implement AD allocations? Or does the coexistence with other assets make money lose its role?

We define a stochastic economy that is a natural generalization of the above deterministic model. Define an event tree such that at each date t and at each node s_t there exist 2 successors of s_t, $s_{t,1}$ and $s_{t,2}$, and denote s_t^- as the predecessor of s_t. Let σ be the root of the event tree and $\mathcal{S} := \{s_t : t \in \mathbb{N}\}$. Denote by P_{s_t} the probability for the successors of s_t.

The utility function of each agent i is a generalization of (3) given by:

$$U^i(x) := \sum_t \zeta_t^i \mathbb{E}_t \left[u^i(x_t)\right] + \beta_i \inf_t \mathbb{E}_t \left[u^i(x_t)\right] \tag{18}$$

where x_t is the consumption of all possible nodes at date t and $\mathbb{E}_t$ is the expected value on $\mathcal{S}_t$, the set of all possible nodes s_t of the date t, with the probability induced by P_{s_t}.

In stochastic economies, wary agents can not be modeled literally as in (3), carrying about the worst outcome on the whole event tree[20]. One possible form, that we follow here, is to suppose that agents are worried about the mean losses at each date, as in (18). Since agents can not know precisely what will be the state that will occur, their concern about losses at distant dates is represented in terms of a concern about the expected value at each date t. This means that there is no aversion to uncertainty among the states, but there is an aversion to ambiguity on the discount factors, as in equation 3.

Let us start by implementing with Lucas trees and I.O.U.s and then we will drop the I.O.U.s and introduce fiat money.

[20]In fact, that might imply that agents would be worried about some states with arbitrarily low probability.

6.2.1 Implementation with Lucas trees and unsecured credit

We now have two Lucas trees in positive net supply and positions $y(j)$, $j = 1, 2$. In the spirit of Remark 8, we allow for trades a in one-period zero-net-supply promises paying an interest rate i_{s_t} in the nodes that immediately follow node s_t. At node s_{t+1} such that $s_{t+1}^{-} = s_t$, the budget constraint and the non-negativity of the Lucas trees constraints are given respectively by:

$$\begin{aligned} x_{s_{t+1}} - \omega_{s_{t+1}} + q_{s_{t+1}} y_{s_{t+1}} + a_{s_{t+1}} + \tau_{s_{t+1}}(y,a) &\leq \left(R_{s_{t+1}} + q_{s_{t+1}}\right) y_{s_t} + (1+i_{s_t}) a_{s_t}, \\ y_{s_{t+1}} &\geq 0, \end{aligned} \tag{19}$$

where $q = (q_{s_t})_{s_t \in \mathcal{S}}$, $(R_{s_t})_{s_t \in \mathcal{S}}$ and $(i_{s_t})_{s_t \in \mathcal{S}}$ are the Lucas trees prices and returns among the set of nodes and the interest rates respectively, $\mathcal{S}$, and τ is the taxation that depends on y and on a if it is used. And denote $B\left(q, y_0^i, \omega^i, \tau\right)$ as the set of (x, y, a) such that satisfy the budget constraint (19).

We define an equilibrium for the economy with Lucas trees and taxes as the natural extension of the Definition 4, with the interest rates i_{s_t} and the promise trades a^i as additional variables, under the condition that the promises' trades clear, $\sum_i a^i_{s_t} = 0$, at each node s_t.

We establish now that we can not implement efficient allocation in a sequential economy with taxes.

Let us reformulate assumption H in the stochastic case.

Assumption H'1: The consumption plan $(x^i)_i$ of agent i is such that $x^i \ggg 0$, $\inf_s \left(\mathbb{E}_s[u^i(x^i_s)]\right) < \mathbb{E}_t[u^i(x^i_t)]$ $\forall t \geq 0$, and there is a subsequence S of dates such that $\mathbb{E}_t[(u^i)'(x_t)(x^i_t - W^i_t)] > 0$ on S and $\limsup_S \mathbb{E}_t[(u^i)'(x_t)(x^i_t - W^i_t)] = \limsup \mathbb{E}_t[(u^i)'(x_t)(x^i_t - W^i_t)]$.

Assumption H'2: $(x^i)_i$ is such that (a) $\liminf_{\left\{t \,:\, \mathbb{E}_t\left[u_i'(x_t)\left(x^i_t - W^i_t\right)\right] > 0\right\}} \mathbb{E}_t[u^i(x^i_t)] = \inf_s \left(\mathbb{E}_s[u^i(x^i_s)]\right)$ and (b) $\lim_t \mathbb{E}_t\left[u^{i\prime}\left(x^i_t\right)\right]$ exists for each i[21].

While H'1+H'2(a) are just the extension of assumption H to the stochastic setting, the hypothesis H'2(b) somehow strengthens it.

The following theorem establishes what can be done with taxes both when the trees are traded alone or together with I.O.U.s that are not secured by the trees,

[21] Part (b) of H'2 can be replaced by the following: there exists $T > 0$ such that for every $t_1,\ t_2 \geq T$ we have that $\zeta^i_{t_1}/\zeta^i_{t_2} = \zeta^j_{t_1}/\zeta^j_{t_2}$ for each pair of agents $i,\ j$.

in which impersonal taxes ensure efficiency. The idea is that equilibrium plans will not taxed but other plans may be penalized. These taxes will eliminate the usual Ponzi schemes (in the zero-net-supply promises) and any other long-run improvement opportunities.

THEOREM 3: ***(Implementability in Unsecured Credit Economies without Money)*** *For preferences given by (3), let $(x^i)_i$ be an efficient allocation such that (i) for each i, x^i satisfies assumptions H'1 and H'2 and (ii) for some agent i, $\mathbb{E}_t\left[u^i\left(x_t^i\right)\left(x_t^i - W_t^i\right)\right]$ does not converge, then, there exist initial holdings of the Lucas trees z_0^i and impersonal taxes that implement $(x^i)_i$ as an equilibrium for the sequential economy, but possibly with trades in the zero-net-supply one period promises (so that short sales of the trees can be avoided).*

Theorem 3 is proven in Appendix C.

REMARK 9: Theorem 3 says that to implement efficient allocations with Lucas trees and taxes, but without money, the Lucas trees would need to be used together with I.O.U. promises (the latter being shorted so that the former are not). Analogously to what was pointed out in the deterministic case, allowing for secured credit, in the form of these promises being collateralized by the Lucas trees, could lead to market incomplete. However, the resulting dependence on unsecured credit, is a fragility of the implementation, due to the full commitment assumed on debtors, which might not be incentive compatible.

6.2.2 Implementation with Fiat Money

Finally, we observe that in stochastic economics, efficient allocations can always be implemented with fiat money. Taxes will be paid in money and markets can be completed sequentially if another assets are added, say two Lucas trees. Money and the Lucas trees have non-negative positions in equilibrium, thanks to the fact that the initial holdings of money can be adjusted. There is no need to allow for trades in zero-net supply promises. Denoting by $y_{s_t} \in \mathbb{R}_+^2$ the positions in the Lucas trees and by z_{s-t} the money balances in state s_t, we write the consumer budget constraint in this state as follows:

$$x_{s_t} - \omega_{s_t} + q_{s_t}^1 y_{s_t} + q_{s_t}^2 z_{s_t} \leq \left(R_{s_t} + q_{s_t}^1\right) y_{s_{t-}} + q_{s_t}^2 z_{s_{t-}} - q_{s_t}^2 \tau_{s_t}(y, z),$$
$$y_{s_t}, z_{s_t} \geq 0,$$

where $q^1_{s_t}, R_{s_t} \in \mathbb{R}^2_+$ are the prices and the returns of the Lucas assets trees, $\tau^i(y,z) \in \mathbb{R}_+$ is the taxation that depends on (y,z) and $q^2_{s_t}$ is the price of money. We suppose that $R = (R^1, R^2)$ is such that for each s_t there exists some s_{t+r} successor of s_t such that $R^1_{s_{t+r}} \neq R^2_{s_{t+r}}$. An equilibrium for this economy is defined analogously to the original deterministic monetary case (again for a government cost assumed to be zero), with market clearing for the two Lucas trees as additional conditions.

THEOREM 4: ***(Coexistence of Fiat Money and Lucas Trees)***

For preferences given by (18), let $(x^i)_i$ be an efficient allocation such that (i) for each i, x^i satisfies assumptions H'1 and H'2 and (ii) for at least one agent i, $\mathbb{E}_t\left[u^{i\prime}\left(x^i_t\right)\left(x^i_t - W^i_t\right)\right]$ does not converge, then, there exist initial holdings y^i_0, z^i_0 of the fiat money and the Lucas trees that manage to implement $(x^i)_i$ as an equilibrium with taxes, non-negative portfolios $(y^i, z^i)_i$ and a non-vanishing money supply.

Theorem 4 is proven in Appendix C.

REMARK 10: Under pure discounting and apart from some special cases, fiat money would lose its efficient role (and its positive price) if other long-lived assets were being added to an economy without frictions that might justify the role of money. Wallace (2014), among many other of his relevant papers on fiat money, addresses the essentiality of money and comments on the difficult coexistence of money and high-return assets. When impatience is replaced by wariness, our results (Proposition 4) show that, coexistence of money and those assets is compatible with efficient monetary equilibria, in robust cases. While no taxes are being levied on the equilibrium money balances, the threat of taxing off-the-equilibrium plans is crucial.

REMARK 11: In stochastic sequential economies as the one that we analyze in this part of the paper, the study of efficient bubbles and the possibility of their crashing in some parts of the tree are quite interesting things to be analyzed. Since the characterization of them can be done in terms of the pure charge of the AD price and the returns of the assets, if some of the latter becomes zero in a subtree, then the former could crash all along that subtree.

7 Concluding Remarks

In this paper we implement sequentially the efficient allocations of economies where wary agents face persistent endowments shocks. These shocks are hedged by trading fiat money (alone in a deterministic setting or together with other long lived assets in the stochastic case). Money balances that deviate from the equilibrium path and would lead to excessive limiting savings are being taxed. If we would dispense with fiat money and implement using Lucas trees as the only long-lived assets, we could face some difficulties. Under non-negative positions in the trees, to get sequential market completeness we might also need zero-net-supply promises. The amount of unsecured credit needed to complete the markets could be quite huge and, presumably, creditors might not be willing to lend it.

Actually, if the implementing asset were a long-lived asset with real returns, there are two extensions that might seem to be natural ways to overcome the dependency on unsecured credit but end up colliding with efficiency. One extension is to allow for the asset to collateralize the short sales of the zero-net-supply promise. The other extension is to allow for short sales of the long lived asset itself in the way that short sales of shares are actually done in financial markets, by borrowing the shares first rather than doing "‘naked"’ short sales. In both cases, it is common to observe frictions that lead to inefficiency. In the former, the collateral constraint could be binding. In the latter, we could have a binding constraint linking the short sale of the shares to the amount of shares that were borrowed[22]. For these reasons, in this paper, by a Lucas tree, we mean the classical notion of a long-lived real asset that can not be shorted and, furthermore, we do not allow it to serve as collateral. In this context, the complementary negative hedging is done through I.O.U. promises.

Fiat money has the merit of dispensing with the problematic role of that unsecured credit (in the form of I.O.U.) in completing the markets. In fact, the initial holdings of money can always be adjusted in order to implement sequentially an efficient allocation using non-negative money balances (alone in a deterministic

[22]Actually, the two cases are often two legs of the same operation and the binding constraint becomes the same. In repo markets, the borrower of shares is a creditor that accepts the shares as collateral for a cash loan.

setting or together with non-negative Lucas tree positions in a stochastic setting). Dispensing with unsecured credit allows us to avoid modeling reputation problems and complex bankruptcy procedures.

Wariness is a lack of impatience that makes consumers care about loses at far away dates. For fiat money to implement sequentially an efficient allocation, the money supply can not go to zero, since wary agents will have a persistent demand for cash to hedge against endowments shocks at far away dates. This optimal positive limit in the money supply is implemented without forcing any money floors or any portfolio constraints at all. We just assume the usual no-short-sales constraint on money together with a tax policy that does not tax the equilibrium plan but taxes plans that lead to excessive savings and, therefore, correct what would be an insatiable demand for precautionary liquidity in a deflationary context (an instance of a problem already noticed by Friedman and Bewley).

APPENDIX

A Notation and Basic Concepts

A.1 The Space ℓ^∞

For $x \in \ell^\infty$, $x \geq 0$ if $x_t \geq 0$ for all $t \in \mathbb{N}$, $x > 0$ if $x \geq 0$ and $x \neq 0$, $x \gg 0$ if $x_t > 0$ for all $t \in \mathbb{N}$, and $x \ggg 0$ if exists $a > 0$ such that $x_t \geq a$ for all $t \in \mathbb{N}$. The space ℓ^∞ is the Banach space *ba* of real bounded sequences equipped with the norm defined by $\|x\| = \sup_t |x_t|$. Its dual is the space ba of bounded finitely additive set functions on $2^{\mathbb{N}}$, also known as charges. Now, *ba* contains strictly ℓ^1, the Banach space of absolutely convergent real sequences equipped with the norm defined by $\|x\|_1 = \sum_{t=1}^{\infty} |x_t|$, since we can associate each $y \in \ell^1$ with some μ in the subspace *ca* of countably additive set functions, by setting $\mu(\{t\}) = y_t$.

A charge $\nu \geq 0$ is a *pure charge* when $[\lambda \in ca_+,\ \nu \geq \lambda \Rightarrow \lambda \equiv 0]$. Denote by pch_+ the set of non-negative pure charges on $(\mathbb{N}, 2^{\mathbb{N}})$. By the Yosida-Hewitt Theorem, any $\mu \in ba_+$ can be written in the form $\mu = \pi + \nu$ where $\mu \in ca_+$ and $\nu \in pch_+$. and this decomposition is unique.

Remark 12: If $\nu > 0$ be a pure charge such that $\nu(\mathbb{1}) = 1$, then, $\nu(x) \in [\liminf x, \limsup x]$, for any $x \in \ell^\infty$. In other words, ν is a generalized limit.

For a supergradient [23] of a concave function $U : \ell^\infty_+ \to I\!R$ at x, which is an element in the dual space, we can actually say more about the norm of its pure charge component: $\|\nu\|_{ba} \equiv \sup\{\nu(x) : \|x\| \leq 1\} = \nu(1\!\!1)$ belongs to $[\lim_n \delta^+ U(x; 1\!\!1^n), \lim_n \delta^- U(x; 1\!\!1^n)]$. The set of all supergradients of U at x is called the superdifferential of U at x and is denoted by $\partial U(x)$.

A.2 General Characterization of Supergradients for the Utility Function (3)

If T is a supporting price of U^i at x^i, then $T(a) = \sum_{t+1}^{\infty} u'(x_t^i)(\zeta_t^i + \gamma_t \beta^i) a_t + \sigma \beta^i u'(\underline{x^i}) \mathrm{LIM}^T(a)$, where (i) $\gamma_t \geq 0$, (ii) $\gamma_t = 0$, if $x_t > \underline{x}$, (iii) $\sigma \geq 0$ is zero when $\underline{x^i}$ is not a cluster point of the sequence x^i and (iv) $\sum_{t=1}^{\infty} \gamma_t + \sigma = 1$. For a proof see Araujo, Novinski and Pascoa (2011).

B On Fiat Money and the Marginal Utility in the Direction of Net Trades

We show here that for a utility function U of the form given by (3), if z^* is an optimal portfolio plan in $B^A(q, y_0^i, \omega^i)$ (defined in Subsection 5.1) such that, at $x^* := x(z^*) \ggg 0$, we have $\inf x^*$ not attained and $\lim_s x_s^* = \inf_s x_s^*$, then

$$\delta^- U(x^*)(x^*; x^* - \omega^i) = \mu(x^* - \omega^i) + \alpha \limsup(x^* - \omega^i)$$

for $\alpha > 0$ equal to the norm of the pure charge component of a supergradient of U at x^* (see Remark 12), where μ is given by $\mu_t = \zeta_t u'(x_t^*)$.

We will estimate $\lim_{r \to 0} \frac{1}{r}[U \circ x(z^* + rz^*) - U \circ x(z^*)]$. Consider the direction $\Delta \in \ell^\infty$ given by $\Delta_t = q_t z_{t-1}^* - q_t z_t^*$. Notice that $\lim_{r \to 0} \frac{1}{r} \sum_{t \geq 1} \zeta_t [u(x_t^* + r\Delta_t) - u(x_t^*)] = \sum_{t \geq 1} \zeta_t \lim_{r \to 0} \frac{1}{r}[u(x_t^* + r\Delta_t) - u(x_t^*)] = \sum_{t \geq 1} \zeta_t u'(x_t^*) \Delta_t$. So, what we still need to do is to estimate $\lim_{r \uparrow 0} \frac{1}{r} \beta[\inf_t u(x_t^* + r\Delta_t) - \inf_s u(x_s^*)]$, which is $\delta^- \inf_t u(x^*, \Delta)$, the left-derivative of the function $\inf_t u(.)$ along the direction Δ evaluated at x^*.

Observe that there exists $\chi > 0$ such that $\forall r \in (-\chi, 0)$ the following holds: $(1 + r)z^* > 0$ is a non-negative plan, $x(z^* + rz^*)$ satisfies (7) and $x(z^* + rz^*) = x^* + r(x^* - \omega) \ggg 0$.

[23] any T such that $U(x + h) - U(x) \leq Th$, for any $h \in \ell^\infty$

CLAIM: $\lim_{r\uparrow 0}\frac{1}{r}[\inf_t u(x_t^*+r\Delta_t)-\inf_t u(x_t^*)] = u'(\underline{x}^*)\limsup_t \Delta_t$

PROOF: Let us denote $\underline{x}^* = \inf x^*$. There exists $\lim_{r\uparrow 0}\frac{1}{r}[\inf_t u(x_t^*+r\Delta_t)-u(\underline{x}^*)]$ since $\inf(.) : \ell^\infty \to I\!R$ is a concave function.

Fixed $r \in (-\chi, 0)$ and given $\epsilon > 0$, it is valid for all τ large enough that $(1/r)[\inf_t u(x_t^*+r\Delta_t)-u(\underline{x}^*)]+\epsilon = (-1/r)[u(\underline{x}^*)-\epsilon r-\inf_t u(x_t^*+r\Delta_t)] \geq (-1/r)[u(x_\tau^*)-u(x_\tau^*+r\Delta_\tau)] \geq u'(x_\tau^*)\Delta_\tau$. Making $\tau\to\infty$ we get $(1/r)[\inf_t u(x_t^*+r\Delta_t)-u(\underline{x}^*)]+\epsilon \geq \limsup_t u'(x_t^*)\Delta_t = u'(\underline{x}^*)\limsup_t\Delta_t$, for an arbitrary $\epsilon > 0$.

To prove the reverse inequality, notice that, under the hypothesis, $\delta U(x^*;1\!\!1^n) = \sum_{t>n}\zeta^t u'(x_t^*)+\beta u'(\underline{x}^*)$ and, therefore, any supergradient has a pure charge component with norm $\beta u'(\underline{x}^*)$ by Remark 12. Hence, for any supergradient T of U at x^* we have $T(\Delta) = \sum_{t\geq 1}\zeta_t u'(x_t^*)\Delta_t + \beta u'(\underline{x}^*)\text{LIM}(\Delta)$, for some generalized limit LIM. So, $\delta^-\inf_t u(x^*,\Delta) \leq u'(\underline{x}^*)\limsup_t\Delta_t$ ∎

Even if x_t^* doesn't converge to $\inf x^*$, the previous claim stills holds if we have $\liminf_{\{t\,:\,x_t^i-W_t^i>0\}} x_t^i = \underline{x}$ and fact that $\limsup_{\{t\,:\,x_t^i-W_t^i>0\}}\Delta_t = \limsup\Delta_t$. We close this subsection with the proof of Proposition 4.

PROOF OF LEMMA 4:

It suffices to find q and $(z_0^i)_i$ so that (13) and $\lim \mu_t^i q_t z_t^i = \tilde{\nu}^i(x^i-\omega^i)$ hold at $(z^i)_i$ implementing $(x^i)_i$ (see Proposition 3) . These hold if (15) holds for any i (as by (14), $\mu^i+\nu^i = \rho^i(p+\nu)$). Let us see that (15) (whose left hand sides are non-negative) has a solution $b \equiv \lim p_t q_t > 0$, for some $(z_0^i)_i > 0$ (allowing us to make $q_t = b(p_t)^{-1}$).

We just have to rule out that $\tilde{\nu}^i(x^i-\omega^i) = \nu^i(x^i-\omega^i)$ for all i, which, implies that $\|\nu^i\|_{ba} = \alpha^i$. Now, $\nu^i = \rho^i\nu$, $\nu = \alpha$LIM and AD prices can be normalized so that $\alpha = 1$. Hence, $\alpha^i = \rho^i$ and we get $\nu(x^i-\omega^i) = \limsup(x^i-\omega^i)$, for any i. Adding across agents, $0 = \sum_i \limsup(x^i-\omega^i)$. Say it is agent 1 whose net trade $x^1-\omega^1$ does not converge. Now, $\limsup(x^1-\omega^1) = -\sum_{i\neq 1}\limsup(x^i-\omega^i) = \sum_{i\neq 1}\liminf(\omega^i-x^i) \leq \liminf(x^1-\omega^1)$, a contradiction. ∎

C PROOFS OF SECTION 6.

PROOF OF PROPOSITION 2:

We start by implementing the efficient allocation in an auxiliary economy without taxes but with portfolio constraints.

Consider $B^A(q, y_0^i, \omega^i)$ the set of plans (x, z) satisfying $x_t - \omega_t^i \leq q_t(z_{t-1} - z_t) + R_t z_{t-1}$ $\forall t \in \mathbb{N}$. By Lemma 3 and assumption H (similarly to what we did in section 5.1), we have $\rho^i \lim p_t q_t z_t^i$ being equal to the highest value that a pure charge $\tilde{\nu}^i$ of a super-gradient can take on $x^i - W^i$ and (normalizing the AD price so that $\alpha = 1$) we use the portfolio constraint $\lim p_t q_t z_t \geq \limsup(x(z) - \omega^i)$, where $x_t(z) = \omega_t^i + q_t(z_{t-1} - z_t) + R_t z_{t-1}$.

Also similarly, $x^i(z)$ satisfies the AD budget equation if and only if $\nu(x^i(z) - \omega^i) - \lim_t p_t q_t z_t^i = z_0^i(\nu(R) - \lim p_t q_t)$, equivalently,

$$(\rho^i)^{-1}\tilde{\nu}^i\left(x^i - \omega^i\right) - \nu\left(x^i - \omega^i\right) = z_0^i\left(p_1 q_1 - \sum_{i=1}^{\infty} p_i R_i - \nu(R)\right) \tag{20}$$

If $x^i - \omega^i$ converges for every agent, we choose q_1 such that $p_1 q_1 = \nu(R) + \sum_{i=1}^{\infty} p_i R_i$ and $z_0^i \geq 0$ such that $M z_0^i < \underline{W}^i$ where $\underline{W}^i = \inf_t\{W_t^i\}$. If $x^i - \omega^i$ does not converge for some i, we choose q_1 big enough so that $p_1 q_1 > \nu(R) + \sum_{i=1}^{\infty} p_i R_i$ and z_0^i satisfying the condition above and $M z_0^i < \underline{W}^i$.

Now let us define a constant sequence of taxes by

$$\tau_t(y) := \left(\frac{1}{1 + \|\tilde{q}\|_1 \gamma}\right)\left(\left(\underline{\beta}^{-1}\limsup(q_t(y_{t-1} - y_t) + R_t y_{t-1}) - \lim_t y_t\right) \vee 0\right)$$

where $\gamma := \prod_{t=1}^{\infty}\left(\frac{R_t}{q_t} + 1\right)$, $(\tilde{q}_t)_t = (1/q_t)_t \in \ell^1$ and $\underline{\beta} := p_1 q_1 - \sum_{t=1}^{\infty} p_t$.

And the relationship between y and z is given by:

$$z_t - y_t = \sum_{r=1}^{t} \frac{\prod_{s=0}^{t-r-1}(q_{t-s} + R_{t-s})}{\prod_{j=0}^{t-s}(q_{t-s})}\tau(y) = \sum_{r=1}^{t} \frac{\tau(y)}{q_r}\left(\prod_{s=0}^{t-r-1}\left(1 + \frac{R_{t-s}}{q_{t-s}}\right)\right)$$

And making the proper substitutions we have $\sum_{r=1}^{\infty} \frac{\beta\tau(y)}{q_r}\gamma \geq \limsup(x(y) - \omega^i) - \lim_t p_t q_t y_t$ with equality with $y = y^i$, where y^i is the asset portfolio, which implements the AD allocation with taxes τ. ■

Proof of Theorem 4:
We start by characterizing the supergradients of the utility (18).

Lemma 5: *Consider a consumption plan $x^* \ggg 0$ such that $\inf_{s\geq 1}\mathbb{E}_s[u(x_s^*)] > \mathbb{E}_t[u(x_t^*)]$ for all $t \geq 0$ and $\mathbb{E}_t[u(x_s^*)] \to \inf_{s\geq 1}\mathbb{E}_s[u(x_s^*)]$.*

$\pi \in \partial U(x^*)$ if and only if it is given by

$$\pi(x) = \sum_{t\geq 0}\zeta^t \mathbb{E}_t[u'(x_t^*)\cdot x_t] + \beta\,\nu\left((\mathbb{E}_t[u'(x_t^*)\cdot x_t])_{t\geq 0}\right)$$

where $\nu \in pch(\ell^\infty)$ such that $\|\nu\|= 1$[24].

PROOF: It is enough to show that, given $x \in \ell^\infty(\mathcal{S})$,

$$\inf_t \mathbb{E}_t[u(x_t)] - \inf_t \mathbb{E}_t[u(x^*_t)] \ \leq \ \nu\left((\mathbb{E}_t[u'(x^*_t)\cdot(x_t - x^*_t)])_{t\geq 0}\right).$$

Given $\varepsilon > 0$, we have, for $t_1 > 0$ large enough,

$$\inf_t \mathbb{E}_t[u(x_t)] - \inf_t \mathbb{E}_t[u(x^*_t)] - \varepsilon \ < \ \mathbb{E}_{t_1}[u(x_{t_1})] - \mathbb{E}_{t_1}[u(x^*_{t_1})] \ \leq \ \mathbb{E}_{t_1}[u'(x^*_{t_1})\cdot(x_{t_1} - x^*_{t_1})].$$

Making $t_1 \to \infty$, we get $\inf_t \mathbb{E}_t[u(x_t)] - \inf_t \mathbb{E}_t[u(x^*_t)] - \varepsilon \ \leq \ \liminf_t \mathbb{E}_t[u'(x^*_t)\cdot(x_t - x^*_t)]$. Now $\|\nu\|= 1$ implies $\nu(z) \geq \liminf z \ \forall z \in \ell^\infty$ and ε is arbitrary.

To prove the other part of the lemma, let us use some results of nonsmooth analysis (see Clarke (1990)) for $U(x) = V \circ \phi(x)$ where $\phi : \ell^\infty(\mathcal{S}) \to \ell^\infty$ and $V : \ell^\infty$ are given by $x \mapsto (\phi_t(x))_{t\in\mathbb{N}} := (\mathbb{E}_t\left[u\left(x_t\right)\right])_{t\in\mathbb{N}}$ and $y \mapsto V(y) := \sum_{t\geq 1}\delta_t y_t + \beta \inf_t y_t$.

Since U is concave and Lipschitz[25] close to x^* (since $x^* \ggg 0$), we have that $\partial_c U\left(x^*\right) = \partial U\left(x^*\right)$ (see page 36 proposition 2.2.7), where $\partial_c F(y)$ is the Clarke subdifferential, see page 10. Notice that V have also the same property.

And since ϕ is Lipschitz close to x^*, we have that ϕ is strictly differentiable (see page 30 proposition 2.2.4). And as a consequence of the Chain Rule (see page 45 proposition 2.3.10), we have that $\partial U(x) \subseteq \partial V(\phi(x^*))\circ\phi'(x^*)$ which concludes the proof. ■

Let us assume that $p_1 q_1 = 1$ and $\alpha = 1$ and we continue by presenting a sufficient condition for individual optimality:

LEMMA 6: *Let $(\tilde{y}^*, \tilde{z}^*)$ be a feasible portfolio and let $x^* = x\left(\tilde{y}^*, \tilde{z}^*\right)$. If there is $T \in \partial U(x^*)$ with $T = \mu + \nu$, $\mu \in \ell^1_+$ and $\nu \in pch_+$ such that for every node s_t and both promises $j = 1, 2$, $\mu_{s_t} q^1_{s_t}(j) = \sum_{s^-_{t+1}=s_t}\mu_{s_{t+1}}\left(R_{s_t}(j) + q^1_{s_t}(j)\right)$ and $\mu_{s_t} q^2_{s_t} = \sum_{s^-_{t+1}=s_t}\mu_{s_{t+1}} q^2_{s_t}$ and $\lim_t\left(\sum_{s_t\in\mathcal{S}_t}\left[\mu_{s_t} q^1_{s_t}\tilde{y}^*_{s_t} + \mu_{s_t} q^2_{s_t}\tilde{z}^*_{s_t}\right]\right) = \nu\left(x^* - \omega\right)$. Suppose also that every feasible portfolio $(\tilde{y}, \tilde{z})$ satisfies the constraint*

$$\lim_t\left(\sum_{s_t\in\mathcal{S}_t}\left[\mu_{s_t} q^1_{s_t}\tilde{y}_{s_t} + \mu_{s_t} q^2_{s_t}\tilde{z}_{s_t}\right]\right) \geq \nu\left(x\left(\tilde{y}, \tilde{z}\right) - \omega\right).$$

Then $(\tilde{y}^*, \tilde{z}^*)$ is an optimal solution for the consumption problem with sequential budget constraints.

[24]The right part of the π is a bounded functional on ℓ^∞ due to $x \ggg 0$.

[25]In the sup-norm.

As usual, implementation follows by imposing the constraint in this proposition and choosing $(z_0^i)_i$ such that the AD budget equation holds, equivalently, $(\rho^i)^{-1}\tilde{\nu}^i\left(x^i-\omega^i\right)-\nu\left(x^i-\omega^i\right)=\tilde{y}_0^i\left(\lim_t \sum_{s_t}\mu_{s_t}q^1_{s_t}-\nu(R)\right)+\tilde{z}_0^i \lim_t \sum_{s_t}{\mu^i}_{s_t}q^2_{s_t}$, where $\tilde{\nu}^i\left(x^i-\omega^i\right)=\limsup_{\left\{\{t_r\}_r\,:\,\lim_r \mathbb{E}_{t_r}\left[u^i\left(x_{s_{t_r}}\right)\right]=\inf_t \mathbb{E}_t\left[u^i\left(x_{s_t}\right)\right]\right\}}\mathbb{E}_t\left[u_i'(x^i_{s_t})\left(x^i_{s_t}-\omega^i_{s_t}\right)\right]$.

Due to H'2, the pure charge $\tilde{\nu}^i$ in the direction of the net trade satisfies

$$
\begin{aligned}
\tilde{\nu}^i\left(x(\tilde{y}^i,\tilde{z}^i)-\omega^i\right) &= \limsup_{\left\{\{t_r\}_r\,:\,\lim_r \mathbb{E}_{t_r}\left[u^i\left(x_{s_{t_r}}\right)\right]=\inf_t \mathbb{E}_t\left[u^i\left(x_{s_t}\right)\right]\right\}} \mathbb{E}_t[(u^{i\prime}(x^i_{s_t})(x_{s_t}(\tilde{y}^i,\tilde{z}^i)-\omega^i_{s_t}))] \\
&= \limsup \mathbb{E}_t[(u^{i\prime}(x^i_{s_t})(x_{s_t}(\tilde{y}^i,\tilde{z}^i)-\omega^i_{s_t}))].
\end{aligned}
$$

From the Euler equations we have, for each i, that $\frac{u^{i\prime}(x^i_{s_t})}{\mathbb{E}_t\left[u^{i\prime}(x^i_{s_t})\right]}=\frac{p_{s_t}}{\mathbb{E}_t\left[p_{s_t}\right]}$ and since $\lim_t \mathbb{E}_t\left[u^{i\prime}\left(x^i_{s_t}\right)\right]$ exists, $\tilde{\nu}^i\left(x(\tilde{y}^i,\tilde{z}^i)-\omega^i\right)=\alpha^i \limsup \mathbb{E}_t\left[\frac{p_{s_t}}{\mathbb{E}_t\left[p_{s_t}\right]}\left(x_{s_t}\left(\tilde{y},\tilde{z}\right)-\omega^i_{s_t}\right)\right]$ where $\alpha^i:=\lim_t \mathbb{E}_t\left[u^{i\prime}\left(x^i_{s_t}\right)\right]$, and without loss of generality we have $\alpha^i=\alpha^j=\alpha=1$.

The no short sale constraints of money and the Lucas trees are satisfied by adding an extra money holding A at $t=0$ to all agents[26].

Therefore, let us define $p_t=\sum_{s_t}p_{s_t}$ and the taxes as

$$
\begin{aligned}
\tau_{s_t}(y,z)=\frac{p_t}{\|p\|_1}\max\Bigg\{0, \limsup \mathbb{E}_t\Bigg[\frac{p_{s_t}}{\mathbb{E}_t\left[p_{s_t}\right]}\Big(q^1_{s_t}\left(y_{s_{t-1}}-y_{s_t}\right)+R_{s_t}y_{s_{t-1}} \\
+q^2_{s_t}\left(z_{s_{t-1}}-z_{s_t}\right)\Big)\Bigg]-\lim_t\left(\sum_{s_t\in\mathcal{S}_t}\left[p_{s_t}q^1_{s_t}y_{s_t}+p_{s_t}q^2_{s_t}z_{s_t}\right]\right)+A\Bigg\}.
\end{aligned}
$$

The argument on non-vanishing money supply replicates the deterministic argument in the proof of Lemma 4[27]. ■

Proof of Theorem 3:

The proof is close to the proof of Theorem 4 in terms of the optimality conditions and close to the proof of Proposition 2 in terms of the constants that are introduced to implement the efficient allocation. Let us look first at the case where the zero-net-supply promises are available. As a preliminary step, we obtain the following

[26] Due to the indeterminacy produced by the three assets in the economy, large money holding at $t=0$ can prevent short sales also for the Lucas tree.

[27] Efficient taxes exist also in the context of the alternative to (b) in H'2 mentioned in the footnote to H'2. In fact, defining $\alpha^i := \frac{u^{i\prime}(x^i_{s_T})}{p_{s_T}}$ for each i and any s_T, we have $\tilde{\nu}^i\left(x(\tilde{y}^i,\tilde{z}^i)-\omega^i\right)=\alpha^i\limsup \mathbb{E}_t\left[p_{s_t}\left(x_{s_t}\left(\tilde{y}^i,\tilde{z}^i\right)-\omega^i_{s_t}\right)\right]$, and then, similarly as before, , we have $\alpha^i=\alpha^j=\alpha=1$ and taxes can be defined.

personal taxes:

$$\tau^i_{s_t}(y) = \frac{1}{\alpha^i \underline{\beta}^i + \gamma^i} \max \left\{ 0, \limsup_{\{\{t_r\}_r \,:\, \lim_r \mathbb{E}_{t_r}\left[u^i\left(x_{s_{t_r}}\right)\right] = \inf_t \mathbb{E}_t\left[u^i\left(x_{s_t}\right)\right]\}} \mathbb{E}_t \Big[u_i'(x^i_t) \Big(q_{s_t} \left(y_{s_{t-1}} - y_{s_t} \right) + R_{s_t} y_{s_{t-1}} \Big) \Big] - \lim_t \left(\sum_{s_t} \mu^i_{s_t} q_{s_t} y_{s_t} \right) \right\}$$

where $\alpha^i = \limsup_{\{\{t_r\}_r \,:\, \lim_r \mathbb{E}_{t_r}\left[u^i\left(x_{s_{t_r}}\right)\right] = \inf_t \mathbb{E}_t\left[u^i\left(x_{s_t}\right)\right]\}} \mathbb{E}_t \left[u^{i'} \left(x^i_{s_t} \right) \right]$, $\underline{\beta}^i = \mu^i_{1,1} q_{1,1} - \sum_{s_t} \mu^i_{s_t} R_{1,s_t}$, and $\gamma^i = \sum_{s_j} \left[\Pi_{i=j+1}^{\infty} \left(1 + \frac{R_{1,s_i}}{q_{1,s_i}} \right) / q_{1,s_j} \left(\lim_{t\to\infty} \sum_{\tilde{s} \,:\, \tilde{s}_t^{-(t-j)} = s_j} \mu^i_{1,s_t} q_{1,s_t} \right) \right]$. Under assumption H'2, these taxes can be rewritten in an impersonal way, by an argument similar to the one used in the proof of Theorem 4.

And if I.O.U.s are added in order to complete markets, we can define optimality conditions, similarly to Lemma 6, which allow us to define a fiscal policy τ to avoid the *long-run improvement opportunities*, under a no short sales constraint on the Lucas trees. ∎

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